SUPERADDITIVITY AND MONOTONICITY OF SOME FUNCTIONALS ASSOCIATED WITH THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS IN LINEAR SPACES

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Abstract. The superadditivity and monotonicity properties of some functionals associated with convex functions and the Hermite-Hadamard inequality in the general setting of linear spaces are investigated. Applications for norms and convex functions of a real variable are given. Some inequalities for arithmetic, geometric, harmonic, logarithmic and identric means are improved.

1. Introduction

For any convex function we can consider the well-known inequality due to Hermite and Hadamard. It was first discovered by Ch. Hermite in 1881 in the journal Mathesis (see [7]). Hermite mentioned that the following inequality holds for any convex function \( f \) defined on \( \mathbb{R} \)

\[
(b - a)f\left(\frac{a + b}{2}\right) < \int_a^b f(x)dx < (b - a)\frac{f(a) + f(b)}{2}, \quad a, b \in \mathbb{R}.
\]

But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result [8]. E.F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D.S. Mitrinović found Hermite’s note in Mathesis [7]. Since (1.1) was known as Hadamard’s inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [8].

Let \( X \) be a vector space, \( x, y \in X, \ x \neq y \). Define the segment \([x, y]\) := \{(1 - t)x + ty, \ t \in [0, 1]\}. We consider the function \( f : [x, y] \to \mathbb{R} \) and the associated function \( g(x, y) : [0, 1] \to \mathbb{R}, \ g(x, y)(t) := f[(1 - t)x + ty], \ t \in [0, 1] \). Note that \( f \) is convex on \([x, y]\) if and only if \( g(x, y) \) is convex on \([0, 1]\).

For any convex function defined on a segment \([x, y] \subset X\), we have the Hermite-Hadamard integral inequality (see [2, p. 2], [3, p. 2])

\[
f\left(\frac{x + y}{2}\right) \leq \int_0^1 f[(1 - t)x + ty]dt \leq \frac{f(x) + f(y)}{2},
\]

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function \( g(x, y) : [0, 1] \to \mathbb{R} \).
Since \( f(x) = \|x\|^p \) (\( x \in X \) and \( 1 \leq p < \infty \)) is a convex function, we have the following norm inequality from (1.2) (see [6, p. 106])

\[
(1.3) \quad \left\| \frac{x + y}{2} \right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2},
\]

for any \( x, y \in X \). Particularly, if \( p = 2 \), then

\[
(1.4) \quad \left\| \frac{x + y}{2} \right\|^2 \leq \int_0^1 \|(1-t)x + ty\|^2 dt \leq \frac{\|x\|^2 + \|y\|^2}{2},
\]

holds for any \( x, y \in X \). We also get the following refinement of the triangle inequality when \( p = 1 \)

\[
(1.5) \quad \left\| \frac{x + y}{2} \right\| \leq \int_0^1 \|(1-t)x + ty\| dt \leq \frac{\|x\| + \|y\|}{2}.
\]

2. Some Functional Properties

Consider a convex function \( f : C \subset X \to \mathbb{R} \) defined on the convex subset \( C \) in the real linear space \( X \) and two distinct vectors \( x, y \in C \). We denote by \( [x, y] \) the closed segment defined by \( \{(1-t)x + ty, \ t \in [0,1]\} \). We also define the functional

\[
(2.1) \quad \Psi_f(x, y; t) := (1-t)f(x) + tf(y) = f((1-t)x + ty) \geq 0
\]

where \( x, y \in C \) and \( t \in [0,1] \).

**Theorem 1.** Let \( f : C \subset X \to \mathbb{R} \) be a convex function on the convex set \( C \). Then for each \( x, y \in C \) and \( z \in [x, y] \) we have

\[
(2.2) \quad (0 \leq) \Psi_f(x, z; t) + \Psi_f(z, y; t) \leq \Psi_f(x, y; t)
\]

for each \( t \in [0,1] \), i.e., the functional \( \Psi_f(\cdot, \cdot; t) \) is superadditive as a function of interval.

If \([z, u] \subset [x, y] \), then

\[
(2.3) \quad (0 \leq) \Psi_f(z, u; t) \leq \Psi_f(x, y; t)
\]

for each \( t \in [0,1] \), i.e., the functional \( \Psi_f(\cdot, \cdot; t) \) is nondecreasing as a function of interval.

**Proof.** Let \( z = (1-s)x + sy \) with \( s \in (0,1) \). For \( t \in (0,1) \) we have

\[
\Psi_f(z, y; t) = (1-t)f((1-s)x + sy) + tf(y) = f((1-t)(1-s)x + sy) + ty)
\]

and

\[
\Psi_f(x, z; t) = (1-t)f(x) + tf((1-s)x + sy) - f((1-t)x + t(1-s)x + sy))
\]

giving that

\[
(2.4) \quad \Psi_f(x, z; t) + \Psi_f(z, y; t) - \Psi_f(x, y; t)
\]

\[
= f((1-s)x + sy) + f((1-t)x + ty)
\]

\[
- f((1-t)(1-s)x + t(1-s)x + sy)) - f((1-t)sx + tsy).
\]

Now, for a convex function \( \phi : I \subset \mathbb{R} \to \mathbb{R} \), where \( I \) is an interval, and any real numbers \( t_1, t_2, s_1 \) and \( s_2 \) from \( I \) and with the properties that \( t_1 \leq s_1 \) and \( t_2 \leq s_2 \) we have that

\[
(2.5) \quad \frac{\phi(t_1) - \phi(t_2)}{t_1 - t_2} \leq \frac{\phi(s_1) - \phi(s_2)}{s_1 - s_2}.
\]
Indeed, since \( \varphi \) is convex on \( I \) then for any \( a \in I \) the function \( \psi : I \setminus \{a\} \to \mathbb{R} \)

\[
\psi(t) := \frac{\varphi(t) - \varphi(a)}{t - a}
\]
is monotonic nondecreasing where is defined. Utilising this property repeatedly we have

\[
\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} \leq \frac{\varphi(s_1) - \varphi(t_2)}{s_1 - t_2} = \frac{\varphi(t_2) - \varphi(s_1)}{t_2 - s_1} \leq \frac{\varphi(s_2) - \varphi(s_1)}{s_2 - s_1} = \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}
\]

which proves the inequality (2.5).

Consider the function \( \varphi : [0, 1] \to \mathbb{R} \) given by \( \varphi(t) := f((1 - t)x + ty) \). Since \( f \) is convex on \( C \) it follows that \( \varphi \) is convex on \([0, 1] \). Now, if we consider for given \( t, s \in (0, 1) \)

\[
t_1 := ts < s =: s_1 \text{ and } t_2 := t < t + (1 - t) s =: s_2,
\]

then we have

\[
\varphi(t_1) = f((1 - ts)x + tsy), \varphi(t_2) = f((1 - t)x + ty)
\]
giving that

\[
\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} = \frac{f((1 - ts)x + tsy) - f((1 - t)x + ty)}{t(s - 1)}.
\]

Also

\[
\varphi(s_1) = f((1 - s)x + sy), \varphi(s_2) = f((1 - t)(1 - s)x + [(1 - t)s + t]y)
\]
giving that

\[
\frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2} = \frac{f((1 - s)x + sy) - f((1 - t)(1 - s)x + [(1 - t)s + t]y)}{t(s - 1)}.
\]

Utilising the inequality (2.5) and multiplying with \( t(s - 1) < 0 \) we deduce the inequality

\[
(2.6) \quad f((1 - ts)x + tsy) - f((1 - t)x + ty) \geq f((1 - s)x + sy) - f((1 - t)(1 - s)x + [(1 - t)s + t]y).
\]

Finally, by (2.4) and (2.6) we get the desired result (2.2).

Applying repeatedly the superadditivity property we have for \([z, u] \subset [x, y]\) that

\[
\Psi_f(x, z; t) + \Psi_f(z, u; t) + \Psi_f(u, y; t) \leq \Psi_f(x, y; t)
\]
giving that

\[
0 \leq \Psi_f(x, z; t) + \Psi_f(u, y; t) \leq \Psi_f(x, y; t) - \Psi_f(z, u; t)
\]

which proves (2.3). \( \square \)

For \( t = \frac{1}{2} \) we consider the functional

\[
\Psi_f(x, y) := \Psi_f \left( x, y; \frac{1}{2} \right) = \frac{f(x) + f(y)}{2} - f \left( \frac{x + y}{2} \right),
\]
which obviously inherits the superadditivity and monotonicity properties of the functional $\Psi_f (\cdot, \cdot; t)$. We are able then to state the following

**Corollary 1.** Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set $C$ and $x, y \in C$. Then we have the bounds

\[
\begin{align*}
\inf_{z \in [x,y]} \left[ f \left( \frac{x + z}{2} \right) + f \left( \frac{z + y}{2} \right) - f(z) \right] &= f \left( \frac{x + y}{2} \right) \\
\sup_{z,u \in [x,y]} \left[ \frac{f(z) + f(u)}{2} - f \left( \frac{z + u}{2} \right) \right] &= \frac{f(x) + f(y)}{2} - f \left( \frac{x + y}{2} \right).
\end{align*}
\]

**Proof.** By the superadditivity of the functional $f (\cdot, \cdot)$ we have for each $z \in [x, y]$ that

\[
\frac{f(x) + f(y)}{2} - f \left( \frac{x + y}{2} \right) \geq \frac{f(x) + f(z)}{2} - f \left( \frac{x + z}{2} \right) + \frac{f(z) + f(y)}{2} - f \left( \frac{z + y}{2} \right),
\]

which is equivalent with

\[
\begin{align*}
f \left( \frac{x + z}{2} \right) + f \left( \frac{z + y}{2} \right) - f(z) &= \frac{f(x) + f(y)}{2}.
\end{align*}
\]

Since the equality case in (2.9) is realized for either $z = x$ or $z = y$ we get the desired bound (2.7).

The bound (2.8) is obvious by the monotonicity of the functional $f (\cdot, \cdot)$ as a function of interval.  

Consider now the following functional

\[
\Gamma_f (x, y; t) := f (x) + f (y) - f \left((1-t)x+ty\right) - f \left((1-t)y+tx\right),
\]

where, as above, $f : C \subset X \to \mathbb{R}$ is a convex function on the convex set $C$ and $x, y \in C$ while $t \in [0, 1]$.

We notice that

\[
\Gamma_f (x, y; t) = \Gamma_f (y, x; t) = \Gamma_f (x, y; 1-t)
\]

and

\[
\Gamma_f (x, y; t) = \Psi_f (x, y; t) + \Psi_f (x, y; 1-t) \geq 0
\]

for any $x, y \in C$ and $t \in [0, 1]$.

Therefore, we can state the following result as well

**Corollary 2.** Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set $C$ and $t \in [0, 1]$. The functional $\Gamma_f (\cdot, \cdot; t)$ is superadditive and monotonic nondecreasing as a function of interval.

In particular, if $z \in [x, y]$ then we have the inequality

\[
\begin{align*}
\frac{1}{2} [f ((1-t)x+ty) + f ((1-t)y+tx)] &\leq \frac{1}{2} [f ((1-t)x+tz) + f ((1-t)z+tx)] \\
&\quad + \frac{1}{2} [f ((1-t)z+ty) + f ((1-t)y+tz)] - f (z)
\end{align*}
\]
Also, if \( z, u \in [x, y] \) then we have the inequality
\[
(2.11) \quad f(x) + f(y) - f((1 - t)x + ty) - f((1 - t)y + tx) \\
\geq f(z) + f(u) - f((1 - t)z + tu) - f((1 - t)z + tu)
\]
for any \( t \in [0, 1] \).

Perhaps the most interesting functional we can consider from the above is the following one:
\[
(2.12) \quad \Theta_f(x, y) := \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty) \, dt \geq 0,
\]
which is related to the second Hermite-Hadamard inequality.

We observe that
\[
(2.13) \quad \Theta_f(x, y) = \int_0^1 \Psi_f(x, y; t) \, dt = \int_0^1 \Psi_f(x, y; 1 - t) \, dt.
\]
Utilising this representation, we can state the following result as well:

**Corollary 3.** Let \( f : C \subset X \to \mathbb{R} \) be a convex function on the convex set \( C \) and \( t \in [0, 1] \). The functional \( \Theta_f(\cdot, \cdot) \) is superadditive and monotonic nondecreasing as a function of interval. Moreover, we have the bounds
\[
(2.14) \quad \inf_{z \in [x, y]} \left[ \int_0^1 [f((1-t)x + tz) + f((1-t)z + ty)] \, dt - f(z) \right] = \int_0^1 f((1-t)x + ty) \, dt
\]
and
\[
(2.15) \quad \sup_{z, u \in [x, y]} \left[ \frac{f(z) + f(u)}{2} - \int_0^1 f((1-t)z + tu) \, dt \right] = \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty) \, dt.
\]

For other functionals associated with the Hermite-Hadamard see the paper [4].

### 3. Applications for Norms

Let \((X, \|\cdot\|)\) be a normed space and \( x, y \) two distinct vectors in \( X \). Then for any \( p \geq 1 \) the function \( f : X \to [0, \infty), f(x) = \|x\|^p \) is convex and utilising the results from the above section we can state the following norm inequalities:
\[
(3.1) \quad \inf_{z \in [x, y]} \left[ \left\| \frac{x + z}{2} \right\|^p + \left\| \frac{z + y}{2} \right\|^p - \|z\|^p \right] = \left\| \frac{x + y}{2} \right\|^p,
\]
and
\[
(3.2) \quad \sup_{z, u \in [x, y]} \left[ \left\| \frac{z}{2} + \frac{u}{2} \right\|^p - \left\| \frac{z + u}{2} \right\|^p \right] = \left\| \frac{x}{2} + \frac{y}{2} \right\|^p - \left\| \frac{x + y}{2} \right\|^p,
\]
Moreover, we can state the following results as well

\[(3.3) \quad \frac{1}{2} \left[ \|(1-t)x + ty\|^p \right. \left. + \|(1-t)y + tx\|^p \right] \]
\[\leq \frac{1}{2} \left[ \|(1-t)x + tz\|^p + \|(1-t)z + tx\|^p \right] + \frac{1}{2} \left[ \|(1-t)z + ty\|^p + \|(1-t)y + tz\|^p \right] - \|z\|^p \]
for any \( z \in [x, y] \) and \( t \in [0, 1] \), and

\[(3.4) \quad \|x\|^p + \|y\|^p - \|(1-t)x + ty\|^p \quad \geq \quad \|x\|^p + \|y\|^p - \|(1-t)z + tx\|^p \]

for any \( z, u \in [x, y] \) and \( t \in [0, 1] \).

In [5] Kikianty & Dragomir have introduced the concept of \( p \)-HH-norm as

\[
\|(x, y)\|_{p\text{-HH}} := \left( \int_0^1 \|(1-t)x + ty\|^p \, dt \right)^{1/p}, \quad p \geq 1
\]

and studied its various properties.

From the integral inequalities established in the above section we can deduce

the following results for the \( p \)-HH-norm of two distinct vectors \( x, y \) in the normed linear space \( (X, \|\cdot\|) \):

\[(3.5) \quad \inf_{z \in [x, y]} \left[ \|(x, z)\|_{p\text{-HH}}^p + \|(z, y)\|_{p\text{-HH}}^p - \|(z, z)\|_{p\text{-HH}}^p \right] = \|(x, y)\|_{p\text{-HH}}^p
\]

and

\[(3.6) \quad \sup_{z, u \in [x, y]} \left[ \frac{\|(z, z)\|_{p\text{-HH}}^p + \|(z, u)\|_{p\text{-HH}}^p}{2} - \|(z, z)\|_{p\text{-HH}}^p \right] = \frac{\|(x, x)\|_{p\text{-HH}}^p + \|(y, y)\|_{p\text{-HH}}^p}{2} - \|(x, y)\|_{p\text{-HH}}^p .
\]

4. Applications for Convex Functions of a Real Variable

Let \( f : I \to \mathbb{R} \) be a convex function on the interval \( I \subset \mathbb{R} \) and \( x, y \in I \) with \( x < y \). Due to the obvious fact that

\[
\int_0^1 f((1-t)x + ty) = \frac{1}{y-x} \int_x^y f(s) \, ds
\]

the functional

\[
\Theta_f(x, y) := \frac{f(x) + f(y)}{2} - \frac{1}{y-x} \int_x^y f(s) \, ds
\]

is superadditive and monotonic nondecreasing as a function of interval. We have also the inequalities

\[(4.1) \quad \inf_{z \in [x, y]} \left[ \frac{1}{z-x} \int_x^z f(s) \, ds + \frac{1}{y-z} \int_z^y f(s) \, ds - f(z) \right] = \frac{1}{y-x} \int_x^y f(s) \, ds
\]
and
\[
\sup_{z,u \in [x,y]} \left[ \frac{f(z) + f(u)}{2} - \frac{1}{z-u} \int_{u}^{z} f(s) \, ds \right] = \frac{f(x) + f(y)}{2} - \frac{1}{y-x} \int_{x}^{y} f(s) \, ds.
\]

The above inequalities may be used to obtain some interesting results for means. For \(0 < x \leq y < \infty\) and \(t \in (0,1)\) consider the weighted arithmetic, geometric and harmonic means defined by
\[
A_t(x,y) := (1-t)x + ty, \quad G_t(x,y) := x^{1-t}y^t \quad \text{and} \quad H_t(x,y) := \frac{1}{\frac{1-t}{x} + \frac{t}{y}}.
\]
For \(t = \frac{1}{2}\) we simply write \(A(x,y), G(x,y)\) and \(H(x,y)\).

It is well known that the following inequality holds
\[
A_t(x,y) \geq G_t(x,y) \geq H_t(x,y).
\]

1. Consider the convex function \(f : (0,\infty) \to (0,\infty), f(s) = s^{-1}\). Then for \(0 < x \leq y < \infty\) and \(t \in (0,1)\) we have
\[
\Psi_{(\cdot)^{-1}}(x,y,t) = (1-t)x^{-1} + ty^{-1} - [(1-t)x + ty]^{-1} = H_t^{-1}(x,y) - A_t^{-1}(x,y) = \frac{A_t(x,y) - H_t(x,y)}{A_t(x,y)H_t(x,y)}.
\]

On making use of Theorem 1 we have for \(0 < x \leq z \leq y < \infty\) and \(t \in (0,1)\) that
\[
(0 \leq) \frac{A_t(x,z) - H_t(x,z)}{A_t(x,z)H_t(x,z)} + \frac{A_t(z,y) - H_t(z,y)}{A_t(z,y)H_t(z,y)} \leq \frac{A_t(x,y) - H_t(x,y)}{A_t(x,y)H_t(x,y)}
\]
and, in particular,
\[
(0 \leq) \frac{A_t(z,u) - H_t(z,u)}{A_t(z,u)H_t(z,u)} \leq \frac{A_t(x,y) - H_t(x,y)}{A_t(x,y)H_t(x,y)}
\]
and for \(0 < x \leq z \leq u \leq y < \infty\) and \(t \in (0,1)\) that
\[
(0 \leq) \frac{A_t(z,u) - H_t(z,u)}{A_t(z,u)H_t(z,u)} \leq \frac{A_t(x,y) - H_t(x,y)}{A_t(x,y)H_t(x,y)}
\]
and, in particular,
\[
(0 \leq) \frac{A(z,u) - H(z,u)}{A(z,u)H(z,u)} \leq \frac{A(x,y) - H(x,y)}{A(x,y)H(x,y)}.
\]

Now, if we consider the logarithmic mean of two positive numbers \(x,y\) defined as
\[
L(x,y) := \left\{ \begin{array}{cl}
\frac{y-x}{\ln y - \ln x} & \text{if } x \neq y \\
x & \text{if } x = y
\end{array} \right.
\]
then
\[
\Theta_{(\cdot)^{-1}}(x,y) := \frac{x^{-1} + y^{-1}}{2} - \frac{1}{y-x} \int_{x}^{y} s^{-1} \, ds = H^{-1}(x,y) - L^{-1}(x,y) = \frac{L(x,y) - H(x,y)}{L(x,y)H(x,y)}.
\]
On making use of the Corollary 3 we have for \(0 < x \leq z \leq y < \infty\) that
\[
(4.9) \quad (0 \leq) \frac{L(x, z) - H(x, z)}{L(x, z) H(x, z)} + \frac{L(z, y) - H(z, y)}{L(z, y) H(z, y)} \leq \frac{L(x, y) - H(x, y)}{L(x, y) H(x, y)}
\]
and for \(0 < x \leq z \leq u \leq y < \infty\) that
\[
(4.10) \quad (0 \leq) \frac{L(z, u) - H(z, u)}{L(z, u) H(z, u)} \leq \frac{L(x, y) - H(x, y)}{L(x, y) H(x, y)}.
\]

2. Consider the convex function \(f : (0, \infty) \to (0, \infty), f(s) = -\ln s.\) Then for
\(0 < x \leq y < \infty\) and \(t \in (0, 1)\) we have
\[
\Psi_{-\ln} (x, y; t) = \ln [(1-t) x + ty] - (1-t) \ln x - t \ln y = \ln \left[ \frac{A_t(x, y)}{G_t(x, y)} \right].
\]
On making use of Theorem 1 we have for \(0 < x \leq z \leq y < \infty\) and \(t \in (0, 1)\) that
\[
(4.11) \quad (1 \leq) \frac{A_t(x, z)}{G_t(x, z)} \cdot \frac{A_t(z, y)}{G_t(z, y)} \leq \frac{A_t(x, y)}{G_t(x, y)}
\]
and, in particular,
\[
(4.12) \quad (1 \leq) \frac{A(x, z)}{G(x, z)} \cdot \frac{A(z, y)}{G(z, y)} \leq \frac{A(x, y)}{G(x, y)}
\]
and for \(0 < x \leq z \leq u \leq y < \infty\) and \(t \in (0, 1)\) that
\[
(4.13) \quad (1 \leq) \frac{A_t(z, u)}{G_t(z, u)} \leq \frac{A_t(x, y)}{G_t(x, y)}
\]
and, in particular,
\[
(4.14) \quad (1 \leq) \frac{A_t(z, u)}{G_t(z, u)} \leq \frac{A_t(x, y)}{G_t(x, y)}.
\]

Now, if we consider the \textit{identric mean} of two positive numbers \(x, y\) defined as
\[
I(x, y) := \begin{cases} 
\frac{1}{e} \cdot \left( \frac{y^x}{x^y} \right)^{y-x} & \text{if } x \neq y \\
\frac{x}{2} & \text{if } x = y
\end{cases}
\]
then
\[
\Theta_{-\ln} (x, y) := \frac{1}{y - x} \int_x^y \ln s \; ds - \ln x + \ln y = \ln \left[ \frac{I(x, y)}{G(x, y)} \right].
\]
On making use of the Corollary 3 we have for \(0 < x \leq z \leq y < \infty\) that
\[
(4.15) \quad (1 \leq) \frac{I(x, z)}{G(x, z)} \cdot \frac{I(z, y)}{G(z, y)} \leq \frac{I(x, y)}{G(x, y)}
\]
and for \(0 < x \leq z \leq u \leq y < \infty\) that
\[
(4.16) \quad (1 \leq) \frac{I(z, u)}{G(z, u)} \leq \frac{I(x, y)}{G(x, y)}.
\]
References


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