ČEBYŠEV'S TYPE INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR

Abstract. Some inequalities for continuous synchronous (asynchronous) functions of selfadjoint linear operators in Hilbert spaces, under suitable assumptions for the involved operators, are given.

1. Introduction

Consider the real sequences ($n-$tuples) $\mathbf{a}=(a_1,\dots,a_n)$, $\mathbf{b}=(b_1,\dots,b_n)$ and the nonnegative sequence $\mathbf{p}=(p_1,\dots,p_n)$ with $P_n:=\sum_{i=1}^n p_i>0$. Define the *weighted Čebyšev's functional*

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b}):=\frac{1}{P_n}\sum_{i=1}^n p_i a_i b_i-\frac{1}{P_n}\sum_{i=1}^n p_i a_i\cdot\frac{1}{P_n}\sum_{i=1}^n p_i b_i. \tag{1.1}$$

In 1882 – 1883, Čebyšev [3] and [4] proved that if $\mathbf{a}$ and $\mathbf{b}$ are monotonic in the same (opposite) sense, then

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b})\geq(\leq)\,0. \tag{1.2}$$

In the special case $\mathbf{p}=\mathbf{a}\geq\mathbf{0}$, it appears that the inequality (1.2) has been obtained by Laplace long before Čebyšev (see for example [15, p. 240]).

The inequality (1.2) was mentioned by Hardy, Littlewood and Pólya in their book [13] in 1934 in the more general setting of synchronous sequences, i.e., if $\mathbf{a}$, $\mathbf{b}$ are synchronous (asynchronous), this means that

$$(a_i-a_j)(b_i-b_j)\geq(\leq)\,0 \text{ for any } i,j\in\{1,\dots,n\}, \tag{1.3}$$

then (1.2) holds true as well.

A relaxation of the synchronicity condition was provided by M. Biernacki in 1951, [1], which showed that, if $\mathbf{a}$, $\mathbf{b}$ are monotonic in mean in the same sense, i.e., for $P_k:=\sum_{i=1}^k p_i$, $k=1,\dots,n-1$;

$$\frac{1}{P_k}\sum_{i=1}^k p_i a_i\leq(\geq)\,\frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i a_i,\quad k\in\{1,\dots,n-1\} \tag{1.4}$$

and

$$\frac{1}{P_k}\sum_{i=1}^k p_i b_i\leq(\geq)\,\frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i b_i,\quad k\in\{1,\dots,n-1\}, \tag{1.5}$$

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then (1.2) holds with " $\geq$ ". If if **a**, **b** are monotonic in mean in the opposite sense then (1.2) holds with " $\leq$ ".

If one would like to drop the assumption of nonnegativity for the components of **p**, then one may state the following inequality obtained by Mitrinović and Pečarić in 1991, [16]: If $0 \leq P_i \leq P_n$ for each $i \in \{1, \ldots, n-1\}$, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq 0 \tag{1.6}$$

provided **a** and **b** are sequences with the same monotonicity.

If **a** and **b** are monotonic in the opposite sense, the sign of the inequality (1.6) reverses.

Similar integral inequalities may be stated, however we do not present them here.

For other recent results on the Čebyšev inequality in either discrete or integral form see [2], [5], [6], [7], [8], [9], [15], [17], [18], [21], [22], [23], and the references therein.

The main aim of the present paper is to provide operator versions for the Čebyšev inequality in different settings. Related results and some particular cases of interest are also given.

2. A Version of the Čebyšev Inequality for One Operator

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [12, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [12] and the references therein.

For other results see [14] and [25].

We say that the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \text{ for each } t, s \in [a, b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [10] and [11].

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators.

Theorem 1. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\langle f(A) g(A) x, x\rangle \geq (\leq) \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. We consider only the case of synchronous functions. In this case we have then

$$f(t) g(t) + f(s) g(s) \geq f(t) g(s) + f(s) g(t) \tag{2.2}$$

for each $t, s \in [a, b]$.

If we fix $s \in [a, b]$ and apply the property (P) for the inequality (2.2) then we have for each $x \in H$ with $\|x\| = 1$ that

$$\langle (f(A) g(A) + f(s) g(s) 1_H) x, x\rangle \geq \langle (g(s) f(A) + f(s) g(A)) x, x\rangle,$$

which is clearly equivalent with

$$\langle f(A) g(A) x, x\rangle + f(s) g(s) \geq g(s) \langle f(A) x, x\rangle + f(s) \langle g(A) x, x\rangle \tag{2.3}$$

for each $s \in [a, b]$.

Now, if we apply again the property (P) for the inequality (2.3), then we have for any $y \in H$ with $\|y\| = 1$ that

$$\langle (\langle f(A) g(A) x, x\rangle 1_H + f(A) g(A)) y, y\rangle$$
$$\geq \langle (\langle f(A) x, x\rangle g(A) + \langle g(A) x, x\rangle f(A)) y, y\rangle,$$

which is clearly equivalent with

$$\begin{aligned} &\langle f(A) g(A) x, x\rangle + \langle f(A) g(A) y, y\rangle \\ &\quad \geq \langle f(A) x, x\rangle \langle g(A) y, y\rangle + \langle f(A) y, y\rangle \langle g(A) x, x\rangle \end{aligned} \tag{2.4}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. This is an inequality of interest in itself.

Finally, on making $y = x$ in (2.4) we deduce the desired result (2.1). ∎

Some particular cases are of interest for applications. In the first instance we consider the case of power functions.

Example 1. *Assume that A is a positive operator on the Hilbert space H and $p, q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle A^{p+q} x, x\right\rangle \geq \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle. \tag{2.5}$$

If A is positive definite then the inequality (2.5) also holds for $p, q < 0$.

If A is positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (2.5).

Another case of interest for applications is the exponential function.

Example 2. *Assume that A is a selfadjoint operator on H. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\langle \exp[(\alpha+\beta) A] x, x\rangle \geq \langle \exp(\alpha A) x, x\rangle \cdot \langle \exp(\beta A) x, x\rangle \tag{2.6}$$

for each $x \in H$ with $\|x\| = 1$.

If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then the reverse inequality holds in (2.6).

The following particular cases may be of interest as well:

Example 3. ***a.*** *Assume that A is positive definite and $p > 0$. Then*

$$\langle A^p \log A x, x\rangle \geq \langle A^p x, x\rangle \cdot \langle \log A x, x\rangle \tag{2.7}$$

for each $x \in H$ with $\|x\| = 1$. If $p < 0$ then the reverse inequality holds in (2.7).

b. *Assume that A is positive definite and $Sp(A) \subset (0,1)$. If $r, s > 0$ or $r, s < 0$ then*

$$\begin{aligned} &\left\langle (1_H - A^r)^{-1} (1_H - A^s)^{-1} x, x \right\rangle \\ &\qquad \geq \left\langle (1_H - A^r)^{-1} x, x \right\rangle \cdot \left\langle (1_H - A^s)^{-1} x, x \right\rangle \end{aligned} \tag{2.8}$$

for each $x \in H$ with $\|x\| = 1$.

If either $r > 0, s < 0$ or $r < 0, s > 0$, then the reverse inequality holds in (2.8).

Remark 1. *We observe, from the proof of the above theorem that, if A and B are selfadjoint operators and $Sp(A), Sp(B) \subseteq [m, M]$, then for any continuous synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the more general result*

$$\begin{aligned} &\langle f(A) g(A) x, x\rangle + \langle f(B) g(B) y, y\rangle \\ &\qquad \geq (\leq) \langle f(A) x, x\rangle \langle g(B) y, y\rangle + \langle f(B) y, y\rangle \langle g(A) x, x\rangle \end{aligned} \tag{2.9}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

If $f : [m, M] \longrightarrow (0, \infty)$ is continuous then the functions f^p, f^q are synchronous in the case when $p, q > 0$ or $p, q < 0$ and asynchronous when either $p > 0, q < 0$ or $p < 0, q > 0$. In this situation if A and B are positive definite operators then we have the inequality

$$\begin{aligned} &\left\langle f^{p+q}(A) x, x\right\rangle + \left\langle f^{p+q}(B) y, y\right\rangle \\ &\qquad \geq \langle f^p(A) x, x\rangle \langle f^q(B) y, y\rangle + \langle f^p(B) y, y\rangle \langle f^q(A) x, x\rangle \end{aligned} \tag{2.10}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$ where either $p, q > 0$ or $p, q < 0$. If $p > 0, q < 0$ or $p < 0, q > 0$ then the reverse inequality also holds in (2.10).

As particular cases, we should observe that for $p = q = 1$ and $f(t) = t$, we get from (2.10) the inequality

$$\left\langle A^2 x, x\right\rangle + \left\langle B^2 y, y\right\rangle \geq 2 \cdot \langle Ax, x\rangle \langle By, y\rangle \tag{2.11}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

For $p = 1$ and $q = -1$ we have from (2.10)

$$\langle Ax, x\rangle \left\langle B^{-1} y, y\right\rangle + \langle By, y\rangle \left\langle A^{-1} x, x\right\rangle \leq 2 \tag{2.12}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

3. A Version of the Čebyšev Inequality for n Operators

The following multiple operator version of Theorem 1 holds:

Theorem 2. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle \geq (\leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle, \tag{3.1}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. As in [12, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle,$$

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \text{ and } \left\langle g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle.$$

Applying Theorem 1 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (3.1). ∎

The following particular cases may be of interest for applications.

Example 4. *Assume that $A_j, j \in \{1, ..., n\}$ are positive operators on the Hilbert space H and $p, q > 0$. Then for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$ we have the inequality*

$$\left\langle \sum_{j=1}^{n} A_j^{p+q} x_j, x_j \right\rangle \geq \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle. \tag{3.2}$$

If A_j are positive definite then the inequality (3.2) also holds for $p, q < 0$.

If A_j are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (3.2).

Another case of interest for applications is the exponential function.

Example 5. *Assume that $A_j, j \in \{1, ..., n\}$ are selfadjoint operators on H. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\left\langle \sum_{j=1}^{n} \exp\left[(\alpha + \beta) A_j\right] x_j, x_j \right\rangle \geq \sum_{j=1}^{n} \langle \exp(\alpha A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle \exp(\beta A_j) x_j, x_j \rangle \tag{3.3}$$

for each $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If either $\alpha > 0, \beta < 0$ *or* $\alpha < 0, \beta > 0$, *then the reverse inequality holds in (3.3).*

The following particular cases may be of interest as well:

Example 6. ***a.*** *Assume that* $A_j, j \in \{1, ..., n\}$ *are positive definite operators and* $p > 0$. *Then*

$$\left\langle \sum_{j=1}^{n} A_j^p \log A_j x_j, x_j \right\rangle \geq \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \log A_j x_j, x_j \right\rangle \tag{3.4}$$

for each $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$. *If* $p < 0$ *then the reverse inequality holds in (3.4).*

b. *If* A_j *are positive definite and* $Sp(A_j) \subset (0, 1)$ *for* $j \in \{1, ..., n\}$ *then for* $r, s > 0$ *or* $r, s < 0$ *we have the inequality*

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} \left(1_H - A_j^r\right)^{-1} \left(1_H - A_j^s\right)^{-1} x_j, x_j \right\rangle \\ &\qquad \geq \sum_{j=1}^{n} \left\langle \left(1_H - A_j^r\right)^{-1} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \left(1_H - A_j^s\right)^{-1} x_j, x_j \right\rangle \end{aligned} \tag{3.5}$$

for each $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If either $r > 0, s < 0$ *or* $r < 0, s > 0$, *then the reverse inequality holds in (3.5).*

4. Another Version of the Čebyšev Inequality for n Operators

The following different version of the Čebyšev inequality for a sequence of operators also holds:

Theorem 3. *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, ..., n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and synchronous (asynchronous) on* $[m, M]$, *then*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle, \tag{4.1}$$

for any $p_j \geq 0, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$.

In particular

$$\begin{aligned} &\left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) g(A_j) x, x \right\rangle \\ &\qquad \geq (\leq) \left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) x, x \right\rangle \cdot \left\langle \frac{1}{n} \sum_{j=1}^{n} g(A_j) x, x \right\rangle, \end{aligned} \tag{4.2}$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. We provide here two proofs. The first is based on the inequality (2.9) and generates as a byproduct a more general result. The second is derived from Theorem 2.

1. If we make use of the inequality (2.9), then we can write

$$\langle f(A_j)\, g(A_j)\, x, x\rangle + \langle f(B_k)\, g(B_k)\, y, y\rangle \geq (\leq) \langle f(A_j)\, x, x\rangle \langle g(B_k)\, y, y\rangle + \langle f(B_k)\, y, y\rangle \langle g(A_j)\, x, x\rangle, \tag{4.3}$$

which holds for any A_j and B_k selfadjoint operators with $Sp(A_j), Sp(B_k) \subseteq [m, M]$, $j, k \in \{1, ..., n\}$ and for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, if $p_j \geq 0, q_k \geq 0, j, k \in \{1, ..., n\}$ and $\sum_{j=1}^n p_j = \sum_{k=1}^n q_k = 1$ then, by multiplying (4.3) with $p_j \geq 0, q_k \geq 0$ and summing over j and k from 1 to n we deduce the following inequality that is of interest in its own right:

$$\begin{aligned} &\left\langle \sum_{j=1}^n p_j f(A_j)\, g(A_j)\, x, x \right\rangle + \left\langle \sum_{k=1}^n q_k f(B_k)\, g(B_k)\, y, y \right\rangle \\ &\quad\geq (\leq) \left\langle \sum_{j=1}^n p_j f(A_j)\, x, x \right\rangle \left\langle \sum_{k=1}^n q_k g(B_k)\, y, y \right\rangle \\ &\qquad + \left\langle \sum_{k=1}^n q_k f(B_k)\, y, y \right\rangle \left\langle \sum_{j=1}^n p_j g(A_j)\, x, x \right\rangle \end{aligned} \tag{4.4}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, the choice $B_k = A_k, q_k = p_k$ and $y = x$ in (4.4) produces the desired result (4.1).

2. In we choose in Theorem 2 $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$, where $p_j \geq 0, j \in \{1, ..., n\}$, $\sum_{j=1}^n p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (3.1) becomes (4.1). The details are omitted. ∎

Remark 2. *We remark that the case $n = 1$ in (4.1) produces the inequality (2.1).*

The following particular cases are of interest:

Example 7. *Assume that $A_j, j \in \{1, ..., n\}$ are positive operators on the Hilbert space H, $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$ and $p, q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle \sum_{j=1}^n p_j A_j^{p+q} x, x \right\rangle \geq \left\langle \sum_{j=1}^n p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j A_j^{q} x, x \right\rangle. \tag{4.5}$$

If $A_j, j \in \{1, ..., n\}$ are positive definite then the inequality (4.5) also holds for $p, q < 0$.

If $A_j, j \in \{1, ..., n\}$ are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (4.5).

Another case of interest for applications is the exponential function.

Example 8. *Assume that $A_j, j \in \{1, ..., n\}$ are selfadjoint operators on H and $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\begin{aligned} &\left\langle \sum_{j=1}^n p_j \exp\left[(\alpha + \beta) A_j\right] x, x \right\rangle \\ &\quad\geq \left\langle \sum_{j=1}^n p_j \exp(\alpha A_j)\, x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j \exp(\beta A_j)\, x, x \right\rangle \end{aligned} \tag{4.6}$$

for each $x \in H$ with $\|x\| = 1$.

If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then the reverse inequality holds in (4.6).

The following particular cases may be of interest as well:

Example 9. ***a.*** *Assume that $A_j, j \in \{1, ..., n\}$ are positive definite operators on the Hilbert space H, $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $p > 0$. Then*

$$\left\langle \sum_{j=1}^{n} p_j A_j^p \log A_j x, x \right\rangle \geq \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle . \tag{4.7}$$

If $p < 0$ then the reverse inequality holds in (4.7).

b. *Assume that $A_j, j \in \{1, ..., n\}$ are positive definite operators on the Hilbert space $H, Sp(A_j) \subset (0, 1)$ and $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$. If $r, s > 0$ or $r, s < 0$ then*

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^r\right)^{-1} \left(1_H - A_j^s\right)^{-1} x, x \right\rangle \\ &\qquad \geq \left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^r\right)^{-1} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^s\right)^{-1} x, x \right\rangle \end{aligned} \tag{4.8}$$

for each $x \in H$ with $\|x\| = 1$.

If either $r > 0, s < 0$ or $r < 0, s > 0$, then the reverse inequality holds in (4.8).

We remark that the following operator norm inequality can be stated as well:

Corollary 1. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, asynchronous on $[m, M]$ and for $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ the operator $\sum_{j=1}^{n} p_j f(A_j) g(A_j)$ is positive, then*

$$\left\| \sum_{j=1}^{n} p_j f(A_j) g(A_j) \right\| \leq \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j g(A_j) \right\| . \tag{4.9}$$

Proof. We have from (4.1) that

$$0 \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum in this inequality over $x \in H$ with $\|x\| = 1$ we deduce the desired result (4.9). ∎

The bove Corollary 1 provides some interesting norm inequalities for sums of positive operators as follows:

Example 10. ***a.*** *If $A_j, j \in \{1, ..., n\}$ are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then for $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have the norm inequality:*

$$\left\| \sum_{j=1}^{n} p_j A_j^{p+q} \right\| \leq \left\| \sum_{j=1}^{n} p_j A_j^p \right\| \cdot \left\| \sum_{j=1}^{n} p_j A_j^q \right\| . \tag{4.10}$$

In particular

$$1 \le \left\| \sum_{j=1}^{n} p_j A_j^r \right\| \cdot \left\| \sum_{j=1}^{n} p_j A_j^{-r} \right\| \tag{4.11}$$

for any $r > 0$.

b. *Assume that* $A_j, j \in \{1, ..., n\}$ *are selfadjoint operators on* H *and* $p_j \ge 0, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$. *If either* $\alpha > 0, \beta < 0$ *or* $\alpha < 0, \beta > 0$, *then*

$$\left\| \sum_{j=1}^{n} p_j \exp\left[(\alpha+\beta) A_j\right] \right\| \le \left\| \sum_{j=1}^{n} p_j \exp(\alpha A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j \exp(\beta A_j) \right\|. \tag{4.12}$$

In particular

$$1 \le \left\| \sum_{j=1}^{n} p_j \exp(\gamma A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j \exp(-\gamma A_j) \right\|. \tag{4.13}$$

for any $\gamma > 0$.

5. Related Results for One Operator

The following result that is related to the Čebyšev inequality may be stated:

Theorem 4. *Let* A *be a selfadjoint operator with* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and synchronous on* $[m, M]$, *then*

$$\begin{aligned} &\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle \\ &\qquad \ge \left[\langle f(A) x, x\rangle - f(\langle Ax, x\rangle)\right] \cdot \left[g(\langle Ax, x\rangle) - \langle g(A) x, x\rangle\right] \end{aligned} \tag{5.1}$$

for any $x \in H$ *with* $\|x\| = 1$.

If f, g *are asynchronous, then*

$$\begin{aligned} &\langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle - \langle f(A) g(A) x, x\rangle \\ &\qquad \ge \left[\langle f(A) x, x\rangle - f(\langle Ax, x\rangle)\right] \cdot \left[\langle g(A) x, x\rangle - g(\langle Ax, x\rangle)\right] \end{aligned} \tag{5.2}$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Since f, g are synchronous and $m \le \langle Ax, x\rangle \le M$ for any $x \in H$ with $\|x\| = 1$, then we have

$$\left[f(t) - f(\langle Ax, x\rangle)\right]\left[g(t) - g(\langle Ax, x\rangle)\right] \ge 0 \tag{5.3}$$

for any $t \in [a, b]$ and $x \in H$ with $\|x\| = 1$.

On utilising the property (P) for the inequality (5.3) we have that

$$\langle\left[f(B) - f(\langle Ax, x\rangle)\right]\left[g(B) - g(\langle Ax, x\rangle)\right] y, y\rangle \ge 0 \tag{5.4}$$

for any B a bounded linear operator with $Sp(B) \subseteq [m, M]$ and $y \in H$ with $\|y\| = 1$.

Since

$$\begin{aligned} &\langle\left[f(B) - f(\langle Ax, x\rangle)\right]\left[g(B) - g(\langle Ax, x\rangle)\right] y, y\rangle \\ &\qquad = \langle f(B) g(B) y, y\rangle + f(\langle Ax, x\rangle) g(\langle Ax, x\rangle) \\ &\qquad\qquad - \langle f(B) y, y\rangle g(\langle Ax, x\rangle) - f(\langle Ax, x\rangle) \langle g(B) y, y\rangle, \end{aligned} \tag{5.5}$$

then from (5.4) we get

$$\langle f(B)\, g(B)\, y, y\rangle + f(\langle Ax, x\rangle)\, g(\langle Ax, x\rangle) \geq \langle f(B)\, y, y\rangle\, g(\langle Ax, x\rangle) + f(\langle Ax, x\rangle)\, \langle g(B)\, y, y\rangle$$

which is clearly equivalent with

$$\langle f(B)\, g(B)\, y, y\rangle - \langle f(A)\, y, y\rangle \cdot \langle g(A)\, y, y\rangle \geq [\langle f(B)\, y, y\rangle - f(\langle Ax, x\rangle)] \cdot [g(\langle Ax, x\rangle) - \langle g(B)\, y, y\rangle] \tag{5.6}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. This inequality is of interest in its own right.

Now, if we choose $B = A$ and $y = x$ in (5.6), then we deduce the desired result (5.1). ∎

The following result which improves the Čebyšev inequality may be stated:

Corollary 2. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, synchronous and one is convex while the other is concave on $[m, M]$, then*

$$\langle f(A)\, g(A)\, x, x\rangle - \langle f(A)\, x, x\rangle \cdot \langle g(A)\, x, x\rangle \geq [\langle f(A)\, x, x\rangle - f(\langle Ax, x\rangle)] \cdot [g(\langle Ax, x\rangle) - \langle g(A)\, x, x\rangle] \geq 0 \tag{5.7}$$

for any $x \in H$ with $\|x\| = 1$.

If f, g are asynchronous and either both of them are convex or both of them concave on $[m, M]$, then

$$\langle f(A)\, x, x\rangle \cdot \langle g(A)\, x, x\rangle - \langle f(A)\, g(A)\, x, x\rangle \geq [\langle f(A)\, x, x\rangle - f(\langle Ax, x\rangle)] \cdot [\langle g(A)\, x, x\rangle - g(\langle Ax, x\rangle)] \geq 0 \tag{5.8}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. The second inequality follows by making use of the result due to Mond & Pečarić, see [19], [20] or [12, p. 5]:

$$\langle h(A)\, x, x\rangle \geq (\leq)\, h(\langle Ax, x\rangle) \tag{MP}$$

for any $x \in H$ with $\|x\| = 1$ provided that A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and h is convex (concave) on the given interval $[m, M]$. ∎

The above Corollary 2 offers the possibility to improve some of the results established before for power function as follows:

Example 11. ***a.*** *Assume that A is a positive operator on the Hilbert space H. If $p \in (0, 1)$ and $q \in (1, \infty)$, then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle A^{p+q} x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \geq [\langle A^q x, x\rangle - \langle Ax, x\rangle^q]\, [\langle Ax, x\rangle^p - \langle A^p x, x\rangle] \geq 0. \tag{5.9}$$

If A is positive definite and $p > 1, q < 0$, then

$$\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle - \left\langle A^{p+q} x, x\right\rangle \geq [\langle A^q x, x\rangle - \langle Ax, x\rangle^q]\, [\langle A^p x, x\rangle - \langle Ax, x\rangle^p] \geq 0 \tag{5.10}$$

for each $x \in H$ with $\|x\| = 1$.

b. *Assume that A is positive definite and $p > 1$. Then*

$$\begin{aligned}\langle A^p \log Ax, x\rangle - \langle A^p x, x\rangle \cdot \langle \log Ax, x\rangle \\ \geq [\langle A^p x, x\rangle - \langle Ax, x\rangle^p] [\log \langle Ax, x\rangle - \langle \log Ax, x\rangle] \geq 0\end{aligned} \tag{5.11}$$

for each $x \in H$ with $\|x\| = 1$.

6. Related Results for n Operators

We can state now the following generalisation of Theorem 4 for n operators:

Theorem 5. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$.*

(i) If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then

$$\begin{aligned}&\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle \\ &\geq \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right)\right] \\ &\quad \cdot \left[g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) - \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle\right]\end{aligned} \tag{6.1}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Moreover, if one function is convex while the other is concave on $[m, M]$, then the right hand side of (6.1) is nonnegative.

(ii) If f, g are asynchronous on $[m, M]$, then

$$\begin{aligned}&\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle - \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle \\ &\geq \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right)\right] \\ &\quad \cdot \left[\sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle - g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right)\right]\end{aligned} \tag{6.2}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Moreover, if either both of them are convex or both of them are concave on $[m, M]$, then the right hand side of (6.2) is nonnegative as well.

Proof. The argument is similar to the one from the proof of Theorem 2 on utilising the results from one operator obtained in Theorem 4.

The nonnegativity of the right hand sides of the inequalities (6.1) and (6.2) follows by the use of the Jensen's type result from [12, p. 5]

$$\sum_{j=1}^{n} \langle h(A_j) x_j, x_j\rangle \geq (\leq)\, h\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) \tag{6.3}$$

for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, which holds provided that A_j are selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$ and h is convex (concave) on $[m,M]$.

The details are omitted. ∎

Example 12. ***a.*** *Assume that $A_j, j \in \{1,...,n\}$ are positive operators on the Hilbert space H. If $p \in (0,1)$ and $q \in (1,\infty)$, then for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$ we have the inequality*

$$\begin{aligned}(6.4)\quad & \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle \\ & \geq \left[\sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^{q} \right] \\ & \cdot \left[\left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^{p} - \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \right] \geq 0.\end{aligned}$$

If A_j are positive definite and $p > 1, q < 0$, then

$$\begin{aligned}(6.5)\quad & \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle \\ & \geq \left[\sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^{q} \right] \\ & \cdot \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^{p} \right] \geq 0\end{aligned}$$

for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

b. *Assume that A_j are positive definite and $p > 1$. Then*

$$\begin{aligned}(6.6)\quad & \sum_{j=1}^{n} \left\langle A_j^{p} \log A x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \log A_j x_j, x_j \right\rangle \\ & \geq \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^{p} \right] \\ & \cdot \left[\sum_{j=1}^{n} \log \left\langle A_j x_j, x_j \right\rangle - \log \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right) \right] \geq 0\end{aligned}$$

for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

The following result may be stated as well:

Theorem 6. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$.*

(i) If $f,g:[m,M]\longrightarrow\mathbb{R}$ are continuous and synchronous on $[m,M]$, then

$$\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle \tag{6.7}$$
$$\geq\left[f\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)-\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\right]$$
$$\cdot\left[\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle-g\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)\right]$$

for any $p_j\geq 0, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$ and $x\in H$ with $\|x\|=1$. Moreover, if one is convex while the other is concave on $[m,M]$, then the right hand side of (6.7) is nonnegative.

(ii) If f,g are asynchronous on $[m,M]$, then

$$\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle \tag{6.8}$$
$$\geq\left[\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle-f\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)\right]$$
$$\cdot\left[\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle-g\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)\right]$$

for any $p_j\geq 0, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$ and $x\in H$ with $\|x\|=1$. Moreover, if either both of them are convex or both of them are concave on $[m,M]$, then the right hand side of (6.8) is nonnegative as well.

Proof. Follows from Theorem 5 on choosing $x_j=\sqrt{p_j}\cdot x$, $j\in\{1,...,n\}$, where $p_j\geq 0, j\in\{1,...,n\}$, $\sum_{j=1}^{n}p_j=1$ and $x\in H$, with $\|x\|=1$.

Also, the positivity of the right hand term in (6.7) follows by the Jensen's type inequality from the inequality (6.3) for the same choices, namely $x_j=\sqrt{p_j}\cdot x$, $j\in\{1,...,n\}$, where $p_j\geq 0, j\in\{1,...,n\}$, $\sum_{j=1}^{n}p_j=1$ and $x\in H$, with $\|x\|=1$. The details are omitted. ∎

Finally, we can list some particular inequalities that may be of interest for applications. They improve some result obtained above:

Example 13. ***a.*** *Assume that $A_j, j\in\{1,...,n\}$ are positive operators on the Hilbert space H and $p_j\geq 0, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$. If $p\in(0,1)$ and $q\in(1,\infty)$,* *then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle \geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{q} \right] \cdot \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{p} - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \right] \geq 0. \tag{6.9}$$

If $A_j, j \in \{1, ..., n\}$ are positive definite and $p > 1, q < 0$, then

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle \geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{q} \right] \cdot \left[\left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{p} \right] \geq 0 \tag{6.10}$$

for each $x \in H$ with $\|x\| = 1$.

b. *Assume that A_j, $j \in \{1, ..., n\}$ are positive definite and $p > 1$. Then*

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p} \log A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{p} \right] \cdot \left[\log \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \right] \geq 0 \tag{6.11}$$

for each $x \in H$ with $\|x\| = 1$.

References

[1] M. Biernacki, Sur une inégalité entre les intégrales due à Tchebyscheff. *Ann. Univ. Mariae Curie-Sklodowska* (Poland), **A5**(1951), 23-29.

[2] K. Boukerrioua, and A. Guezane-Lakoud, On generalization of Čebyšev type inequalities. *J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 55, 4 pp.

[3] P.L. Čebyšev, O približennyh vyraženijah odnih integralov čerez drugie. *Soobšćenija i protokoly zasedaniĭ Matemmatičeskogo občestva pri Imperatorskom Har'kovskom Universitete* No. 2, 93–98; *Polnoe sobranie sočineniĭ P. L. Čebyševa.* Moskva–Leningrad, 1948a, (1882), 128-131.

[4] P.L. Čebyšev, Ob odnom rjade, dostavljajušćem predel'nye veličiny integralov pri razloženii podintegral'noĭ funkcii na množeteli. *Priloženi k 57 tomu Zapisok Imp. Akad. Nauk,* No. 4; *Polnoe sobranie sočineniĭ P. L. Čebyševa.* Moskva–Leningrad, 1948b, (1883),157-169.

[5] S.S. Dragomir, On some improvements of Čebyšev's inequality for sequences and integrals, *Studia Univ. Babeş-Bolyai, Mathematica* (Romania), **XXXV** (4)(1990), 35-40.

[6] S.S. Dragomir, Some improvement of Čebyšev's inequality for isotonic functionals, *Atti. Sem. Mat. Fis. Univ. Modena* (Italy), **41** (1993), 473-481.

[7] S. S. Dragomir, On the Čebyšev's inequality for weighted means. *Acta Math. Hungar.* **104** (2004), no. 4, 345–355.

[8] S. S. Dragomir and B. Mond, Some mappings associated with Čebyšev's inequality for sequences of real numbers. *Bull. Allahabad Math. Soc.* **8/9** (1993/94), 37–55

[9] S.S. Dragomir and J.E. Pečarić, Refinements of some inequalities for isotonic linear functionals, *L'Anal. Num. Théor de L'Approx.* (Romania) **18**(1989) (1), 61-65.

[10] S. S. Dragomir and J. Sándor, The Chebyshev inequality in pre-Hilbertian spaces. I. *Proceedings of the Second Symposium of Mathematics and its Applications (Timişoara, 1987)*, 61–64, Res. Centre, Acad. SR Romania, Timişoara, 1988. MR1006000 (90k:46048).

[11] S. S. Dragomir, J. Pečarić and J. Sándor, The Chebyshev inequality in pre-Hilbertian spaces. II. *Proceedings of the Third Symposium of Mathematics and its Applications (Timişoara, 1989)*, 75–78, Rom. Acad., Timişoara, 1990. MR1266442 (94m:46033)

[12] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[13] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities,* 1st Ed. and 2nd Ed. Cambridge University Press, (1934, 1952) Cambridge, England.

[14] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), No. 2-3, 551–564.

[15] D.S. Mitrinović, J. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic, Dordrecht, 1993.

[16] D.S. Mitrinović and J.E. Pečarić, On an identity of D.Z. Djoković, *Prilozi Mak. Akad.Nauk. Umj. (Skopje),* **12**(1)(1991), 21-22.

[17] D.S. Mitrinović and J.E. Pečarić, History, variations and generalizations of the Čebyšev inequality and question of some priorities. II. *Rad Jugoslav. Akad. Znan. Umjet. No.* **450** (1990), 139–156.

[18] D.S. Mitrinović and P.M. Vasić, History, variations and generalisations of the Čebyšev inequality and the question of some priorities. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **No. 461–497** (1974), 1–30.

[19] B. Mond and J. Pečarić, Convex inequalities in Hilbert spaces, *Houston J. Math.*, **19**(1993), 405-420.

[20] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.

[21] B. G. Pachpatte, New Čebyšev type inequalities involving functions of two and three variables. *Soochow J. Math.* **33** (2007), no. 4, 569–577.

[22] B. G. Pachpatte, A note on Čebyšev type inequalities. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.* (N.S.)(Romania), **53** (2007), no. 1, 97–102.

[23] J. Pečarić, Remarks on Biernacki's generalization of Čebyšev's inequality. *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **47** (1993), 116–122.

[24] J.E. Pečarić and S.S. Dragomir, Some remarks on Čebyšev's inequality, *L'Anal. Num. Théor de L'Approx.* **19** (1)(1990), 58-65.

[25] J. Pečarić, J. Mićić and Y. Seo, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, 191–207.

Research Group in Mathematical Inequalities & Applications, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://www.staff.vu.edu.au/rgmia/dragomir/`