INEQUALITIES FOR THE NUMERICAL RADIUS IN UNITAL
NORMED ALGEBRAS

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Abstract. In this paper, some inequalities between the numerical radius of an element from a unital normed algebra and certain semi-inner products involving that element and the unity are given.

1. Introduction

Let $A$ be a unital normed algebra over the complex number field $\mathbb{C}$ and let $a \in A$. Recall that the numerical radius of $a$ is given by (see [2, p. 15])

\begin{equation}
(1.1) \quad v(a) = \sup \{ |f(a)| : f \in A', \|f\| \leq 1 \text{ and } f(1) = 1 \},
\end{equation}

where $A'$ denotes the dual space of $A$, i.e., the Banach space of all continuous linear functionals on $A$.

It is known that $v(\cdot)$ is a norm on $A$ that is equivalent to the given norm $\|\cdot\|$. More precisely, the following double inequality holds true:

\begin{equation}
(1.2) \quad \frac{1}{e} \|a\| \leq v(a) \leq \|a\|
\end{equation}

for any $a \in A$.

Following [2], we notice that this crucial result appears slightly hidden in Bohnenblust and Karlin [1, Theorem 1] together with the inequality $\|x\| \leq e\Phi (x)$, which occurs on page 219. A simpler proof was given by Lumer [5], though with the constant $\frac{1}{e}$ in place of $\frac{1}{2}$. For a simple proof of (1.2) that borrows ideas from Lumer and from Glickfeld [6], see [2, p. 34].

A generalisation of (1.2) for powers has been obtained by M.J. Crabb [3] which proved that

\begin{equation}
(1.3) \quad \|a^n\| \leq n! \left( \frac{e}{n} \right)^n [v(a)]^n, \quad n = 1, 2, \ldots
\end{equation}

for any $a \in A$.

In this paper, some inequalities between the numerical radius of an element and the superior semi-inner product of that element and the unity in the normed algebra $A$ are given via the celebrated representation result of Lumer from [5].

2. Some Subsets in $A$

Let $D(1) := \{ f \in A' : \|f\| \leq 1 \text{ and } f(1) = 1 \}$. For $\lambda \in \mathbb{C}$ and $r > 0$, we define the subset of $A$ by

\begin{equation}
\Delta(\lambda, r) := \{ a \in A | |f(a) - \lambda| \leq r \text{ for each } f \in D(1) \}.
\end{equation}
The following result holds.

**Proposition 1.** Let $\lambda \in \mathbb{C}$ and $r > 0$. Then $\bar{\Delta} (\lambda, r)$ is a closed convex subset of $A$ and

$$\bar{B} (\lambda, r) \subseteq \bar{\Delta} (\lambda, r),$$

where $\bar{B} (\lambda, r) := \{ a \in A \mid \| a - \lambda \| \leq r \}$.

Now, for $\gamma, \Gamma \in \mathbb{C}$, define the set

$$\bar{U} (\gamma, \Gamma) := \{ a \in A \mid \Re \left[ (\Gamma - f(a)) \left( \overline{f(a)} - \gamma \right) \right] \geq 0 \text{ for each } f \in D(1) \}.$$

The following representation result may be stated.

**Proposition 2.** For any $\gamma, \Gamma \in \mathbb{C}$, we have:

$$\bar{U} (\gamma, \Gamma) = \bar{\Delta} \left( \frac{\gamma + \Gamma}{2}, \frac{1}{2} |\Gamma - \gamma| \right).$$

**Proof.** We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\Re [(\Gamma - z) (\overline{\varepsilon - \tilde{\gamma}})] \geq 0.$$ 

This follows by the equality

$$\frac{1}{4} |\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \Re [(\Gamma - z) (\overline{\varepsilon - \tilde{\gamma}})]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.2) is thus a simple conclusion of this fact. \hfill \Box

Making use of some obvious properties in $\mathbb{C}$ and for continuous linear functionals, we can state the following corollary as well.

**Corollary 1.** For any $\gamma, \Gamma \in \mathbb{C}$, we have

$$\bar{U} (\gamma, \Gamma) = \{ a \in A \mid \Re \left[ f \left( \Gamma - a \right) \overline{f(a - \gamma)} \right] \geq 0 \text{ for each } f \in D(1) \} = \{ a \in A \mid (\Re \Gamma - \Re f(a)) (\Re f(a) - \Re \gamma) + (\Im \Gamma - \Im f(a)) (\Im f(a) - \Im \gamma) \geq 0 \text{ for each } f \in D(1) \}.$$

Now, if we assume that $\Re (\Gamma) \geq \Re (\gamma)$ and $\Im (\Gamma) \geq \Im (\gamma)$, then we can define the following subset of $A$:

$$\bar{S} (\gamma, \Gamma) := \{ a \in A \mid \Re (\Gamma) \geq \Re f(a) \geq \Re (\gamma) \text{ and } \Im (\Gamma) \geq \Im f(a) \geq \Im (\gamma) \text{ for each } f \in D(1) \}.$$

One can easily observe that $\bar{S} (\gamma, \Gamma)$ is closed, convex and

$$\bar{S} (\gamma, \Gamma) \subseteq \bar{U} (\gamma, \Gamma).$$
3. Semi-Inner Products and Lumer’s Theorem

Let \((X, \|\cdot\|)\) be a normed linear space over the real of complex number field \(K\). The mapping \(f : X \to \mathbb{R}, f(x) = \frac{1}{2} \|x\|^2\) is obviously convex and then there exists the following limits:

\[
\langle x, y \rangle_{t} = \lim_{t \to 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t},
\]

\[
\langle x, y \rangle_{t} = \lim_{t \to 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t}
\]

for every two elements \(x, y \in X\). The mapping \(\langle \cdot, \cdot \rangle_{s} (\langle \cdot, \cdot \rangle_{i})\) will be called the \textit{superior semi-inner product} (the \textit{interior semi-inner product}) associated to the norm \(\|\cdot\|\).

We list some properties of these semi-inner products that can be easily derived from the definition (see for instance \([4]\)):

(i) \(\langle x, x \rangle_{p} = \|x\|^2; \langle ix, x \rangle_{p} = \langle x, ix \rangle_{p} = 0, x \in X\);

(ii) \(\langle \lambda x, y \rangle_{p} = \lambda \langle x, y \rangle_{p}; \langle x, \lambda y \rangle_{p} = \lambda \langle x, y \rangle_{p}\) for \(\lambda \geq 0, x, y \in X\);

(iii) \(\langle \lambda x, y \rangle_{p} = \lambda \langle x, y \rangle_{p}; \langle x, \lambda y \rangle_{p} = \lambda \langle x, y \rangle_{p}\) for \(\lambda < 0, x, y \in X\);

(iv) \(\langle ix, y \rangle_{p} = -\langle x, iy \rangle_{p}; \langle \alpha x, \beta y \rangle = \alpha \beta \langle x, y \rangle\) if \(\alpha \beta \geq 0, x, y \in X\);

(v) \(\langle -x, y \rangle_{p} = \langle x, -y \rangle_{p} = -\langle x, y \rangle_{q}, x, y \in X\);

(vi) \(\langle x, y \rangle_{p} \leq \|x\| \|y\|, x, y \in X\);

(vii) \(\langle x_{1} + x_{2}, y \rangle_{s(i)} \leq (\geq) \langle x_{1}, y \rangle_{s(i)} + \langle x_{2}, y \rangle_{s(i)}\) for \(x_{1}, x_{2}, y \in X\);

(ix) \(\langle \alpha x + y, x \rangle_{p} = \alpha \|x\|^2 + \langle y, x \rangle_{p}, \alpha \in \mathbb{R}, x, y \in X\);

(x) \(\langle y + z, x \rangle_{p} - \langle z, x \rangle_{p} \leq \|y\| \|x\|, x, y, z \in X\);

(xi) The mapping \(\langle \cdot, x \rangle_{p}\) is continuous on \((X, \|\cdot\|)\) for each \(x \in X\), where \(p, q \in \{s, i\}\) and \(p \neq q\).

The following result essentially due to Lumer \([5]\) (see \([2, p. 17]\)) can be stated.

**Theorem 1.** Let \(A\) be a unital normed algebra over \(K (K = \mathbb{C}, \mathbb{R})\). For each \(a \in A\),

\[
\max \{\text{Re} \lambda |\lambda \in V(a)\} = \inf_{\alpha > 0} \frac{1}{\alpha} ||1 + \alpha a|| - 1 = \lim_{\alpha \to 0^+} \frac{1}{\alpha} ||1 + \alpha a|| - 1,
\]

where \(V(a)\) is the numerical range of \(a\) (see for instance \([2, p. 15]\)).

**Remark 1.** In terms of semi-inner products, the above identity can be stated as:

\[
\max \{\text{Re} f(a) | f \in D(1)\} = \langle a, 1 \rangle_{s}.
\]

The following result that provides more information may be stated.

**Theorem 2.** For any \(a \in A\), we have:

\[
\langle a, 1 \rangle_{v,s} = \langle a, 1 \rangle_{s},
\]

where

\[
\langle a, b \rangle_{v,s} := \lim_{t \to 0^+} \frac{v^2(b + ta) - v^2(b)}{2t}
\]

is the superior semi-inner product associated with the numerical radius.
Proof. Since \( v(a) \leq \|a\| \), we have:

\[
\langle a, 1 \rangle_{v,s} = \lim_{t \to 0^+} \frac{v^2(1 + ta) - v^2(1)}{2t} = \lim_{t \to 0^+} \frac{v^2(1 + ta) - 1}{2t} \leq \lim_{t \to 0^+} \frac{\|1 + ta\|^2 - 1}{2t} = \langle a, 1 \rangle_s .
\]

Now, let \( f \in D(1) \). Then, for each \( \alpha > 0 \),

\[
f(a) = \frac{1}{\alpha} [f(1 + \alpha a) - f(1)] = \frac{1}{\alpha} [f(1 + \alpha a) - 1],
\]
giving

\[
\Re f(a) = \frac{1}{\alpha} [\Re f(1 + \alpha a) - f(1)] \leq \frac{1}{\alpha} [\|f(1 + \alpha a)\| - 1]
\]

\[
\leq \frac{1}{\alpha} [v(1 + \alpha a) - 1].
\]

Taking the infimum over \( \alpha > 0 \), we deduce

\[
\Re f(a) \leq \inf_{\alpha > 0} \left[ \frac{1}{\alpha} [v(1 + \alpha a) - 1] \right] = \lim_{\alpha \to 0^+} \left[ \frac{v^2(1 + \alpha a) - 1}{2\alpha} \right] = \lim_{\alpha \to 0^+} \frac{v(1 + \alpha a) - 1}{\alpha} = \langle a, 1 \rangle_{v,s}.
\]

If we now take the supremum over \( f \in D(1) \) in (3.4), we obtain:

\[
\sup \{ \Re f(a) | f \in D(1) \} \leq \langle a, 1 \rangle_{v,s}
\]

which gives, by Lumer’s identity that \( \langle a, 1 \rangle_s \leq \langle a, 1 \rangle_{v,s} \). \( \square \)

Corollary 2. We have the inequality

\[
\langle a, 1 \rangle_s \leq v(a) \quad (\leq \|a\|).
\]

Proof. Schwarz’s inequality for the norm \( v(.) \) gives that

\[
\langle a, 1 \rangle_{v,s} \leq v(a) v(1) = v(a),
\]

and by (3.3), the inequality (3.5) is proved. \( \square \)

4. REVERSE INEQUALITIES FOR THE NUMERICAL RADIUS

Utilising the inequality (3.5) we observe that for any complex number \( \beta \) located in the closed disc centered in 0 and with radius 1 we have \( \|\beta a, 1\|_s \) as a lower bound for the numerical radius \( v(a) \). Therefore, it is a natural question to ask how far these quantities are from each other under various assumptions for the element \( a \) in the unital normed algebra \( A \) and the scalar \( \beta \). A number of results answering this question are incorporated in the following theorems.

Theorem 3. Let \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( r > 0 \). If \( a \in \mathbb{A}(\lambda, r) \), then

\[
v(a) \leq \left\langle \frac{\lambda}{|\lambda|} a, 1 \right\rangle_s + \frac{1}{2} \cdot \frac{r^2}{|\lambda|}.
\]
Proof. Since $a \in \Delta (\lambda, 1)$, then $|f(a) - \lambda|^2 \leq r^2$, for each $f \in D(1)$, giving that
\begin{equation}
|f(a)|^2 + |\lambda|^2 \leq 2 \text{Re}[f(\lambda a)] + r^2
\end{equation}
for each $f \in D(1)$.

Taking the supremum of $f \in D(1)$ in (4.2) and utilising the representation (3.2), we deduce
\begin{equation}
v^2(a) + |\lambda|^2 \leq 2 \langle \lambda a, 1 \rangle_s + r^2
\end{equation}
which is an inequality of interest in itself.

On the other hand, we have the elementary inequality
\begin{equation}
2v(a)|\lambda| \leq v^2(a) + |\lambda|^2,
\end{equation}
which, together with (4.3) implies the desired result (4.1). \hfill \Box

**Remark 2.** Notice that, by the inclusion (2.1) a sufficient condition for (4.1) to holds is that $a \in \bar{B}(\lambda, r)$.

**Corollary 3.** Let $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \pm \gamma$. If $a \in \bar{U}(\gamma, \Gamma)$, then
\begin{equation}
v(a) \leq \left\langle \frac{\Gamma + \gamma}{|\Gamma + \gamma|} a, 1 \right\rangle_s + \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}.
\end{equation}

**Remark 3.** If $M > m \geq 0$ and $a \in \bar{U}(m, M)$, then
\begin{equation}
(0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M}.
\end{equation}
Observe that, due to the inclusion (2.5), a sufficient condition for (4.6) to holds is that $M \geq \text{Re} f(a), \text{Im} f(a) \geq m$ for any $f \in D(1)$.

The following result may be stated as well.

**Theorem 4.** Let $\lambda \in \mathbb{C}$ and $r > 0$ with $|\lambda| > r$. If $a \in \Delta (\lambda, r)$, then
\begin{equation}
v(a) \leq \left\langle \frac{\lambda}{|\lambda|^2 - r^2} a, 1 \right\rangle_s
\end{equation}
and, equivalently,
\begin{equation}
v^2(a) \leq \left\langle \frac{\lambda}{|\lambda|^2 - r^2} a, 1 \right\rangle^2_s + \frac{r^2}{|\lambda|^2} \cdot v^2(a).
\end{equation}

**Proof.** Since $|\lambda| > r$, hence by (4.3) we have, on dividing by $\sqrt{|\lambda|^2 - r^2} > 0$, that
\begin{equation}
\frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leq \frac{2}{\sqrt{|\lambda|^2 - r^2}} \langle \lambda a, 1 \rangle_s.
\end{equation}
On the other hand, we also have
\[2v(a) \leq \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2}\]
which, together with (4.9), gives
\begin{equation}
v(a) \leq \frac{1}{\sqrt{|\lambda|^2 - r^2}} \langle \lambda a, 1 \rangle_s.
\end{equation}
Taking the square in (4.10), we have
\[ v^2(a) \left( |\lambda|^2 - r^2 \right) \leq \langle \lambda a, 1 \rangle^2, \]
which is clearly equivalent to (4.7).

**Corollary 4.** Let \( \gamma, \Gamma \in \mathbb{C} \) with \( \text{Re} (\Gamma \bar{\gamma}) > 0 \). If \( a \in \bar{U}(\gamma, \Gamma) \), then,
\[ v(a) \leq \left\langle \frac{\Gamma + \bar{\gamma}}{2 \sqrt{\text{Re} (\Gamma \bar{\gamma})}} a, 1 \right\rangle_s. \]

**Remark 4.** If \( M \geq m > 0 \) and \( a \in \bar{U}(m, M) \), then
\[ v(a) \leq \frac{M + m}{2 \sqrt{mM}} \langle a, 1 \rangle_s, \]
or, equivalently,
\[ (0 \leq v(a) - \langle a, 1 \rangle_s \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2 \sqrt{mM}} \langle a, 1 \rangle_s \left( \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2 \sqrt{mM}} \|a\| \right). \]

The following result may be stated as well.

**Theorem 5.** Let \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( r > 0 \) with \(|\lambda| > r\). If \( a \in \bar{\Delta}(\lambda, r) \), then
\[ v^2(a) \leq \left\langle \frac{1}{|\lambda|} a, 1 \right\rangle_s^2 + 2 \left( |\lambda| - \sqrt{|\lambda|^2 - r^2} \right) \left\langle \frac{1}{|\lambda|} a, 1 \right\rangle_s. \]

**Proof.** Since (by (4.2)) \( \text{Re} [f(\bar{\lambda}a)] > 0 \), then dividing by it in (4.2) gives:
\[ \frac{|f(a)|^2}{\text{Re} [f(\bar{\lambda}a)]} + \frac{|\lambda|^2}{\text{Re} [f(\bar{\lambda}a)]} \leq 2 + \frac{r^2}{\text{Re} [f(\bar{\lambda}a)]}, \]
which is clearly equivalent to:
\[ \frac{|f(a)|^2}{\text{Re} [f(\bar{\lambda}a)]} - \frac{\text{Re} [f(\bar{\lambda}a)]}{|\lambda|^2} \leq 2 + \frac{r^2}{\text{Re} [f(\bar{\lambda}a)]} - \frac{\text{Re} [f(\bar{\lambda}a)]}{|\lambda|^2} =: I. \]

Since
\[ I = 2 - \frac{\text{Re} [f(\bar{\lambda}a)]}{|\lambda|^2} - \frac{(|\lambda|^2 - r^2)}{\text{Re} [f(\bar{\lambda}a)]} \]
\[ = 2 - 2 \frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|} - \left[ \frac{\sqrt{\text{Re} [f(\bar{\lambda}a)]}}{|\lambda|} - \frac{\sqrt{|\lambda|^2 - r^2}}{\text{Re} [f(\bar{\lambda}a)]} \right]^2 \]
\[ \leq 2 \left( 1 - \sqrt{1 - \left( \frac{r}{|\lambda|} \right)^2} \right), \]
hence by (4.14) and (4.15) we have
\begin{equation}
\tag{4.16}
|f(a)|^2 \leq \frac{\text{Re}[f(\tilde{\lambda}a)]}{|\lambda|^2} + 2 \left( 1 - \sqrt{1 - \left( \frac{r}{|\lambda|} \right)^2} \right) \text{Re}[f(\tilde{\lambda}a)].
\end{equation}

Taking the supremum in $f \in D(1)$ and utilising Lumer’s result, we deduce the desired inequality (4.13).

\textbf{Corollary 5.} Let $\gamma, \Gamma \in \mathbb{C}$ with $\text{Re}(\Gamma\bar{\gamma}) > 0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,
\begin{equation}
v^2(a) \leq \left( \frac{\bar{\Gamma} + \gamma}{|\Gamma + \gamma|} a, 1 \right)_s + 2 \left( \frac{\gamma + \Gamma}{2} - \sqrt{\text{Re}(\Gamma\bar{\gamma})} \right) \left( \frac{\bar{\Gamma} + \gamma}{|\Gamma + \gamma|} a, 1 \right)_s.
\end{equation}

\textbf{Remark 5.} If $M > m \geq 0$ and $a \in \bar{U}(m, M)$, then
\begin{equation}
(0 \leq) v^2(a) - (a, 1)_s^2 \leq \left( \sqrt{M} - \sqrt{m} \right)^2 (a, 1)_s \left( \text{Re}(\Gamma\bar{\gamma}) \right) \|a\|.
\end{equation}

Finally, the following result can be stated as well.

\textbf{Theorem 6.} Let $\lambda \in \mathbb{C}$ and $r > 0$ with $|\lambda| > r$. If $a \in \Delta(\lambda, r)$, then
\begin{equation}
\tag{4.17}
v(a) \leq \left( |\lambda| + \sqrt{|\lambda|^2 - r^2} \right) \left( \frac{\lambda}{r^2} a, 1 \right)_s + \left( |\lambda| \left( |\lambda| + \sqrt{|\lambda|^2 - r^2} \right) \left( |\lambda| - 2\sqrt{|\lambda|^2 - r^2} \right) \right)_{2r^2}.
\end{equation}

\textbf{Proof.} From the proof of Theorem 3 above, we have
\begin{equation}
|f(a)|^2 + |\lambda|^2 \leq 2 \text{Re}[f(\tilde{\lambda}a)] + r^2
\end{equation}
which is equivalent with
\begin{equation}
\tag{4.19}
|f(a)|^2 + \left( |\lambda| + \sqrt{|\lambda|^2 - r^2} \right)^2 \leq 2 \text{Re}[f(\tilde{\lambda}a)] + r^2 - |\lambda|^2 + \left( |\lambda| - \sqrt{|\lambda|^2 - r^2} \right)^2 = 2 \text{Re}[f(\tilde{\lambda}a)] + |\lambda|^2 - 2|\lambda| \sqrt{|\lambda|^2 - r^2}.
\end{equation}

Taking the supremum in (4.19) over $f \in D(1)$ and utilising Lumer’s representation theorem, we get:
\begin{equation}
\tag{4.20}
v^2(a) + \left( |\lambda| - \sqrt{|\lambda|^2 - r^2} \right)^2 \leq 2 \langle \lambda a, 1 \rangle_s + |\lambda| \left( |\lambda| - 2\sqrt{|\lambda|^2 - r^2} \right).
\end{equation}

Since $r \neq 0$, then $|\lambda| - \sqrt{|\lambda|^2 - r^2} > 0$, giving
\begin{equation}
\tag{4.21}
2 \left( |\lambda| - \sqrt{|\lambda|^2 - r^2} \right) v(a) \leq v^2(a) + \left( |\lambda| - \sqrt{|\lambda|^2 - r^2} \right)^2.
\end{equation}
Now, utilising (4.20) and (4.21), we deduce

\[
v(a) \leq \frac{1}{|\lambda| - \sqrt{|\lambda|^2 - r^2}} \langle \lambda a, 1 \rangle_s + \frac{|\lambda| \left( |\lambda| - 2\sqrt{|\lambda|^2 - r^2} \right)}{2 \left( |\lambda| - \sqrt{|\lambda|^2 - r^2} \right)},
\]

which is clearly equivalent with the desired result (4.17).

\[\square\]

**Remark 6.** If \( M > m \geq 0 \) and \( a \in \bar{U} (m, M) \), then

\[
v(a) \leq \frac{M + m}{\left( \sqrt{M} - \sqrt{m} \right)^2} \left[ \langle a, 1 \rangle_s + \frac{1}{2} \left( \frac{m + M}{2} - 2\sqrt{mm} \right) \right].
\]

In particular, if \( a \in \bar{U} (0, \delta) \) with \( \delta > 0 \), then we have the following reverse inequality as well

\[
(0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{1}{4} \delta.
\]

REFERENCES


