Inequalities for the Numerical Radius in Unital Normed Algebras

INEQUALITIES FOR THE NUMERICAL RADIUS IN UNITAL NORMED ALGEBRAS

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Abstract. In this paper, some inequalities between the numerical radius of an element from a unital normed algebra and certain semi-inner products involving that element and the unity are given.

1. Introduction

Let A be a unital normed algebra over the complex number field $\mathbb{C}$ and let $a \in A$. Recall that the numerical radius of a is given by (see [2, p. 15])

$$v(a) = \sup\{|f(a)|, \ f \in A', \ \|f\| \leq 1 \ \text{ and } \ f(1) = 1\}, \tag{1.1}$$

where A' denotes the dual space of A, i.e., the Banach space of all continuous linear functionals on A.

It is known that $v(\cdot)$ is a norm on A that is equivalent to the given norm $\|\cdot\|$. More precisely, the following double inequality holds true:

$$\frac{1}{e}\|a\| \leq v(a) \leq \|a\| \tag{1.2}$$

for any $a \in A$.

Following [2], we notice that this crucial result appears slightly hidden in Bohnenblust and Karlin [1, Theorem 1] together with the inequality $\|x\| \leq e\Phi(x)$, which occurs on page 219. A simpler proof was given by Lumer [5], though with the constant $\frac{1}{4}$ in place of $\frac{1}{e}$. For a simple proof of (1.2) that borrows ideas from Lumer and from Glickfeld [6], see [2, p. 34].

A generalisation of (1.2) for powers has been obtained by M.J. Crabb [3] which proved that

$$\|a^n\| \leq n!\left(\frac{e}{n}\right)^n [v(a)]^n, \qquad n = 1, 2, \ldots \tag{1.3}$$

for any $a \in A$.

In this paper, some inequalities between the numerical radius of an element and the superior semi-inner product of that element and the unity in the normed algebra A are given via the celebrated representation result of Lumer from [5].

2. Some Subsets in A

Let $D(1) := \{f \in A' \,|\, \|f\| \leq 1 \ \text{ and } \ f(1) = 1\}$. For $\lambda \in \mathbb{C}$ and $r > 0$, we define the subset of A by

$$\bar{\Delta}(\lambda, r) := \{a \in A \,|\ |f(a) - \lambda| \leq r \ \text{ for each } \ f \in D(1)\}.$$

Date: 16 April, 2008.

2000 *Mathematics Subject Classification.* 46H05, 46K05, 47A10.

Key words and phrases. Unital normed algebras, Numerical radius, Semi-inner products.

The following result holds.

Proposition 1. *Let $\lambda \in \mathbb{C}$ and $r > 0$. Then $\bar{\Delta}(\lambda, r)$ is a closed convex subset of A and*

$$\bar{B}(\lambda, r) \subseteq \bar{\Delta}(\lambda, r), \tag{2.1}$$

where $\bar{B}(\lambda, r) := \{a \in A | \, \|a - \lambda\| \leq r\}$.

Now, for $\gamma, \Gamma \in \mathbb{C}$, define the set

$$\bar{U}(\gamma, \Gamma) := \left\{ a \in A | \operatorname{Re}\left[(\Gamma - f(a)) \left(\overline{f(a)} - \bar{\gamma} \right) \right] \geq 0 \text{ for each } f \in D(1) \right\}.$$

The following representation result may be stated.

Proposition 2. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have:*

$$\bar{U}(\gamma, \Gamma) = \bar{\Delta}\left(\frac{\gamma + \Gamma}{2}, \frac{1}{2}|\Gamma - \gamma| \right). \tag{2.2}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.2) is thus a simple conclusion of this fact. □

Making use of some obvious properties in $\mathbb{C}$ and for continuous linear functionals, we can state the following corollary as well.

Corollary 1. *For any $\gamma, \Gamma \in \mathbb{C}$, we have*

$$\begin{aligned} \bar{U}(\gamma, \Gamma) &= \left\{ a \in A \mid \operatorname{Re}\left[f(\Gamma - a) \overline{f(a - \gamma)} \right] \geq 0 \text{ for each } f \in D(1) \right\} \\ &= \{ a \in A \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(a))(\operatorname{Re} f(a) - \operatorname{Re}\gamma) \\ &\quad + (\operatorname{Im}\Gamma - \operatorname{Im} f(a))(\operatorname{Im} f(a) - \operatorname{Im}\gamma) \geq 0 \text{ for each } f \in D(1) \}. \end{aligned} \tag{2.3}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following subset of A:

$$\begin{aligned} \bar{S}(\gamma, \Gamma) := \{ a \in A \mid \ & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(a) \geq \operatorname{Re}(\gamma) \quad \text{and} \\ & \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(a) \geq \operatorname{Im}(\gamma) \quad \text{for each } f \in D(1) \}. \end{aligned} \tag{2.4}$$

One can easily observe that $\bar{S}(\gamma, \Gamma)$ is closed, convex and

$$\bar{S}(\gamma, \Gamma) \subseteq \bar{U}(\gamma, \Gamma). \tag{2.5}$$

3. Semi-Inner Products and Lumer's Theorem

Let $(X, \|\cdot\|)$ be a normed linear space over the real of complex number field $\mathbb{K}$. The mapping $f : X \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2$ is obviously convex and then there exists the following limits:

$$\langle x, y\rangle_i = \lim_{t\to 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t},$$
$$\langle x, y\rangle s = \lim_{t\to 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t}$$

for every two elements $x, y \in X$. The mapping $\langle \cdot, \cdot\rangle_s$ ($\langle \cdot, \cdot\rangle_i$) will be called the *superior semi-inner product* (*the interior semi-inner product*) associated to the norm $\|\cdot\|$.

We list some properties of these semi-inner products that can be easily derived from the definition (see for instance [4]):

(i) $\langle x, x\rangle_p = \|x\|^2$; $\langle ix, x\rangle_p = \langle x, ix\rangle_p = 0$, $x \in X$;
(ii) $\langle \lambda x, y\rangle_p = \lambda \langle x, y\rangle_p$; $\langle x, \lambda y\rangle_p = \lambda \langle x, y\rangle_p$ for $\lambda \geq 0$, $x, y \in X$;
(iii) $\langle \lambda x, y\rangle_p = \lambda \langle x, y\rangle_q$; $\langle x, \lambda y\rangle_p = \lambda \langle x, y\rangle_q$ for $\lambda < 0$, $x, y \in X$;
(iv) $\langle ix, y\rangle_p = -\langle x, iy\rangle_p$; $\langle \alpha x, \beta y\rangle = \alpha\beta \langle x, y\rangle$ if $\alpha\beta \geq 0$, $x, y \in X$;
(v) $\langle -x, y\rangle_p = \langle x, -y\rangle_p = -\langle x, y\rangle_q$, $x, y \in X$;
(vi) $\left|\langle x, y\rangle_p\right| \leq \|x\| \|y\|$, $x, y \in X$;
(vii) $\langle x_1 + x_2, y\rangle_{s(i)} \leq (\geq) \langle x_1, y\rangle_{s(i)} + \langle x_2, y\rangle_{s(i)}$ for $x_1, x_2, y \in X$;
(ix) $\langle \alpha x + y, x\rangle_p = \alpha \|x\|^2 + \langle y, x\rangle_p$, $\alpha \in \mathbb{R}$, $x, y \in X$;
(x) $\left|\langle y + z, x\rangle_p - \langle z, x\rangle_p\right| \leq \|y\| \|x\|$, $x, y, z \in X$;
(xi) The mapping $\langle \cdot, x\rangle_p$ is continuous on $(X, \|\cdot\|)$ for each $x \in X$, where $p, q \in \{s, i\}$ and $p \neq q$.

The following result essentially due to Lumer [5] (see [2, p. 17]) can be stated.

Theorem 1. *Let A be a unital normed algebra over $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$. For each $a \in A$,*

$$\max\{\operatorname{Re}\lambda | \lambda \in V(a)|\} = \inf_{\alpha>0} \frac{1}{\alpha}[\|1 + \alpha a\| - 1] = \lim_{\alpha\to 0^+} \frac{1}{\alpha}[\|1 + \alpha a\| - 1], \tag{3.1}$$

where $V(a)$ is the numerical range of a (see for instance [2, p. 15]*).*

Remark 1. *In terms of semi-inner products, the above identity can be stated as:*

$$\max\{\operatorname{Re} f(a) | f \in D(1)\} = \langle a, 1\rangle_s. \tag{3.2}$$

The following result that provides more information may be stated.

Theorem 2. *For any $a \in A$, we have:*

$$\langle a, 1\rangle_{v,s} = \langle a, 1\rangle_s, \tag{3.3}$$

where

$$\langle a, b\rangle_{v,s} := \lim_{t\to 0^+} \frac{v^2(b + ta) - v^2(b)}{2t}$$

is the superior semi-inner product associated with the numerical radius.

Proof. Since $v(a) \leq \|a\|$, we have:

$$\begin{aligned}\langle a, 1\rangle_{v,s} &= \lim_{t\to 0^+} \frac{v^2(1+ta) - v^2(1)}{2t} = \lim_{t\to 0^+} \frac{v^2(1+ta) - 1}{2t} \\ &\leq \lim_{t\to 0^+} \frac{\|1+ta\|^2 - 1}{2t} = \langle a, 1\rangle_s .\end{aligned}$$

Now, let $f \in D(1)$. Then, for each $\alpha > 0$,

$$f(a) = \frac{1}{\alpha}[f(1+\alpha a) - f(1)] = \frac{1}{\alpha}[f(1+\alpha a) - 1],$$

giving

$$\begin{aligned}\operatorname{Re} f(a) &= \frac{1}{\alpha}[\operatorname{Re} f(1+\alpha a) - f(1)] \leq \frac{1}{\alpha}[|f(1+\alpha a)| - 1] \\ &\leq \frac{1}{\alpha}[v(1+\alpha a) - 1].\end{aligned}$$

Taking the infimum over $\alpha > 0$, we deduce

$$\begin{aligned}\operatorname{Re} f(a) &\leq \inf_{\alpha>0}\left[\frac{1}{\alpha}[v(1+\alpha a) - 1]\right] = \lim_{\alpha\to 0^+}\left[\frac{v^2(1+\alpha a) - 1}{2\alpha}\right] \\ &= \lim_{\alpha\to 0^+} \frac{v(1+\alpha a) - 1}{\alpha} = \langle a, 1\rangle_{v,s} .\end{aligned} \tag{3.4}$$

If we now take the supremum over $f \in D(1)$ in (3.4), we obtain:

$$\sup\{\operatorname{Re} f(a) \,|\, f \in D(1)\} \leq \langle a, 1\rangle_{v,s}$$

which gives, by Lumer's identity that $\langle a, 1\rangle_s \leq \langle a, 1\rangle_{v,s}$. □

Corollary 2. *We have the inequality*

$$|\langle a, 1\rangle_s| \leq v(a) \quad (\leq \|a\|). \tag{3.5}$$

Proof. Schwarz's inequality for the norm $v(.)$ gives that

$$\left|\langle a, 1\rangle_{v,s}\right| \leq v(a)\, v(1) = v(a),$$

and by (3.3), the inequality (3.5) is proved. □

4. Reverse Inequalities for the Numerical Radius

Utilising the inequality (3.5) we observe that for any complex number β located in the closed disc centered in 0 and with radius 1 we have $|\langle \beta a, 1\rangle_s|$ as a lower bound for the numerical radius $v(a)$. Therefore, it is a natural question to ask how far these quantities are from each other under various assumptions for the element a in the unital normed algebra A and the scalar β. A number of results answering this question are incorporated in the following theorems.

Theorem 3. *Let* $\lambda \in \mathbb{C}\setminus\{0\}$ *and* $r > 0$. *If* $a \in \bar{\Delta}(\lambda, r)$, *then*

$$v(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1\right\rangle_s + \frac{1}{2}\cdot\frac{r^2}{|\lambda|}. \tag{4.1}$$

Proof. Since $a \in \bar{\Delta}(\lambda, 1)$, then $|f(a) - \lambda|^2 \leq r^2$, for each $f \in D(1)$, giving that

$$|f(a)|^2 + |\lambda|^2 \leq 2\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right] + r^2 \tag{4.2}$$

for each $f \in D(1)$.

Taking the supremum of $f \in D(1)$ in (4.2) and utilising the representation (3.2), we deduce

$$v^2(a) + |\lambda|^2 \leq 2\left\langle \bar{\lambda}a, 1\right\rangle_s + r^2 \tag{4.3}$$

which is an inequality of interest in itself.

On the other hand, we have the elementary inequality

$$2v(a)|\lambda| \leq v^2(a) + |\lambda|^2, \tag{4.4}$$

which, together with (4.3) implies the desired result (4.1). □

Remark 2. *Notice that, by the inclusion (2.1) a sufficient condition for (4.1) to holds is that* $a \in \bar{B}(\lambda, r)$.

Corollary 3. *Let* $\gamma, \Gamma \in \mathbb{C}$ *with* $\Gamma \neq \pm\gamma$. *If* $a \in \bar{U}(\gamma, \Gamma)$, *then*

$$v(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|}a, 1\right\rangle_s + \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \tag{4.5}$$

Remark 3. *If* $M > m \geq 0$ *and* $a \in \bar{U}(m, M)$, *then*

$$(0 \leq)\, v(a) - \langle a, 1\rangle_s \leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M}. \tag{4.6}$$

Observe that, due to the inclusion (2.5), a sufficient condition for (4.6) to holds is that $M \geq \operatorname{Re} f(a), \operatorname{Im} f(a) \geq m$ *for any* $f \in D(1)$.

The following result may be stated as well.

Theorem 4. *Let* $\lambda \in \mathbb{C}$ *and* $r > 0$ *with* $|\lambda| > r$. *If* $a \in \bar{\Delta}(\lambda, r)$, *then*

$$v(a) \leq \left\langle \frac{\bar{\lambda}}{\sqrt{|\lambda|^2 - r^2}}a, 1\right\rangle_s \tag{4.7}$$

and, equivalently,

$$v^2(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|}a, 1\right\rangle_s^2 + \frac{r^2}{|\lambda|^2} \cdot v^2(a). \tag{4.8}$$

Proof. Since $|\lambda| > r$, hence by (4.3) we have, on dividing by $\sqrt{|\lambda|^2 - r^2} > 0$, that

$$\frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leq \frac{2}{\sqrt{|\lambda|^2 - r^2}}\left\langle \bar{\lambda}a, 1\right\rangle_s. \tag{4.9}$$

On the other hand, we also have

$$2v(a) \leq \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2}$$

which, together with (4.9), gives

$$v(a) \leq \frac{1}{\sqrt{|\lambda|^2 - r^2}}\left\langle \bar{\lambda}a, 1\right\rangle_s. \tag{4.10}$$

Taking the square in (4.10), we have

$$v^2(a)\left(|\lambda|^2 - r^2\right) \le \left\langle \bar{\lambda} a, 1\right\rangle_s^2,$$

which is clearly equivalent to (4.7). □

Corollary 4. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,*

$$v(a) \le \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} a, 1\right\rangle_s. \tag{4.11}$$

Remark 4. *If $M \ge m > 0$ and $a \in \bar{U}(m, M)$, then*

$$v(a) \le \frac{M+m}{2\sqrt{mM}} \langle a, 1\rangle_s, \tag{4.12}$$

or, equivalently,

$$(0 \le)\, v(a) - \langle a, 1\rangle_s \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \langle a, 1\rangle_s \quad \left(\le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \|a\| \right).$$

The following result may be stated as well.

Theorem 5. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$v^2(a) \le \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1\right\rangle_s^2 + 2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)\left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1\right\rangle_s. \tag{4.13}$$

Proof. Since (by (4.2)) $\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right] > 0$, then dividing by it in (4.2) gives:

$$\frac{|f(a)|^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} + \frac{|\lambda|^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} \le 2 + \frac{r^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]},$$

which is clearly equivalent to:

$$\begin{aligned} &\frac{|f(a)|^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} - \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{|\lambda|^2} \\ &\qquad \le 2 + \frac{r^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} - \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{|\lambda|^2} - \frac{|\lambda|^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} =: I. \end{aligned} \tag{4.14}$$

Since

$$\begin{aligned} I &= 2 - \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{|\lambda|^2} - \frac{\left(|\lambda|^2 - r^2\right)}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} \\ &= 2 - 2\frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|} - \left[\frac{\sqrt{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}}{|\lambda|} - \frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}}\right]^2 \\ &\le 2\left(1 - \sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}\right), \end{aligned} \tag{4.15}$$

hence by (4.14) and (4.15) we have

$$|f(a)|^2 \le \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{|\lambda|^2} + 2\left(1-\sqrt{1-\left(\frac{r}{|\lambda|}\right)^2}\right)\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]. \tag{4.16}$$

Taking the supremum in $f \in D(1)$ and utilising Lumer's result, we deduce the desired inequality (4.13). □

Corollary 5. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,*

$$v^2(a) \le \left\langle \frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}a, 1\right\rangle_s^2 + 2\left(\left|\frac{\gamma+\Gamma}{2}\right| - \sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right)\left\langle \frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}a, 1\right\rangle_s.$$

Remark 5. *If $M > m \ge 0$ and $a \in \bar{U}(m, M)$, then*

$$(0 \le)\, v^2(a) - \langle a, 1\rangle_s^2 \le \left(\sqrt{M}-\sqrt{m}\right)^2 \langle a, 1\rangle_s \left(\le \left(\sqrt{M}-\sqrt{m}\right)^2 \|a\|\right).$$

Finally, the following result can be stated as well.

Theorem 6. *Let $\lambda \in \mathbb{C}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$\begin{aligned} v(a) \le{}& \left(|\lambda| + \sqrt{|\lambda|^2 - r^2}\right)\left\langle \frac{\bar{\lambda}}{r^2}a, 1\right\rangle_s \\ &+ \frac{|\lambda|\left(|\lambda| + \sqrt{|\lambda|^2 - r^2}\right)\left(|\lambda| - 2\sqrt{|\lambda|^2 - r^2}\right)}{2r^2}. \end{aligned} \tag{4.17}$$

Proof. From the proof of Theorem 3 above, we have

$$|f(a)|^2 + |\lambda|^2 \le 2\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right] + r^2 \tag{4.18}$$

which is equivalent with

$$\begin{aligned} &|f(a)|^2 + \left(|\lambda| + \sqrt{|\lambda|^2 - r^2}\right)^2 \\ &\le 2\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right] + r^2 - |\lambda|^2 + \left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)^2 \\ &= 2\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right] + |\lambda|^2 - 2|\lambda|\sqrt{|\lambda|^2 - r^2}. \end{aligned} \tag{4.19}$$

Taking the supremum in (4.19) over $f \in D(1)$ and utilising Lumer's representation theorem, we get:

$$v^2(a) + \left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)^2 \le 2\left\langle \bar{\lambda}a, 1\right\rangle_s + |\lambda|\left(|\lambda| - 2\sqrt{|\lambda|^2 - r^2}\right). \tag{4.20}$$

Since $r \neq 0$, then $|\lambda| - \sqrt{|\lambda|^2 - r^2} > 0$, giving

$$2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)v(a) \le v^2(a) + \left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)^2. \tag{4.21}$$

Now, utilising (4.20) and (4.21), we deduce

$$v(a) \le \frac{1}{|\lambda| - \sqrt{|\lambda|^2 - r^2}} \left\langle \bar{\lambda} a, 1 \right\rangle_s + \frac{|\lambda| \left(|\lambda| - 2\sqrt{|\lambda|^2 - r^2} \right)}{2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right)},$$

which is clearly equivalent with the desired result (4.17). □

Remark 6. *If $M > m \ge 0$ and $a \in \bar{U}(m, M)$, then*

$$v(a) \le \frac{M+m}{\left(\sqrt{M} - \sqrt{m}\right)^2} \left[\langle a, 1 \rangle_s + \frac{1}{2} \left(\frac{m+M}{2} - 2\sqrt{mM} \right) \right].$$

In particular, if $a \in \bar{U}(0, \delta)$ with $\delta > 0$, then we have the following reverse inequality as well

$$(0 \le)\, v(a) - \langle a, 1 \rangle_s \le \frac{1}{4}\delta.$$

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