Some Slater's Type Inequalities for Convex Functions of Selfadjoint Operators in Hilbert Spaces

SOME SLATER'S TYPE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities of the Slater type for convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following result is well known in the literature as *the Slater inequality:*

Theorem 1 (Slater, 1981, [15]). *If $f : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $x_i \in I, p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$ and $\sum_{i=1}^n p_i\varphi(x_i) \neq 0$, where $\varphi \in \partial f$, then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \leq f\left(\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)}\right). \tag{1.1}$$

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As pointed out in [1, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I, \tag{1.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

2. Some Operator Inequalities for Convex Functions

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [7, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [7] and the references therein. For other results, see [14], [8] and [10].

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [12] (see also [7, p. 5]):

Theorem 2 (Mond-Pečarić, 1993, [12]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle, \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 2 we have the following Hölder-McCarthy inequality:

Theorem 3 (Hölder-McCarthy, 1967, [9]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
(iii) If A is invertible, then $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

The following result that provides a reverse of the Mond & Pečarić has been obtained in [4]:

Theorem 4 (Dragomir, 2008, [4])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle, \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$.

Perhaps more convenient reverses of the Mond & Pečarić result are the following inequalities that have been obtained in the same paper [4]:

Theorem 5 (Dragomir, 2008, [4])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$\begin{aligned}(0 \leq) & \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ & \leq \begin{cases} \frac{1}{2} \cdot (M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\ & \leq \frac{1}{4}(M-m)(f'(M) - f'(m)),\end{aligned} \tag{2.2}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$\begin{aligned}(0 \leq) & \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \frac{1}{4}(M-m)(f'(M) - f'(m)) \\ & - \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x, x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \\ & \leq \frac{1}{4}(M-m)(f'(M) - f'(m)),\end{aligned} \tag{2.3}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$\begin{aligned}(0 \leq) & \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ & \leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M) - f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x\rangle \langle f'(A)x, x\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right)[\langle Ax, x\rangle \langle f'(A)x, x\rangle]^{\frac{1}{2}}, \end{cases}\end{aligned} \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

For generalisations to n-tuples of operators as well as for some particular cases of interest, see [4].

The main aim of the present paper is to provide some Slater's type vector inequalities for convex functions whose derivatives are continuous.

3. Some Slater's Type Inequalities

The following result holds:

Theorem 6. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive definite operator on H then*

$$0 \leq f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \leq f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)\left[\frac{\langle Af'(A)x, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right], \tag{3.1}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s)\cdot(t-s) \leq f(t) - f(s) \leq f'(t)\cdot(t-s) \tag{3.2}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then for any $x \in H$ with $\|x\| = 1$ we have

$$\langle f'(A)\cdot(t\cdot 1_H - A)x, x\rangle \leq \langle [f(t)\cdot 1_H - f(A)]x, x\rangle \leq \langle f'(t)\cdot(t\cdot 1_H - A)x, x\rangle \tag{3.3}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (3.3) is equivalent with

$$t\langle f'(A)x, x\rangle - \langle f'(A)Ax, x\rangle \leq f(t) - \langle f(A)x, x\rangle \leq f'(t)t - f'(t)\langle Ax, x\rangle \tag{3.4}$$

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

Now, since A is selfadjoint with $mI \leq A \leq MI$ and $f'(A)$ is positive definite, then $mf'(A) \leq Af'(A) \leq Mf'(A)$, i.e., $m\langle f'(A)x, x\rangle \leq \langle Af'(A)x, x\rangle \leq M\langle f'(A)x, x\rangle$ for any $x \in H$ with $\|x\| = 1$, which shows that

$$t_0 := \frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} \in [m, M] \text{ for any } x \in H \text{ with } \|x\| = 1.$$

Finally, if we put $t = t_0$ in the equation (3.4), then we get the desired result (3.1). ∎

Remark 1. *It is important to observe that, the condition that $f'(A)$ is a positive definite operator on H can be replaced with the more general assumption that*

$$\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} \in \mathring{I} \text{ for any } x \in H \text{ with } \|x\| = 1, \tag{3.5}$$

which may be easily verified for particular convex functions f.

Remark 2. *Now, if the functions is concave on $\mathring{I}$ and the condition (3.5) holds, then we have the inequality*

$$0 \le \langle f(A)x,x\rangle - f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) \le f'\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)\left[\frac{\langle Ax,x\rangle\langle f'(A)x,x\rangle - \langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right], \tag{3.6}$$

for any $x \in H$ with $\|x\| = 1$.

The following examples are of interest:

Example 1. *If A is a positive definite operator on H, then*

$$(0 \le) \langle \ln Ax,x\rangle - \ln\left(\left\langle A^{-1}x,x\right\rangle^{-1}\right) \le \langle Ax,x\rangle \cdot \left\langle A^{-1}x,x\right\rangle - 1, \tag{3.7}$$

for any $x \in H$ with $\|x\| = 1$.

Indeed, we observe that if we consider the concave function $f : (0,\infty) \to \mathbb{R}$, $f(t) = \ln t$, then

$$\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle} = \frac{1}{\langle A^{-1}x,x\rangle} \in (0,\infty), \text{ for any } x \in H \text{ with } \|x\| = 1$$

and by the inequality (3.6) we deduce the desired result (3.7).

The following example concerning powers of operators is of interest as well:

Example 2. *If A is a positive definite operator on H, then for any $x \in H$ with $\|x\| = 1$ we have*

$$0 \le \langle A^p x,x\rangle^{p-1} - \left\langle A^{p-1}x,x\right\rangle^p \le p\langle A^p x,x\rangle^{p-2}\left[\langle A^p x,x\rangle - \langle Ax,x\rangle\left\langle A^{p-1}x,x\right\rangle\right] \tag{3.8}$$

for $p \ge 1$,

$$0 \le \left\langle A^{p-1}x,x\right\rangle^p - \langle A^p x,x\rangle^{p-1} \le p\langle A^p x,x\rangle^{p-2}\left[\langle Ax,x\rangle\left\langle A^{p-1}x,x\right\rangle - \langle A^p x,x\rangle\right] \tag{3.9}$$

for $0 < p < 1$, and

$$0 \le \langle A^p x,x\rangle^{p-1} - \left\langle A^{p-1}x,x\right\rangle^p \le (-p)\langle A^p x,x\rangle^{p-2}\left[\langle Ax,x\rangle\left\langle A^{p-1}x,x\right\rangle - \langle A^p x,x\rangle\right] \tag{3.10}$$

for $p < 0$.

The proof follows from the inequalities (3.1) and (3.6) for the convex (concave) function $f(t) = t^p, p \in (-\infty,0) \cup [1,\infty)$ $(p \in (0,1))$ by performing the required calculation. The details are omitted.

4. Further Reverses

The following results that provide perhaps more useful upper bounds for the nonnegative quantity

$$f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) - \langle f(A)x,x\rangle \text{ for } x \in H \text{ with } \|x\| = 1,$$

can be stated:

Theorem 7. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. Assume that A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive definite operator on H. If we define*

$$B(f', A; x) := \frac{1}{\langle f'(A)x, x\rangle} . f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)$$

then

$$(4.1)\quad (0 \le) f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle$$
$$\le B(f', A; x) \times \begin{cases} \frac{1}{2} \cdot (M - m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases}$$
$$\le \frac{1}{4}(M - m)(f'(M) - f'(m))\, B(f', A; x)$$

and

$$(4.2)\quad (0 \le) f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle$$
$$\le B(f', A; x) \times \left[\frac{1}{4}(M - m)(f'(M) - f'(m))\right.$$
$$\left. - \begin{cases} \left[\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x, x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \right]$$
$$\le \frac{1}{4}(M - m)(f'(M) - f'(m))\, B(f', A; x),$$

for any $x \in H$ with $\|x\| = 1$, respectively.

Moreover, if A is a positive definite operator, then

$$(4.3)\quad (0 \le) f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle$$
$$\le B(f', A; x) \times \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x\rangle \langle f'(A)x, x\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right)\left[\langle Ax, x\rangle \langle f'(A)x, x\rangle\right]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. We use the following Grüss' type result we obtained in [2]:

Let A be a selfadjoint operator on the Hilbert space $(H; \langle ., .\rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If h and g are continuous on $[m, M]$ and $\gamma := \min_{t\in[m,M]} h(t)$ and $\Gamma := \max_{t\in[m,M]} h(t)$, then

$$(4.4)\quad |\langle h(A) g(A)x, x\rangle - \langle h(A)x, x\rangle \cdot \langle g(A)x, x\rangle|$$
$$\le \frac{1}{2} \cdot (\Gamma - \gamma)\left[\|g(A)x\|^2 - \langle g(A)x, x\rangle^2\right]^{1/2} \left(\le \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta)\right),$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Therefore, we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \leq \frac{1}{2} \cdot (M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x,x\rangle^2\right]^{1/2} \leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \tag{4.5}$$

and

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \leq \frac{1}{2} \cdot (f'(M)-f'(m))\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2} \leq \frac{1}{4}(M-m)(f'(M)-f'(m)), \tag{4.6}$$

for each $x \in H$ with $\|x\| = 1$, which together with (3.1) provide the desired result (4.1).

On making use of the inequality obtained in [3]

$$|\langle h(A)g(A)x,x\rangle - \langle h(A)x,x\rangle\langle g(A)x,x\rangle| \leq \frac{1}{4} \cdot (\Gamma-\gamma)(\Delta-\delta) - \begin{cases} [\langle \Gamma x - h(A)x, f(A)x - \gamma x\rangle \langle \Delta x - g(A)x, g(A)x - \delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle h(A)x,x\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\langle g(A)x,x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \tag{4.7}$$

for each $x \in H$ with $\|x\| = 1$, we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \leq \frac{1}{4}(M-m)(f'(M)-f'(m)) - \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \left|\langle Ax,x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x,x\rangle - \frac{f'(M)+f'(m)}{2}\right|, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (3.1) provide the desired result (4.2).

Further, in order to proof the third inequality, we make use of the following result of Grüss' type we obtained in [3]:

If γ and δ are positive, then

$$|\langle h(A)g(A)x,x\rangle - \langle h(A)x,x\rangle\langle g(A)x,x\rangle| \leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \langle h(A)x,x\rangle\langle g(A)x,x\rangle, \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)[\langle h(A)x,x\rangle\langle g(A)x,x\rangle]^{\frac{1}{2}}, \end{cases} \tag{4.8}$$

for each $x \in H$ with $\|x\| = 1$.

Now, on making use of (4.8) we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \le \begin{cases} \frac{1}{4}\cdot\frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}}\langle Ax,x\rangle\langle f'(A)x,x\rangle, \\ \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)\left[\langle Ax,x\rangle\langle f'(A)x,x\rangle\right]^{\frac{1}{2}}, \end{cases}$$

for each $x\in H$ with $\|x\|=1$, which together with (3.1) provide the desired result (4.3). ∎

Remark 3. *We observe, from the first inequality in (4.3), that*

$$(1\le)\ \frac{\langle Af'(A)x,x\rangle}{\langle Ax,x\rangle\langle f'(A)x,x\rangle}\le\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1$$

which implies that

$$f'\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)\le f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right),$$

for each $x\in H$ with $\|x\|=1$, since f' is monotonic nondecreasing and A is positive definite.

Now, the first inequality in (4.3) implies the following result

$$(0\le)\ f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle \le\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \times f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right)\langle Ax,x\rangle, \tag{4.9}$$

for each $x\in H$ with $\|x\|=1$.

From the second inequality in (4.3) we also have

$$(0\le)\ f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle \le\left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right) \times f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right)\left[\frac{\langle Ax,x\rangle}{\langle f'(A)x,x\rangle}\right]^{\frac{1}{2}}, \tag{4.10}$$

for each $x\in H$ with $\|x\|=1$.

Remark 4. *If the condition that $f'(A)$ is a positive definite operator on H from the Theorem 7 is replaced by the condition (3.5), then the inequalities (4.1) and (4.4) will still hold. Similar inequalities for concave functions can be stated. However, the details are not provided here.*

5. Multivariate Versions

The following result for sequences of operators can be stated.

Theorem 8. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If $A_j, j \in \{1, ..., n\}$ are selfadjoint operators on the Hilbert space H with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$ and*

$$\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle} \in \mathring{I} \tag{5.1}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$\begin{aligned} 0 &\le f\left(\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}\right) - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \\ &\le f'\left(\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}\right) \\ &\times \left[\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}\right], \end{aligned} \tag{5.2}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. As in [7, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix},$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) g x_j, x_j \rangle, \left\langle f'\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle,$$

$$\left\langle \widetilde{A} f'\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle$$

and so on.

Applying Theorem 6 under the condition (3.5) for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result. The details are omitted. ∎

The following particular case is of interest

Corollary 1. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If $A_j, j \in \{1, ..., n\}$ are selfadjoint operators on the Hilbert space H with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$*

and for $p_j \geq 0$ with $\sum_{j=1}^{n} p_j = 1$ if we also assume that

$$\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle} \in \mathring{I} \tag{5.3}$$

for each $x \in H$ with $\|x\| = 1$, then

$$\begin{aligned} 0 &\leq f\left(\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle} \right) - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \\ &\leq f'\left(\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle} \right) \\ &\times \left[\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle} \right], \end{aligned} \tag{5.4}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Follows from Theorem 8 on choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$, where $p_j \geq 0, j \in \{1, ..., n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

The following examples are interesting in themselves:

Example 3. *If A_j, $j \in \{1, ..., n\}$ are positive definite operators on H, then*

$$\begin{aligned} (0 \leq) \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle - \ln \left[\left(\sum_{j=1}^{n} \langle A_j^{-1} x_j, x_j \rangle \right)^{-1} \right] \\ \leq \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^{-1} x_j, x_j \rangle - 1, \end{aligned} \tag{5.5}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then we also have the inequality

$$\begin{aligned} (0 \leq) \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle - \ln \left[\left(\left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle \right)^{-1} \right] \\ \leq \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle - 1, \end{aligned} \tag{5.6}$$

for each $x \in H$ with $\|x\| = 1$.

The following inequalities for powers also hold:

Example 4. *If A_j, $j \in \{1,...,n\}$ are positive definite operators on H, then for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$ we have*

$$
\begin{aligned}
(5.7) \quad 0 &\le \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right)^{p-1} - \left(\sum_{j=1}^{n} \left\langle A_j^{p-1} x_j, x_j\right\rangle\right)^{p} \\
&\le p \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right)^{p-2} \\
&\quad \times \left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j\rangle \sum_{j=1}^{n} \left\langle A_j^{p-1} x_j, x_j\right\rangle\right]
\end{aligned}
$$

for $p \ge 1$,

$$
\begin{aligned}
(5.8) \quad 0 &\le \left(\sum_{j=1}^{n} \left\langle A_j^{p-1} x_j, x_j\right\rangle\right)^{p} - \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right)^{p-1} \\
&\le p \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right)^{p-2} \\
&\quad \times \left[\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle \sum_{j=1}^{n} \left\langle A_j^{p-1} x_j, x_j\right\rangle - \sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right]
\end{aligned}
$$

for $0 < p < 1$, and

$$
\begin{aligned}
(5.9) \quad 0 &\le \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right)^{p-1} - \left(\sum_{j=1}^{n} \left\langle A_j^{p-1} x_j, x_j\right\rangle\right)^{p} \\
&\le (-p) \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right)^{p-2} \\
&\quad \times \left[\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle \sum_{j=1}^{n} \left\langle A_j^{p-1} x_j, x_j\right\rangle - \sum_{j=1}^{n} \langle A_j^p x_j, x_j\rangle\right]
\end{aligned}
$$

for $p < 0$.

Now, for any $p_j \geq 0$ *with* $\sum_{j=1}^n p_j = 1$ *and for any* $x \in H$ *with* $\|x\| = 1$ *we also have the inequalities*

$$0 \leq \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle^{p-1} - \left\langle \sum_{j=1}^n p_j A_j^{p-1} x, x \right\rangle^p \leq p \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle^{p-2} \times \left[\left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n p_j A_j^{p-1} x, x \right\rangle \right] \tag{5.10}$$

for $p \geq 1$,

$$0 \leq \left\langle \sum_{j=1}^n p_j A_j^{p-1} x, x \right\rangle^p - \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle^{p-1} \leq p \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle^{p-2} \times \left[\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n p_j A_j^{p-1} x, x \right\rangle - \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle \right] \tag{5.11}$$

for $0 < p < 1$, *and*

$$0 \leq \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle^{p-1} - \left\langle \sum_{j=1}^n p_j A_j^{p-1} x, x \right\rangle^p \leq (-p) \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle^{p-2} \times \left[\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n p_j A_j^{p-1} x, x \right\rangle - \left\langle \sum_{j=1}^n p_j A_j^p x, x \right\rangle \right] \tag{5.12}$$

for $p < 0$.

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