

INTEGRAL INEQUALITIES AND APPLICATIONS FOR BOUNDING THE ČEBYŠEV FUNCTIONAL

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ABSTRACT. Some inequalities related to the Hölder integral inequality and applications for bounding the Čebyšev functional are given.

1. INTRODUCTION

The integral Hölder inequality, namely

$$(1.1) \quad \left| \int_a^b f(t)g(t) dt \right| \leq \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q dt \right)^{\frac{1}{q}},$$

plays an important role in Mathematical Analysis and its applications. Here the complex-valued functions $f, g : [a, b] \rightarrow \mathbb{C}$ are p and q -integrable respectively on $[a, b]$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In order to provide sharper bounds for the Hölder inequality, Abramovich, Mond and Pečarić considered in [1] the function $\Phi : [a, b] \rightarrow \mathbb{R}$ given by

$$(1.2) \quad \Phi(x) := \left| \int_a^x f(t)g(t) dt \right| + \left(\int_x^b |f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_x^b |g(t)|^q dt \right)^{\frac{1}{q}}$$

and proved that $\Phi(\cdot)$ is nondecreasing on $[a, b]$. As a consequence we can observe that

$$\inf_{x \in [a, b]} \Phi(x) = \left| \int_a^b f(t)g(t) dt \right|$$

and

$$\sup_{x \in [a, b]} \Phi(x) = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q dt \right)^{\frac{1}{q}}.$$

Using geometrical arguments, G.S. Mahajani [8] obtained the following results for the absolute value of the integral $\int_a^x f(t) dt$:

1. *If f has a bounded derivative on $[a, b]$, namely $|f'(t)| \leq M$ ($M > 0$) and if $\int_a^b f(t) dt = 0$, then*

$$\left| \int_a^x f(t) dt \right| \leq \frac{1}{8} \cdot M \cdot (b - a)^2,$$

for any $x \in [a, b]$.

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2. If, additional to the conditions given above, $f(a) = f(b) = 0$, then

$$\left| \int_a^x f(t) dt \right| \leq \frac{1}{16} \cdot M \cdot (b-a)^2.$$

Analytic proofs of these results were given by P.R. Beesack, [9, p. 474]. For other results related to the Mahajani inequality see Chapter XV of [9].

In this paper some similar results are obtained and applied in obtaining bounds for the quantities $\left| \int_a^x f(t) dt \right|$,

$$\int_a^b \left| \int_a^x f(t) dt - \frac{x-a}{b-a} \int_a^b f(t) dt \right|^r dx, \quad r \in [1, \infty)$$

and

$$\sup_{x \in [a, b]} \left| \int_a^x f(t) dt - \frac{x-a}{b-a} \int_a^b f(t) dt \right|,$$

under various assumptions for the function $f : [a, b] \rightarrow \mathbb{R}$.

These results are also utilized to provide bounds for the Čebyšev functional

$$T(f, g) := \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx,$$

where $f, g : [a, b] \rightarrow \mathbb{C}$ are Lebesgue integrable functions, in terms of the *shifted integral means*:

$$\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^r dt$$

with $r \in [1, \infty)$. This is possible due to the following representation result obtained in [2]:

$$T(f, g) = -\frac{1}{b-a} \int_a^b \left(\int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right) f'(x) dx,$$

that holds for g Lebesgue integrable and f absolutely continuous on $[a, b]$.

For recent results on bounding the Čebyšev functional $T(\cdot, \cdot)$ see [2], [3] and [5] where further references are provided.

2. THE RESULTS

The following result may be stated:

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{C}$ be a Lebesgue measurable function and $p : [a, b] \rightarrow [0, \infty)$ a Lebesgue integrable weight with $\int_a^b p(t) dt = 1$. For any $r > 1$ and $x \in [a, b]$, we have the inequality:

$$(2.1) \quad \int_a^x p(t) |f(t)|^r dt + \frac{\left| \int_a^b p(t) f(t) dt - \int_a^x p(t) f(t) dt \right|^r}{\left[1 - \int_a^x p(t) dt \right]^{r-1}} \leq \int_a^b p(t) |f(t)|^r dt.$$

In particular,

$$(2.2) \quad \int_a^x |f(t)|^r dt + \frac{\left| \int_a^b f(t) dt - \int_a^x f(t) dt \right|^r}{(b-x)^{r-1}} \leq \int_a^b |f(t)|^r dt.$$

Proof. Obviously,

$$(2.3) \quad \int_a^b p(t) f(t) dt - \int_a^x p(t) f(t) dt = \int_x^b p(t) f(t) dt$$

for any $x \in [a, b]$.

Utilising the Hölder inequality, we have for $r > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ that

$$(2.4) \quad \begin{aligned} & \left| \int_x^b p(t) f(t) dt \right| \\ & \leq \left(\int_x^b p(t) dt \right)^{\frac{1}{q}} \left(\int_x^b p(t) |f(t)|^r dt \right)^{\frac{1}{r}} \\ & = \left(\int_a^b p(t) dt - \int_a^x p(t) dt \right)^{\frac{1}{q}} \left(\int_a^b p(t) |f(t)|^r dt - \int_a^x p(t) |f(t)|^r dt \right)^{\frac{1}{r}} \\ & = \left(1 - \int_a^x p(t) dt \right)^{\frac{1}{q}} \left(\int_a^b p(t) |f(t)|^r dt - \int_a^x p(t) |f(t)|^r dt \right)^{\frac{1}{r}} \end{aligned}$$

for each $x \in [a, b]$.

Utilising (2.3) and (2.4) and taking the power r , we get

$$\begin{aligned} & \left| \int_a^b p(t) f(t) dt - \int_a^x p(t) f(t) dt \right|^r \\ & \leq \left(1 - \int_a^x p(t) dt \right)^{\frac{r}{q}} \left(\int_a^b p(t) |f(t)|^r dt - \int_a^x p(t) |f(t)|^r dt \right), \end{aligned}$$

which gives the desired inequality (2.1). \square

Corollary 1. *With the assumptions of Theorem 1 and if $\int_a^b p(t) f(t) dt = 0$, then*

$$(2.5) \quad \frac{1 + \left(\int_a^x p(t) dt \right)^{1-r} \left[1 - \int_a^x p(t) dt \right]^{r-1}}{\left[1 - \int_a^x p(t) dt \right]^{r-1}} \cdot \left| \int_a^x p(t) f(t) dt \right|^r \leq \int_a^b p(t) |f(t)|^r dt$$

for any $x \in [a, b]$.

Proof. Since $\int_a^b p(t) f(t) dt = 0$, then, by (2.1), we have

$$(2.6) \quad \int_a^x p(t) |f(t)|^r dt + \frac{\left| \int_a^x p(t) f(t) dt \right|^r}{\left[1 - \int_a^x p(t) dt \right]^{r-1}} \leq \int_a^b p(t) |f(t)|^r dt$$

for any $x \in [a, b]$.

Utilising Hölder's inequality for $r > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, we have

$$(2.7) \quad \begin{aligned} \left| \int_a^x p(t) f(t) dt \right|^r & \leq \left(\int_a^x p(t) dt \right)^{\frac{r}{q}} \int_a^x p(t) |f(t)|^r dt \\ & = \left(\int_a^x p(t) dt \right)^{r-1} \int_a^x p(t) |f(t)|^r dt. \end{aligned}$$

Combining (2.6) with (2.7), we get the desired result (2.5). \square

Remark 1. If $\int_a^b f(t) dt = 0$, then from inequality (2.2) we get the following result as well:

$$(2.8) \quad \frac{1 + (x-a)^{1-r} (b-x)^{r-1}}{(b-x)^{r-1}} \cdot \left| \int_a^x f(t) dt \right|^r \leq \int_a^b |f(t)|^r dt,$$

or, equivalently

$$(2.9) \quad \left| \int_a^x f(t) dt \right| \leq \left(\frac{(b-x)^{r-1}}{1 + (x-a)^{1-r} (b-x)^{r-1}} \right)^{1/r} \cdot \left(\int_a^b |f(t)|^r dt \right)^{1/r},$$

which is a Mahajani type result.

The following result is of interest:

Corollary 2. Let $g : [a, b] \rightarrow \mathbb{C}$ be a Lebesgue integrable function and $p : [a, b] \rightarrow [0, \infty)$ an integrable weight with $\int_a^b p(t) dt = 1$. Then

$$(2.10) \quad \frac{1 + \left(\int_a^x p(t) dt \right)^{1-r} \left[1 - \int_a^x p(t) dt \right]^{r-1}}{\left[1 - \int_a^x p(t) dt \right]^{r-1}} \times \left| \int_a^x p(t) g(t) dt - \int_a^x p(t) dt \cdot \int_x^b p(t) g(t) dt \right|^r \leq \int_a^b p(t) \left| g(t) - \int_a^b p(s) g(s) ds \right|^r dt,$$

for any $x \in [a, b]$.

In particular,

$$(2.11) \quad \frac{1 + (x-a)^{1-r} (b-x)^{r-1}}{(b-x)^{r-1}} \cdot \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right|^r \leq \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^r dt,$$

for each $x \in [a, b]$.

The proof is by Corollary 1 applied for $f(t) = g(t) - \int_a^b p(s) g(s) ds$, $t \in [a, b]$. Then inequality (2.11) follows by (2.8) on choosing $f(t) = g(t) - \frac{1}{b-a} \int_a^b g(s) ds$.

A similar result concerning the supremum of the weight can be stated as well:

Proposition 1. Let p, f be as in Theorem 1. Then we have the inequality

$$(2.12) \quad \int_a^x |f(t)| dt + \frac{\left| \int_a^b p(t) f(t) dt - \int_a^x p(t) f(t) dt \right|}{\sup_{t \in [x, b]} p(t)} \leq \int_a^b |f(t)| dt$$

for any $x \in [a, b]$.

Proof. We have

$$\begin{aligned} \left| \int_a^b p(t) f(t) dt - \int_a^x p(t) f(t) dt \right| &= \left| \int_x^b p(t) f(t) dt \right| \\ &\leq \sup_{t \in [x, b]} p(t) \int_x^b |f(t)| dt \\ &= \sup_{t \in [x, b]} p(t) \left[\int_a^b |f(t)| dt - \int_a^x |f(t)| dt \right], \end{aligned}$$

which easily implies (2.12). \square

Corollary 3. *If p and f are as in Corollary 1, then we have*

$$(2.13) \quad \frac{\sup_{t \in [a, x]} p(t) + \sup_{t \in [x, b]} p(t)}{\sup_{t \in [x, b]} p(t)} \left| \int_a^x p(t) f(t) dt \right| \leq \int_a^b |f(t)| dt,$$

for each $x \in [a, b]$.

Proof. Since

$$(2.14) \quad \left| \int_a^x p(t) f(t) dt \right| \leq \sup_{t \in [a, x]} p(t) \int_a^x |f(t)| dt$$

for any $x \in [a, b]$ and (2.12) becomes, under the assumption that $\int_a^b p(t) f(t) dt = 0$,

$$(2.15) \quad \int_a^x |f(t)| dt + \frac{\left| \int_a^x p(t) f(t) dt \right|}{\sup_{t \in [x, b]} p(t)} \leq \int_a^b |f(t)| dt,$$

hence by (2.14) and (2.4) we deduce the desired result (2.13). \square

Corollary 4. *If p, g are as in Corollary 2, then*

$$(2.16) \quad \frac{\sup_{t \in [a, x]} p(t) + \sup_{t \in [x, b]} p(t)}{\sup_{t \in [x, b]} p(t)} \left| \int_a^x p(t) g(t) dt - \int_a^x p(t) dt \cdot \int_x^b p(t) g(t) dt \right| \\ \leq \int_a^b \left| g(t) - \int_a^b p(s) g(s) ds \right| dt,$$

for any $x \in [a, b]$.

The following result holds as well.

Proposition 2. *With the above assumptions for f and p we have*

$$(2.17) \quad \int_a^x p(t) |f(t)| dt + \left| \int_a^b p(t) f(t) dt - \int_a^x p(t) f(t) dt \right| \leq \int_a^b p(t) |f(t)| dt$$

for any $x \in [a, b]$.

Proof. We have

$$\begin{aligned} \left| \int_a^b p(t) f(t) dt - \int_a^x p(t) f(t) dt \right| &= \left| \int_x^b p(t) f(t) dt \right| \\ &\leq \int_x^b p(t) |f(t)| dt \\ &= \int_a^b p(t) |f(t)| dt - \int_a^x p(t) |f(t)| dt, \end{aligned}$$

which is clearly equivalent to (2.17). \square

Corollary 5. *If p and f are as in Corollary 1, then*

$$(2.18) \quad 2 \left| \int_a^x p(t) f(t) dt \right| \leq \int_a^b p(t) |f(t)| dt$$

for any $x \in [a, b]$.

Corollary 6. *If p, g are as in Corollary 2, then*

$$(2.19) \quad \left| \int_a^x p(t) g(t) dt - \int_a^x p(t) dt \cdot \int_x^b p(t) g(t) dt \right| \leq \frac{1}{2} \int_a^b p(t) \left| g(t) - \int_a^b p(s) g(s) ds \right| dt.$$

Remark 2. *If in Corollary 6 we choose the uniform weight $p(t) = \frac{1}{b-a}$, $t \in [a, b]$, then (2.19) becomes:*

$$(2.20) \quad \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right| \leq \frac{1}{2} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt,$$

for each $x \in [a, b]$.

The inequality (2.20) can be seen as the limiting case of (2.11) where $r \rightarrow 1$, $r > 1$.

Remark 3. *We observe that (2.20) produces the following Mahajani type inequality, which is, in a sense, the limiting case of (2.9) for $r \rightarrow 1, r > 1$:*

$$(2.21) \quad \left| \int_a^x f(t) dt \right| \leq \frac{1}{2} \int_a^b |f(t)| dt,$$

provided $\int_a^b f(t) dt = 0$.

3. APPLICATIONS FOR GRÜSS TYPE INEQUALITIES

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$ consider the Čebyšev functional:

$$(3.1) \quad T(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

In 1934, G. Grüss [6] showed that

$$(3.2) \quad |T(f, g)| \leq \frac{1}{4} (M - m) (N - n),$$

provided m, M, n, N are real numbers with the property

$$(3.3) \quad -\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b].$$

The constant $\frac{1}{4}$ is best possible in (3.2) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $T(f, g)$ was derived in 1882 by Čebyšev [4] under the assumption that f', g' exist and are continuous on $[a, b]$, and is given by

$$(3.4) \quad |T(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2,$$

where $\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general.

Čebyšev's inequality (3.4) also holds if $f, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$.

In 1970, A.M. Ostrowski [11] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$(3.5) \quad |T(f, g)| \leq \frac{1}{8} (b-a) (M-m) \|g'\|_\infty,$$

provided f is Lebesgue integrable on $[a, b]$ and satisfying (3.3) while $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a, b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [7] (see also [10, p. 210]) obtained the following result as well:

$$(3.6) \quad |T(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a),$$

provided f, g are absolutely continuous and $f', g' \in L_2[a, b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [2], P. Cerone and S.S. Dragomir proved the following inequalities:

$$(3.7) \quad |T(f, g)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b |\bar{f}(t)| dt \\ \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b |\bar{f}(t)|^p dt \right)^{\frac{1}{p}} \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

where

$$\bar{f}(t) := f(t) - \frac{1}{b-a} \int_a^b f(s) ds, \quad t \in [a, b].$$

For $\gamma = 0$, we get from the first inequality

$$(3.8) \quad |T(f, g)| \leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b |\bar{f}(t)| dt$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a, b]$, then $\|g - \frac{m+M}{2}\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (3.7) we can deduce the following result obtained by Cheng and Sun [5]

$$(3.9) \quad |T(f, g)| \leq \frac{1}{2} (M-m) \cdot \frac{1}{b-a} \int_a^b |\bar{f}(t)| dt.$$

The constant $\frac{1}{2}$ is best in (3.9) as shown by Cerone and Dragomir in [3].

For $r > 1$, we define

$$I(r) := \int_a^b \frac{[(b-x)(x-a)]^{r-1}}{(b-x)^{r-1} + (x-a)^{r-1}} dx.$$

For $r = 2$, we have

$$(3.10) \quad I(2) = \frac{1}{b-a} \int_a^b (b-x)(x-a) dx = \frac{(b-a)^2}{6}.$$

For $r > 2$, since the inequality

$$\begin{aligned} \frac{(b-x)^{r-1} + (x-a)^{r-1}}{2} &\geq \left[\frac{(b-x) + (x-a)}{2} \right]^{r-1} \\ &= \frac{1}{2^{r-1}} (b-a)^{r-1} \end{aligned}$$

holds, then

$$(b-x)^{r-1} + (x-a)^{r-1} \geq 2^{2-r} (b-a)^{r-1}, \quad x \in [a, b],$$

and so

$$\begin{aligned} (3.11) \quad I(r) &\leq \frac{2^{r-2}}{(b-a)^{r-1}} \int_a^b [(b-x)(x-a)]^{r-1} dx \\ &\leq \frac{2^{r-2}}{(b-a)^{r-1}} \int_0^1 (b-(1-t)a-tb)^{r-1} ((1-t)a+tb-a)^{r-1} (b-a) dt \\ &= \frac{2^{r-2}}{(b-a)^{r-1}} \int_0^1 (b-a)^{r-1} (1-t)^{r-1} (b-a)^{r-1} t^{r-1} (b-a) dt \\ &= 2^{r-2} (b-a)^r B(r, r), \quad r \geq 2, \end{aligned}$$

where $B(\cdot, \cdot)$ is the well known Euler beta function.

A different possibility to bound $I(r)$ is by utilising the inequality between the harmonic and geometric means, namely

$$\frac{2\alpha\beta}{\alpha+\beta} \leq \sqrt{\alpha\beta}, \quad \alpha, \beta > 0.$$

Therefore

$$\frac{(b-x)^{r-1} (x-a)^{r-1}}{(b-x)^{r-1} + (x-a)^{r-1}} \leq \frac{1}{2} \sqrt{(b-x)^{r-1} (x-a)^{r-1}}, \quad r > 1$$

for $x \in [a, b]$, which gives by integration

$$\begin{aligned} (3.12) \quad I(r) &\leq \frac{1}{2} \int_a^b (b-x)^{\frac{r-1}{2}} (x-a)^{\frac{r-1}{2}} dx \\ &= \frac{1}{2} (b-a)^r \int_0^1 t^{\frac{r-1}{2}} (1-t)^{\frac{r-1}{2}} dt \\ &= \frac{1}{2} (b-a)^r B\left(\frac{r+1}{2}, \frac{r+1}{2}\right) \end{aligned}$$

for $r > 1$.

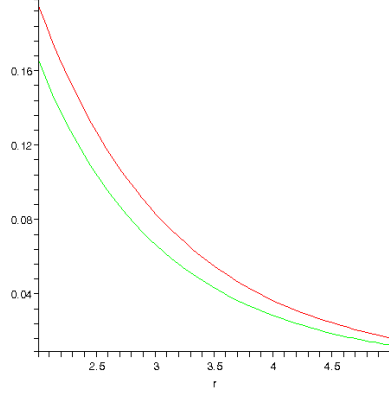


FIGURE 1. The plot of $2^{r-2}B(r, r)$ and $\frac{1}{2}B\left(\frac{r+1}{2}, \frac{r+1}{2}\right)$ for $r \geq 2$.

Remark 4. If we compare $2^{r-2}B(r, r)$ with $\frac{1}{2}B\left(\frac{r+1}{2}, \frac{r+1}{2}\right)$ for $r \geq 2$, using the Maple computer package, we observe that the bound provided by (3.11) for $I(r)$ is better than the one provided for (3.12). However, the second one is also valid for $r \in (0, 1)$. The plot concerning the variations of $2^{r-2}B(r, r)$ and $\frac{1}{2}B\left(\frac{r+1}{2}, \frac{r+1}{2}\right)$ on $[2, \infty)$ is depicted in Figure 1. However, we do not have an analytic proof to show that

$$2^{r-2}B(r, r) \leq \frac{1}{2}B\left(\frac{r+1}{2}, \frac{r+1}{2}\right)$$

for any $r \in [2, \infty)$.

The following lemma may be stated.

Lemma 1. For $r > 1$ we have the inequality

$$(3.13) \quad \int_a^b \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right|^r dt \leq I(r) \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^r dt.$$

In particular:

$$(3.14) \quad \int_a^b \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right|^2 dt \leq \frac{(b-a)^2}{6} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^2 dt.$$

The proof follows by the inequality (2.17) which is equivalent with

$$(3.15) \quad \int_a^b \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right|^r dx \\ \leq \frac{(b-x)^{r-1} (x-a)^{r-1}}{(b-x)^{r-1} + (x-a)^{r-1}} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^r dt$$

for any $x \in [a, b]$.

Also, if we take the supremum over $x \in [a, b]$ in (3.15) for $r = 2$, then we get

$$(3.16) \quad \sup_{x \in [a, b]} \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right|^2 \\ \leq \frac{b-a}{4} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^2 dt.$$

Therefore the following lemma may be stated:

Lemma 2. *With the above assumptions, we have*

$$(3.17) \quad \sup_{x \in [a, b]} \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{(b-a)^{\frac{1}{2}}}{2} \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^2 dt \right)^{\frac{1}{2}}.$$

Also, on utilising the inequality (2.20), we get the following result as well:

Lemma 3. *With the above assumptions, we have*

$$(3.18) \quad \int_a^b \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right| dx \\ \leq \frac{1}{2} (b-a) \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt.$$

We can now state the following result that provides upper bounds for the absolute value of the Čebyšev functional $T(f, g)$.

Theorem 2. Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function and $g : [a, b] \rightarrow \mathbb{C}$ a Lebesgue integrable function on $[a, b]$. Then:

$$(3.19) \quad |T(f, g)| \leq \begin{cases} \frac{1}{2} \int_a^b |f'(x)| dx \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^2 dt \right)^{\frac{1}{2}} \\ \frac{1}{2} \cdot \sup_{x \in [a, b]} |f'(x)| \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ [I(r)]^{\frac{1}{r}} \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^r dt \right)^{\frac{1}{r}} \\ \quad \times \left(\frac{1}{b-a} \int_a^b |f'(x)|^q dx \right)^{\frac{1}{q}} \\ \quad \text{where } r > 1, \frac{1}{r} + \frac{1}{q} = 1. \end{cases}$$

In particular, for $r = 2$, we have

$$(3.20) \quad |T(f, g)| \leq \frac{\sqrt{6}}{6} (b-a) \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^2 dt \right)^{\frac{1}{2}} \\ \times \left(\frac{1}{b-a} \int_a^b |f'(x)|^2 dx \right)^{\frac{1}{2}}.$$

Proof. Utilising the identity (2.13) from [2], namely,

$$T(f, g) = -\frac{1}{b-a} \int_a^b \bar{G}(x) f'(x) dx,$$

where

$$\bar{G}(x) = \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt, \quad x \in [a, b],$$

we have

$$|T(f, g)| \leq \frac{1}{b-a} \int_a^b |\bar{G}(x)| |f'(x)| dx =: T.$$

Using Lemma 2, we have

$$T \leq \sup_{x \in [a, b]} |\bar{G}(x)| \cdot \frac{1}{b-a} \int_a^b |f'(x)| dx \\ \leq \frac{(b-a)^{\frac{1}{2}}}{2} \cdot |f'(x)| dx \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^2 dt \right)^{\frac{1}{2}} \\ = \frac{1}{2} \cdot \int_a^b |f'(x)| dx \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^2 dt \right)^{\frac{1}{2}},$$

and the first inequality in (3.19) is proved.

Utilising Lemma 3, we have

$$\begin{aligned} T &\leq \sup_{x \in [a, b]} |f'(x)| \cdot \frac{1}{b-a} \int_a^b |\bar{G}(x)| dx \\ &\leq \frac{1}{2} \sup_{x \in [a, b]} |f'(x)| \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt, \end{aligned}$$

which proves the second inequality in (3.19).

Now, from Hölder's inequality and Lemma 1, we also have

$$\begin{aligned} T &\leq \frac{1}{b-a} \left(\int_a^b |\bar{G}(x)|^r dx \right)^{\frac{1}{r}} \left(\int_a^b |f'(x)|^q dx \right)^{\frac{1}{q}} \\ &\leq \frac{1}{b-a} \left[I(r) \cdot \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^r dt \right]^{\frac{1}{r}} \left(\int_a^b |f'(x)|^q dx \right)^{\frac{1}{q}} \\ &= \frac{1}{b-a} [I(r)]^{\frac{1}{r}} \left[\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^r dt \right]^{\frac{1}{r}} \left(\int_a^b |f'(x)|^q dx \right)^{\frac{1}{q}} \end{aligned}$$

and the last part of (3.19) is also proved. \square

Remark 5. *It is an open question whether or not the constants $\frac{1}{2}$ in (3.19) and $\frac{\sqrt{6}}{6}$ in (3.20) are best possible.*

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