SOME COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA AND POLYGAMMA FUNCTIONS

FENG QI, BAI-NI GUO, AND CHAO-PING CHEN

Abstract. The function \( \frac{\Gamma(x+1)^{1/x}}{x} (1 + \frac{1}{x})^x \) is strictly logarithmically completely monotonic in \((0, \infty)\). The function \( \psi''(x + 2) + \frac{1+x^2}{x^2(1+x)^2} \) is strictly completely monotonic in \((0, \infty)\).

1. Introduction

It is well known that the gamma function \( \Gamma(z) \) is defined for \( \Re z > 0 \) as
\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt. \tag{1}
\]
The psi or digamma function \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \), the logarithmic derivative of the gamma function, and the polygamma functions can be expressed for \( x > 0 \) and \( k \in \mathbb{N} \) as
\[
\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{1+n} - \frac{1}{x+n} \right), \tag{2}
\]
\[
\psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}}, \tag{3}
\]
\[
\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, dt, \tag{4}
\]
\[
\psi^{(k)}(x) = (-1)^k k! \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \, dt, \tag{5}
\]
where \( \gamma = 0.57721566490153286 \cdots \) is the Euler-Mascheroni constant.

2000 Mathematics Subject Classification. Primary 33B15; Secondary 26D07.

Key words and phrases. strictly completely monotonic function, logarithmically completely monotonic function, gamma function, polygamma function.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, Doctor Fund of Jiaozuo Institute of Technology, CHINA.

This paper was typeset using \texttt{AMSTeX}.  

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A function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geq 0$$

(6)

for $x \in I$ and $n \geq 0$. If inequality (6) is strict for all $x \in I$ and for all $n \geq 0$, then $f$ is said to be strictly completely monotonic.

For $x > 0$ and $s \geq 0$, we have

$$\frac{1}{(x+s)^n} = \frac{1}{(n-1)!} \int_{0}^{\infty} t^{n-1}e^{-(x+s)t} \, dt, \quad n \in \mathbb{N}. \quad (7)$$

A function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0$$

(8)

for $k \in \mathbb{N}$ on $I$. If inequality (8) is strict for all $x \in I$ and for all $k \in \mathbb{N}$, then $f$ is said to be strictly logarithmically completely monotonic.

In [4] it is proved that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic. But not conversely, since a convex function may not be logarithmically convex (see Remark. 1.16 at page 7 in [3]).

Completely monotonic functions have applications in many branches. For example, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics. Some related references are listed in [1].

It is well known that the function $(1 + \frac{1}{x})^{-x}$ is strictly completely monotonic in $(0, \infty)$. In [1], it is proved that the function $(1 + \frac{a}{x})^{x+b} - e^{a}$ is completely monotonic with $x \in (0, \infty)$ if and only if $a \leq 2b$, where $a > 0$ and $b$ are real numbers.

Among other things, the following completely monotonic properties are obtained in [4]: For $\alpha \leq 0$, the function $\frac{x^\alpha}{[\Gamma(x+1)]^{1/\alpha}}$ is strictly completely monotonic in $(0, \infty)$. For $\alpha \geq 1$, the function $\frac{[\Gamma(x+1)]^{1/\alpha}}{x^\alpha}$ is strictly completely monotonic in $(0, \infty)$.

In [2] the following two inequalities are presented: For $x \in (0, 1)$, we have

$$\frac{x}{[\Gamma(x+1)]^{1/\alpha}} < \left(1 + \frac{1}{x}\right)^x < \frac{x + 1}{[\Gamma(x+1)]^{1/\alpha}}. \quad (9)$$
For $x \geq 1$,
\[
\left(1 + \frac{1}{x}\right)^x \geq \frac{x + 1}{\Gamma(x + 1)^{1/x}}.
\]
Equality in (10) occurs for $x = 1$.

It is easy to see that
\[
\lim_{x \to \infty} \frac{\Gamma(x + 1)^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x = 1.
\]

The main purpose of this paper is to give a strictly logarithmically completely monotonic property of the function $\frac{\Gamma(x + 1)^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x$ in $(0, \infty)$ as follows.

**Theorem 1.** The function $\frac{\Gamma(x + 1)^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x$ is strictly logarithmically completely monotonic in $(0, \infty)$.

As a direct consequence of the proof of Theorem 1, we have the following

**Corollary 1.** The function
\[
\psi''(x) + \frac{x^4 + 5x^3 + 7x^2 + 7x + 2}{x^3(x + 1)^3} = \psi''(x + 2) + \frac{1 + x^2}{x^2(1 + x)^2}
\]

is strictly completely monotonic in $(0, \infty)$.

**2. Proof of Theorem 1**

Define
\[
F(x) = \frac{\Gamma(x + 1)^{1/x}}{x^c} \left(1 + \frac{a}{x}\right)^{x+b}
\]
for $x > 0$ and some fixed real numbers $a$, $b$ and $c$.

Taking the logarithm of $F(x)$ defined by (13) and differentiating yields
\[
\ln F(x) = (x + b) \ln \left(1 + \frac{a}{x}\right) + \ln \frac{\Gamma(x + 1)}{x} - c \ln x,
\]
\[
[\ln F(x)]' = \ln \left(1 + \frac{a}{x}\right) - \frac{a(x + b)}{x(x + a)} + \frac{x \psi(x + 1) - \ln \Gamma(x + 1)}{x^2} - \frac{c}{x},
\]

and
\[
[\ln F(x)](n) = (-1)^n(n - 1)! \frac{1}{(x + a)n} - \frac{1}{x^n}
\]
\[+ (-1)^n(n - 2)! \ln \left[\frac{1}{(x + a)^{n-1}} - \frac{1}{x^{n-1}}\right]
\]
\[+ h_\alpha(x) \frac{x}{x^n} + (-1)^n(n - 1)! \frac{c}{x^n}
\]
where \( n \geq 2 \), \( \psi^{(-1)}(x + 1) = \ln \Gamma(x + 1) \), \( \psi^{(0)}(x + 1) = \psi(x + 1) \), and

\[
h_n(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k}x^k\psi(k-1)(x + 1)}{k!},
\]

\[
h'_n(x) = x^n\psi^{(n)}(x + 1) \begin{cases} > 0, & \text{if } n \text{ is odd,} \\ < 0, & \text{if } n \text{ is even.} \end{cases}
\]

Therefore, we have

\[
(-1)^{n}x^{n+1}[\ln F(x)]^{(n)}
\]

\[
= (n-2)! \left\{ (n-1)(b+c) - x + \frac{x^n[x + na - (n-1)b]}{(x + a)^{n} } \right\} x + (-1)^{n}h_n(x)
\]

and

\[
\frac{d\{ (-1)^{n}x^{n+1}[\ln F(x)]^{(n)} \}}{dx}
\]

\[
= (-1)^{n}x^n\psi^{(n)}(x + 1) + (n-2)! \left\{ (n-1)(b+c) - 2x \\
+ \frac{x^n[a(b+an + an^2 - bn) + (2a + b + 2an - bn) x + 2x^2]}{(x + a)^{n+1} } \right\}
\]

\[
= x^n \left\{ (-1)^{n}\psi^{(n)}(x + 1) + (n-2)! \left[ \frac{(n-1)(b+c) - 2x}{x^n} \\
+ \frac{a(b+an + an^2 - bn) + (2a + b + 2an - bn) x + 2x^2}{(x + a)^{n+1} } \right] \right\}
\]

\[
= x^n \left\{ (-1)^{n}\psi^{(n)}(x) + \frac{n!}{x^{n+1}} + (n-2)! \left[ \frac{(n-1)(b+c) - 2x}{x^n} \\
+ \frac{a(b+an + an^2 - bn) + (2a + b + 2an - bn) x + 2x^2}{(x + a)^{n+1} } \right] \right\}.
\]

By letting \( a = c = 1 \) and \( b = 0 \), we have

\[
\frac{d\{ (-1)^{n}x^{n+1}[\ln F(x)]^{(n)} \}}{dx} = x^n \left\{ (-1)^{n}\psi^{(n)}(x) + \frac{n!}{x^{n+1}} \\
+ (n-2)! \left[ \frac{n-1 - 2x}{x^n} + \frac{n(n+1) + 2(n+1)x + 2x^2}{(x + 1)^{n+1} } \right] \right\}
\]

\[
= x^n \left\{ (-1)^{n}\psi^{(n)}(x) + (n-2)! \left[ \frac{n(n-1) + (n-1)x - 2x^2}{x^{n+1}} \\
+ \frac{n(n+1) + 2(n+1)x + 2x^2}{(x + 1)^{n+1} } \right] \right\}
\]
\[ \triangleq x^n \{( -1)^n \psi^{(n)}(x) + (n - 2)! g_n(x) + (n - 2)! h_n(x) \} \]

By induction, it follows that

\[ g'_n(x) = -(n - 1) g_{n+1}(x) \quad \text{and} \quad h'_n(x) = -(n - 1) h_{n+1}(x), \quad (20) \]

this implies

\[ g^{(n-2)}_2(x) = -(1)^n(n - 2)! g_n(x) \quad \text{and} \quad h^{(n-2)}_2(x) = -(1)^n(n - 2)! h_n(x), \quad (21) \]

therefore

\[ \frac{d}{dx} \{( -1)^n x^{n+1} [\ln F(x)]^{(n)} \} = ( -1)^n x^n \left[ \psi''(x) + g_2(x) + h_2(x) \right]^{(n-2)}. \quad (22) \]

From formulas (3), (5) and (7), for \( x \in (0, \infty) \) and any nonnegative integer \( i \), we have

\[
\phi(x) \triangleq \psi''(x) + g_2(x) + h_2(x) \\
= \psi''(x) + \frac{2 + x - 2x^2}{x^3} + \frac{2(3 + 3x + x^2)}{(x + 1)^3} \\
= \psi''(x) + \frac{x^4 + 5x^3 + 7x^2 + 7x + 2}{x^3(x + 1)^3} \\
= \psi''(x) + \frac{1}{x^2} - \frac{2}{1 + x} + \frac{1}{x} - \sum_{i=2}^{\infty} \frac{1}{(x + i)^3} \\
= \psi''(x) + \frac{1}{x^2} - \frac{2}{1 + x} + \frac{2}{1 + x} + \frac{2}{1 + x} \\
= \psi''(x) + \frac{1 + x^2}{x^2(1 + x)^2} \\
= \int_0^\infty te^{-xt} dt - 2 \int_0^\infty e^{-xt} dt + 2 \int_0^\infty te^{-(x+1)t} dt \\
+ 2 \int_0^\infty e^{-(x+1)t} dt - \int_0^\infty \frac{t^2 e^{-(x+2)t}}{1 - e^{-t}} dt \\
= \int_0^\infty [t - 2 + (t + 4)e^{-t} - (t^2 + 2t + 2)e^{-2t}] e^{-xt} dt \\
\triangleq \int_0^\infty q(t)e^{-xt} dt, \\
\phi^{(i)}(x) = (-1)^i \int_0^\infty q(t)t^i e^{-xt} dt,
\]
and

\[ q'(t) = (2 + 2t + 2t^2 - 3e^t + e^{2t} - te^t)e^{-2t} \]
\[ \equiv p(t)e^{-2t}, \]
\[ p'(t) = 2 + 4t - 4e^t + 2e^{2t} - te^t, \]
\[ p''(t) = 4 - 5e^t + 4e^{2t} - te^t, \]
\[ p'''(t) = (8e^t - t - 6)e^t \]
\[ > 0. \]

Hence, \( p''(t) \) increases in \((0, \infty)\). Since \( p''(0) = 3 > 0 \), we have \( p''(t) > 0 \) and \( p'(t) \) is increasing. Because of \( p'(0) = 0 \), it follows that \( p'(t) > 0 \) in \((0, \infty)\), and then \( p(t) \) is increasing. From \( p(0) = 0 \), it is deduced that \( p(t) > 0 \) and \( q'(t) > 0 \) in \((0, \infty)\), then \( q(t) \) increases. As a result of \( q(0) = 0 \), we obtain \( q(t) > 0 \) in \((0, \infty)\).

Therefore, we have \( \phi(x) > 0 \) in \((0, \infty)\), and then for all nonnegative integer \( i \), we have \( (-1)^i \phi^{(i)}(x) > 0 \) in \((0, \infty)\). This means that the function \( \psi''(x) + g_2(x) + h_2(x) \) is strictly completely monotonic on \((0, \infty)\).

Thus the function \( (-1)^nx^{n+1}[\ln F(x)]^{(n)} \) is increasing in \( x \in (0, \infty) \). Since

\[ \lim_{x \to \infty} \{-(-1)^nx^{n+1}[\ln F(x)]^{(n)}\} = 0, \]

we have \( (-1)^nx^{n+1}[\ln F(x)]^{(n)} > 0 \), then \( (-1)^n[\ln F(x)]^{(n)} > 0 \) for \( n \geq 2 \) in \((0, \infty)\).

Since \( [\ln F(x)]'' > 0 \), the function \( [\ln F(x)]' \) is increasing. It is not difficult to obtain \( \lim_{x \to \infty} [\ln F(x)]' = 0 \), so \( [\ln F(x)]' < 0 \) and \( \ln F(x) \) is decreasing in \((0, \infty)\). In conclusion, the function \( \ln F(x) \) is strictly completely monotonic in \((0, \infty)\). The proof is complete.

### 3. An open problem

**Open Problem.** Under what conditions on \( a, b \) and \( c \) the function \( F(x) \) defined by (13) is strictly logarithmically completely monotonic in \((0, \infty)\)?

### References

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