A Note on Short Intervals Containing Primes

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Abstract

In 1999, P. Dusart showed that for \( x \geq 3275 \), there exists at least a prime number in the interval \( (x, x(1 + \frac{1}{2\ln^2 x})) \) and in 2003, O. Ramaré and Y. Saouter showed that for \( x \geq 10726905041 \) there exists at least a prime number in the interval \( (x(1 - \Delta^{-1}), x] \), in which \( \Delta = 28314000 \). In this note, we show that for \( x \geq 1.17 \times 10^{1634} \), we can yield Ramaré-Saouter's result from Dusart's result.

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As usual, suppose \( \mathbb{P} \) be the set of all prime numbers. In 1999, P. Dusart [1] showed that for every \( x \geq 3275 \), we have

\[ \mathbb{P} \cap \left( x, x(1 + \frac{1}{2\ln^2 x}) \right) \neq \phi. \]

In 2003, O. Ramaré and Y. Saouter [2] showed that for \( x \geq 10726905041 \), we have

\[ \mathbb{P} \cap (x(1 - \Delta^{-1}), x] \neq \phi, \]

in which \( \Delta = 28314000 \). If \( L_D(x) \) denotes the length of Dusart’s interval, \( (x, x(1 + \frac{1}{2\ln^2 x})) \), and \( L_{RS}(x) \) denotes the length of Ramaré-Saouter’s interval, \( (x(1 - \Delta^{-1}), x] \), then clearly we have:

\[ L_D(x) = \frac{x}{2\ln^2 x} = O\left( \frac{x}{\ln^2 x} \right), \]

and

\[ L_{RS}(x) = \frac{x}{\Delta} = O(x). \]
Also, we observe that
\[ \lim_{x \to \infty} \frac{L_D(x)}{L_{RS}(x)} = 0, \]
and this suggest that for sufficiently large values of x’s, Dusart’s interval become shorter than Ramaré-Saouter’s interval. In this research report, we show that for \( x \geq 1.17 \times 10^{1634} \), we can yield Ramaré-Saouter’s interval from Dusart’s interval.

Let \( d(x) = x(1 + \frac{1}{2 \ln^2 x}) \). For \( x > 0 \), \( d'(x) > 0 \); so, \( d^{-1}(x) \), the inverse of the function \( d(x) \), is well-defined. According to Dusart, for \( x \geq 3275 \), \( \mathbb{P} \cap (x, d(x)) \neq \phi \), and so for such x’s that \( d^{-1}(x) \geq 3275 \), we have
\[ \mathbb{P} \cap (d^{-1}(x), x] \neq \phi. \]
Therefore, \( \mathbb{P} \cap (d^{-1}(x), x] \neq \phi \) holds for \( x \geq d(3275) \) or for \( x \geq 3300 \). Now, we search such x’s that \( x(1-\Delta^{-1}) \leq d^{-1}(x) \) or \( d(x(1-\Delta^{-1})) \leq x \) and this is equivalent to
\[ x(1-\Delta^{-1}) \left( 1 + \frac{1}{2 \ln^2 (x(1-\Delta^{-1}))} \right) \leq x, \]
and since \( x > 0 \), we yield that for \( x \geq e^{\sqrt{\frac{\pi-1}{1-\frac{1}{2}}} \approx 1.167417545 \times 10^{1634} \), Dusart’s interval yields Ramaré-Saouter’s interval. This prove our claim at above.

We end this short note with a question about Dusart’s interval:

**Question.** For every \( x \in \mathbb{R} \), let
\[ n(x) := \# \mathbb{P} \cap \left( x, x(1 + \frac{1}{2 \ln^2 x}) \right). \]
Is there some elementary function \( f(x) \) such that \( n(x) \sim f(x) \), when \( x \to \infty \)? More generally study of \( n(x) \) is a nice subject.

**Note.** All computations in this note done by Maple software.

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**References**
