APPROSSIMAZIONE DEL PRODOTTO $p_1 p_2 \cdots p_n$

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Abstract. In this research report, we find some lower and upper bounds for the product $p_1 p_2 \cdots p_n$.

1. INTRODUCTION AND APPROXIMATION OF THE PRODUCT $p_1 p_2 \cdots p_n$

As usual, let $p_n$ be the $n$th prime. The Mondl’s inequality (see [1] and [8]) asserts that for every $n \geq 9$, we have $\sum_{i=1}^{n} p_i < \frac{n}{2}p_n$. In [2], one has get the following refinement of this inequality:

$$\sum_{i=1}^{n} p_i < \frac{n}{2}p_n - 0.01659n^2 \quad (n \geq 9).$$

Considering the AGM Inequality (see [7]) and (1.1), for every $n \geq 9$, we obtain:

$$p_1 p_2 \cdots p_n < \left(\frac{p_n}{2} - 0.01659n\right)^n \quad (n \geq 9).$$

Note that (1.2) holds also for $5 \leq n \leq 8$. This yields an upper bound for the product $p_1 p_2 \cdots p_n$. About lower bound, one can get a trivial one for that product, using Euclid’s proof of infinity of primes; Letting $E_n = p_1 p_2 \cdots p_n - 1$, for every $n \geq 2$, it is clear that $p_n < E_n$. So, if $p_n < E_n < p_{n+1}$, then $E_n$ should has a prime factor among $p_1, p_2, \ldots, p_n$, which isn’t possible. Thus, $E_n \geq p_{n+1}$, and therefore, for every $n \geq 2$, we have:

$$p_1 p_2 \cdots p_n > p_{n+1}.$$ 

In 1957 in [5], Bonse used elementary methods to show that:

$$p_1 p_2 \cdots p_n > p_{n+1}^2 \quad (n \geq 4),$$

and

$$p_1 p_2 \cdots p_n > p_{n+1}^3 \quad (n \geq 5).$$

In 1960 Pósa [4] proved that for every $k > 1$, there exists an $n_k$, such that for all $n \geq n_k$, we have:

$$p_1 p_2 \cdots p_n > p_{n+1}^k.$$ 

In 1988, J. Sandó found some inequalities of similar type; For example he showed that for every $n \geq 24$, we have:

$$p_1 p_2 \cdots p_n > p_{n+5}^2 + p_{[\frac{n}{2}]}^2.$$ 

In 2000, Panaitopol [3] showed that in Pósa’s result, we can get $n_k = 2k$. More precisely, he proved that for every $n \geq 2$, we have:

$$p_1 p_2 \cdots p_n > p_{n+1}^{n-\pi(n)},$$

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in which \( \pi(x) \) = the number of primes \( \leq x \). In this note, we refine Panaitopol’s result by proving

\[
p_1 p_2 \cdots p_n > p_{n+1}^{\left(1 - \frac{1}{\log x}\right)\left(n - \pi(n)\right)} \quad (n \geq 101).
\]

During proofs, we will need some known results, which we review them briefly; we have the following bound [1]:

\[
\pi(x) \geq \frac{x}{\log x} \left(1 + \frac{1}{\log x}\right) \quad (x \geq 599).
\]

For every \( n \geq 53 \), we have [3]:

\[
\log p_{n+1} < \log n + \log \log n + \frac{\log \log n - 0.4}{\log n}.
\]

Also, for every \( n \geq 3 \), we have [6]:

\[
\theta(p_n) > n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2.1454}{\log n}\right),
\]

in which \( \theta(x) = \sum_{p \leq x} \log p \). Specially, \( \theta(p_n) = \log(p_1 p_2 \cdots p_n) \) and this will act as a key for approximating \( p_1 p_2 \cdots p_n \). Finally, just for insisting, we note that base of all logarithms are \( e \). To prove (1.3), considering (1.4), (1.5) and (1.6), it is enough to prove that:

\[
\left(1 - \frac{1 - \frac{1}{\log n}}{\log n} - \frac{1 - \frac{1}{\log n}}{\log^2 n}\right) \left(\log n + \log \log n + \frac{\log \log n - 0.4}{\log n}\right) < \log n + \log \log n - 1 + \frac{\log \log n - 2.1454}{\log n} \quad (n \geq 599),
\]

which putting \( x = \log n \), is equivalent with:

\[
\frac{1.7454x^3 + 1.4x^2 - 0.4}{x^3 + x^2 - x - 1} < \log x \quad x \geq \log 599,
\]

and trivially, this holds true; because for \( x \geq \log 599 \) we have \( \frac{1.7454x^3 + 1.4x^2 - 0.4}{x^3 + x^2 - x - 1} < 1.7454 \), and \( 1.85 < \log x \). Therefore, we yield (1.3) for all \( n \geq 599 \). For \( 101 \leq n \leq 598 \), we get it by a computer.

References

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