ON INTEGRAL VERSION OF ALZER’S INEQUALITY AND MARTINS’ INEQUALITY

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Abstract. Let \( c > b > a \) and \( r \) be real numbers, and let \( f \) be a positive, twice differentiable function and satisfy \( f'(t) > 0 \) and \( (\ln f(t))'' \geq 0 \) on \((a, +\infty)\). Then

\[
\sup_{x \in [a,b]} f(x) < \left( \frac{\frac{1}{b-a} \int_a^b f'(x) \, dx}{\frac{1}{c-a} \int_a^c f'(x) \, dx} \right)^{1/r} < 1
\]

for all real \( r \), according as \( r \neq 0 \). This solves a recently open problem of B.-N. Guo and F. Qi.

1. Introduction

It was shown in [1, 2, 8, 13, 17] that let \( n \) be a positive integer, then for \( r > 0 \),

\[
\frac{n}{n+1} < \left( \frac{1}{n} \sum_{i=1}^n i^r \right) < \frac{\sqrt{n}!}{\sqrt{n+1}!}
\]

(1)

We call the left-hand side of (1) H. Alzer’s inequality [1], and the right-hand side of (1) J. S. Martins’ inequality [8]. In [3, 14] Alzer’s inequality is extended to all real \( r \). In [5] it was proved that Martins’ inequality is reversed for \( r < 0 \).

F. Qi and B.-N. Guo [10, 11] presented an integral version of inequality (1) as follows: Let \( b > a > 0 \) and \( \delta > 0 \), then for \( r > 0 \),

\[
\frac{b}{b + \delta} < \left( \frac{\frac{1}{b-a} \int_a^b x^r \, dx}{\frac{1}{b+\delta-a} \int_a^{b+\delta} x^r \, dx} \right)^{1/r} < \frac{b^r/a^r}{b^r/a^r + \delta^r/a^r}
\]

(2)

We note that the inequality (4) can be written for \( r > 0 \) as

\[
\frac{b}{b + \delta} < \frac{L_r(a, b)}{L_r(a, b + \delta)} < \frac{I(a, b)}{I(a, b + \delta)};
\]

(3)

where \( L_r(a, b) \) and \( I(a, b) \) are respectively the generalized logarithmic mean and the exponential mean of two positive numbers \( a, b \), defined in [6, 15, 16] by, for \( a = b \) by \( L_r(a, b) = a \) and for \( a \neq b \) by

\[
L_r(a, b) = \left( \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right)^{1/r}, \quad r \neq -1, 0;
\]

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Then we have

\[ L(a, b) = \frac{b - a}{\ln b - \ln a} = L(a, b); \]

\[ L_0(a, b) = \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{1/(b-a)} = I(a, b). \]

\[ L(a, b) \] is the logarithmic mean of two positive numbers \( a, b \). When \( a \neq b \), \( L_r(a, b) \) is a strictly increasing function of \( r \). In particular,

\[ \lim_{r \to -\infty} L_r(a, b) = \min\{a, b\}, \quad \lim_{r \to +\infty} L_r(a, b) = \max\{a, b\}. \]

In [4], it was indirectly shown that the function \( r \mapsto L_r(a, b)/L_r(a, b + \delta) \) is strictly decreasing with \( r \in (-\infty, +\infty) \). This yields that

\[ \frac{b}{b + \delta} \left( \frac{1}{b-a} \int_a^b x^r \, dx \right)^{1/r} \leq \left( \frac{1}{b + \delta - a} \int_a^{b+\delta} x^r \, dx \right)^{1/r} \]

for all real \( r \), (4)

\[ \frac{[b^{b/a}]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} \] according as \( r \geq 0 \). (5)

In [7], B.-N. Guo and F. Qi ask under which conditions the inequality

\[ \sup_{x \in [a, b]} f(x) \left( \frac{1}{b-a} \int_a^b f'(x) \, dx \right)^{1/r} \leq \frac{\exp \left( \frac{1}{c-a} \int_a^c \ln f(x) \, dx \right)}{\exp \left( \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right)} \]

holds for \( b > a > 0, \delta > 0 \) and \( r > 0 \).

V. Mascioni [9] found the sufficient conditions on the function \( f \), and proved the right-hand inequality of (6) for \( r > 0 \). Motivated by the paper of Mascioni [9], we establish the following

**Theorem.** Let \( c > b > a \) and \( r \) be real numbers, and let \( f \) be a positive, twice differentiable function and satisfy \( f'(t) > 0 \) and \( (\ln f(t))'' \geq 0 \) on \((a, +\infty)\). Then

\[ \sup_{x \in [a, b]} f(x) \left( \frac{1}{b-a} \int_a^b f'(x) \, dx \right)^{1/r} < 1 \quad \text{for all real } r \], (7)

\[ \left( \frac{1}{b-a} \int_a^b f'(x) \, dx \right)^{1/r} \leq \frac{\exp \left( \frac{1}{c-a} \int_a^c \ln f(x) \, dx \right)}{\exp \left( \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right)} \quad \text{according as } r \geq 0. \] (8)

Both bounds in (7) are best possible.

**2. LEMMAS**

**Lemma 1.** Let the function \( f \) be a positive and twice differentiable on \((a, +\infty)\), where \( a \) is a given real number, and let

\[ G(t) = \frac{1}{t-a} \int_a^t f(x) \, dx/f(t), \quad t > a. \]

Then we have

(i) If \( f'(t) > 0 \) and \( (\ln f(t))'' \geq 0 \), then the function \( G \) is strictly decreasing on \((a, +\infty)\).
(ii) If \( f'(t) < 0 \) and \((\ln f(t))'' \leq 0 \), then the function \( G \) is strictly increasing on \((a, +\infty)\).

**Proof.** Easy calculation reveals that

\[
\frac{[(t-a)f(t)]^2G'(t)}{f(t) + (t-a)f'(t)} = \frac{(t-a)f^2(t)}{f(t) + (t-a)f'(t)} - \int_a^t f(x) \triangleq H(t),
\]

\[
\frac{[f(t)+(t-a)f'(t)]^2H'(t)}{(t-a)f^3(t)} = -(t-a)\frac{f''(t)f(t) - [f'(t)]^2}{f^2(t)} - \frac{f'(t)}{f(t)}
\]

\[
= -(t-a)(\ln(f(t))'' + (\ln f(t))').
\]

If \((\ln f(t))' > (\leq 0) \) and \((\ln f(t))'' \geq (\leq 0) \) for \( t > a \), then \( H'(t) < (\geq 0) \) for \( t > a \), and then, \( H(t) < (\geq 0) \) for \( t > a \). The proof is complete. \( \square \)

**Lemma 2** ([12]). If \( F(t) \) is a strictly increasing (decreasing) integrable function on an interval \( I \subseteq \mathbb{R} \), then the arithmetic mean \( G(r,s) \) of function \( F(t) \),

\[
G(r,s) = \begin{cases} 
\frac{1}{s-r} \int_r^s F(t) \, dt, & r \neq s, \\
F(r), & r = s,
\end{cases}
\]

is also strictly increasing (decreasing) with both \( r \) and \( s \) on \( I \).

### 3. Proof of Theorem

For \( r = 0 \), (7) can be interpreted as

\[
\frac{f(b)}{f(c)} < \exp\left(\frac{1}{b-a} \int_a^b \ln f(x) \, dx\right) < \frac{1}{c-a} \int_a^c \ln f(x) \, dx < 1. \tag{9}
\]

Define for \( t > a \),

\[
P(t) = \frac{\exp\left(\frac{1}{t-a} \int_a^t \ln f(x) \, dx\right)}{f(t)}.
\]

A simple computation yields

\[
(t-a)^2 \frac{P'(t)}{P(t)} = (t-a) \ln f(t) - \int_a^t \ln f(x) \, dx - (t-a)(\ln f(t))' \triangleq Q(t),
\]

\[
Q'(t) = -(t-a) [(\ln f(t))' + (t-a)(\ln f(t))''] < 0.
\]

Hence, we have \( Q(t) < Q(a) = 0 \) and \( P'(t) < 0 \) for \( t > a \). This means the left-hand inequality of (9) holds for \( c > b > a \). By Lemma 2, the right-hand inequality of (9) holds clearly.

For \( r \neq 0 \), (7) is equivalent to

\[
\frac{f^r(b)}{f^r(c)} \leq \frac{1}{b-a} \int_a^b f^r(x) \, dx \leq \frac{1}{c-a} \int_a^c f^r(x) \, dx \leq 1, \quad \text{according as } r \geq 0. \tag{10}
\]

Define for \( t > a \),

\[
G_r(t) = \frac{\frac{1}{t-a} \int_a^t f^r(x) \, dx}{f^r(t)}.
\]
It is easy to see that
\[
\ln f'(t)'_t \geq 0 \quad \text{and} \quad \ln(f'(t))''_t \leq 0, \quad \text{according as} \quad r \geq 0. \quad (11)
\]
By Lemma 1, the function \( t \mapsto G_r(t) \) strictly decreases with respect to \( t \in (a, +\infty) \) according as \( r \geq 0 \). This produces the left-hand inequality of (10). By Lemma 2, the right-hand inequality of (10) holds clearly.

Both bounds in (7) are best possible because of
\[
\lim_{r \to +\infty} \left( \frac{1}{b-a} \int_a^b f''(x) \, dx \right)^{1/r} = \sup_{x \in [a,b]} \frac{f(x)}{\sup_{x \in [a,c]} f(x)},
\]
\[
\lim_{r \to -\infty} \left( \frac{1}{b-a} \int_a^b f''(x) \, dx \right)^{1/r} = \inf_{x \in [a,b]} \frac{f(x)}{\inf_{x \in [a,c]} f(x)} = 1.
\]
The inequality (8) is equivalent to
\[
\frac{1}{b-a} \int_a^b f''(x) \, dx < \frac{\exp\left( \frac{1}{b-a} \int_a^b \ln f''(x) \, dx \right)}{\exp\left( \frac{1}{c-a} \int_c^a \ln f''(x) \, dx \right)} \quad \text{for} \quad r \neq 0. \quad (12)
\]
Define for \( t > a \),
\[
F_(t) = \frac{1}{b-a} \int_a^t f''(x) \, dx \quad \frac{1}{c-a} \int_t^c f''(x) \, dx.
\]
It is easy to see from the proof of Theorem 1 of [9] that if \( f''(t) > 0 \) and \( (\ln f(t))'' \geq 0 \), then the function \( F_(t) \) is strictly increasing on \((a, +\infty)\); If \( f''(t) < 0 \) and \( (\ln f(t))'' \leq 0 \), then the function \( F_(t) \) is strictly decreasing on \((a, +\infty)\). Applying this result, together with (11), we obviously imply the function \( t \mapsto F_(t) \) strictly increases with respect to \( t \in (a, +\infty) \) according as \( r \geq 0 \). This produces (12). The proof is complete.

References

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