ON SOME INEQUALITIES IN NORMED ALGEBRAS

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Abstract. Some inequalities in normed algebras that provides lower and upper bounds for the norm of \( \sum_{j=1}^{n} \alpha_j x_j \) are obtained. Applications for estimating the quantities \( \|x^{-1}\| x \pm \|y^{-1}\| y \) and \( \|y^{-1}\| x \pm \|x^{-1}\| y \) for invertible elements \( x, y \) in unital normed algebras are also given.

1. Introduction

In [1], in order to provide a generalisation of a norm inequality for \( n \) vectors in a normed linear space obtained by Pečarić and Rajić in [2], the author obtained the following result:

\[
\begin{align*}
\max_{k \in \{1, \ldots, n\}} \left\{ \frac{1}{\|x_k\|} \left[ \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| - \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\| \right] \right\} & \leq \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \leq \\
\min_{k \in \{1, \ldots, n\}} \left\{ \frac{1}{\|x_k\|} \left[ \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| + \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\| \right] \right\},
\end{align*}
\]

where \( x_j, j \in \{1, \ldots, n\} \) are vectors in the normed linear space \((X, \|\cdot\|)\) over \( K \) while \( \alpha_j, j \in \{1, \ldots, n\} \) are scalars in \( K (K = \mathbb{C}, \mathbb{R}) \).

For \( \alpha_k = \frac{1}{\|x_k\|}, \) with \( x_k \neq 0, k \in \{1, \ldots, n\} \) the above inequality produces the following result established by Pečarić and Rajić in [2]:

\[
\begin{align*}
\max_{k \in \{1, \ldots, n\}} \left\{ \frac{1}{\|x_k\|} \left[ \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| - \sum_{j=1}^{n} \|x_j\| - \|x_k\| \right] \right\} & \leq \\
\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \leq \min_{k \in \{1, \ldots, n\}} \left\{ \frac{1}{\|x_k\|} \left[ \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| + \sum_{j=1}^{n} \|x_j\| - \|x_k\| \right] \right\},
\end{align*}
\]

which implies the following refinement and reverse of the generalised triangle inequality due to M. Kato et al. [3]:

\[
\begin{align*}
\min_{k \in \{1, \ldots, n\}} \left\{ \|x_k\| \right\} & \leq \\
\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leq n - \sum_{j=4}^{n} \left\| \frac{x_j}{\|x_j\|} \right\|
\end{align*}
\]

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\[ \leq \sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{n} x_j \leq \max_{k \in \{1, \ldots, n\}} \{\|x_k\|\} \left( n - \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right). \]

The other natural choice, \( \alpha_k = \|x_k\|, k \in \{1, \ldots, n\} \) in (1.1) produces the result

\[
(1.4) \quad \max_{k \in \{1, \ldots, n\}} \left\{ \|x_k\| \left| \sum_{j=1}^{n} \|x_j\| - \|x_k\| - \|x_j\| \right| \right\} 
\leq \left| \sum_{j=1}^{n} \|x_j\| x_j \right| \leq \min_{k \in \{1, \ldots, n\}} \left\{ \|x_k\| \left| \sum_{j=1}^{n} \|x_j\| + \sum_{j=1}^{n} \|x_j\| - \|x_k\| - \|x_j\| \right| \right\},
\]

which in its turn implies another refinement and reverse of the generalised triangle inequality:

\[
(1.5) \quad 0 \leq \frac{\sum_{j=1}^{n} \|x_j\|^{2} - \sum_{j=1}^{n} \|x_j\| x_j}{\max_{k \in \{1, \ldots, n\}} \{\|x_k\|\}} \leq \sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{n} x_j \leq \frac{\sum_{j=1}^{n} \|x_j\|^{2} - \sum_{j=1}^{n} \|x_j\| x_j}{\min_{k \in \{1, \ldots, n\}} \{\|x_k\|\}},
\]

provided \( x_k \neq 0, k \in \{1, \ldots, n\} \).

In [2], the authors have shown that the case \( n = 2 \) in (1.2) produces the Maligranda-Mercer inequality:

\[
(1.6) \quad \frac{\|x - y\| - \|x\| - \|y\|}{\min \{\|x\|, \|y\|\}} \leq \frac{\|x - y\|}{\|x\|} - \frac{\|x\| - \|y\|}{\|y\|} \leq \frac{\|x - y\|}{\max \{\|x\|, \|y\|\}} \leq \frac{\|x\| - \|y\|}{\min \{\|x\|, \|y\|\}},
\]

for any \( x, y \in X \setminus \{0\} \).

We notice that Maligranda proved the right inequality in [5] while Mercer proved the left inequality in [4].

We have shown in [1] that the following dual result for two vectors is also valid:

\[
(1.7) \quad 0 \leq \frac{\|x - y\|}{\min \{\|x\|, \|y\|\}} - \frac{\|x\| - \|y\|}{\max \{\|x\|, \|y\|\}} \leq \frac{\|x - y\|}{\|y\|} - \frac{\|x\| - \|y\|}{\|x\|} \leq \frac{\|x - y\|}{\max \{\|x\|, \|y\|\}} + \frac{\|x\| - \|y\|}{\min \{\|x\|, \|y\|\}},
\]

for any \( x, y \in X \setminus \{0\} \).

Motivated by the above results, the aim of the present paper is to establish lower and upper bounds for the norm of \( \sum_{j=1}^{n} a_j x_j \), where \( a_j, x_j, j \in \{1, \ldots, n\} \) are elements in a normed algebra \((A, \|\|)\) over the real or complex number field \( K \). In the case where \((A, \|\|)\) is a unital algebra and \( x, y \) are invertible, lower and upper bounds for the quantities

\[ \| x^{-1} \| x \pm y^{-1} \| y \| \quad \text{and} \quad \| y^{-1} \| x + \| x^{-1} \| y \| \]

are provided as well.
2. Inequalities for \( n \) Pairs of Elements

Let \((A, \|\|)\) be a normed algebra over the real or complex number field \( K \).

**Theorem 1.** If \((a_j, x_j) \in A^2, j \in \{1, \ldots, n\}\), then

\[
\max_{k \in \{1, \ldots, n\}} \left\{ \| a_k \left( \sum_{j=1}^{n} x_j \right) \| - \sum_{j=1}^{n} \| a_j - a_k \| \| x_j \| \right\} \\
\leq \max_{k \in \{1, \ldots, n\}} \left\{ \| a_k \left( \sum_{j=1}^{n} x_j \right) \| - \sum_{j=1}^{n} \| (a_j - a_k) x_j \| \right\} \\
\leq \| \sum_{j=1}^{n} a_j x_j \| \\
\leq \min_{k \in \{1, \ldots, n\}} \left\{ \| a_k \left( \sum_{j=1}^{n} x_j \right) \| + \sum_{j=1}^{n} \| (a_j - a_k) x_j \| \right\} \\
\leq \min_{k \in \{1, \ldots, n\}} \left\{ \| a_k \left( \sum_{j=1}^{n} x_j \right) \| + \sum_{j=1}^{n} \| a_j - a_k \| \| x_j \| \right\}.
\]

**Proof.** Observe that for any \( k \in \{1, \ldots, n\}\) we have

\[
\sum_{j=1}^{n} a_j x_j = a_k \left( \sum_{j=1}^{n} x_j \right) + \sum_{j=1}^{n} (a_j - a_k) x_j.
\]

Taking the norm and utilizing the triangle inequality and the normed algebra properties, we have

\[
\left\| \sum_{j=1}^{n} a_j x_j \right\| \leq \left\| a_k \left( \sum_{j=1}^{n} x_j \right) \right\| + \left\| \sum_{j=1}^{n} (a_j - a_k) x_j \right\| \\
\leq \left\| a_k \left( \sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \| (a_j - a_k) x_j \| \\
\leq \left\| a_k \left( \sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \| a_j - a_k \| \| x_j \|,
\]

for any \( k \in \{1, \ldots, n\}\), which implies the second part in (2.1). Observing that

\[
\sum_{j=1}^{n} a_j x_j = a_k \left( \sum_{j=1}^{n} x_j \right) - \sum_{j=1}^{n} (a_k - a_j) x_j
\]
and utilising the continuity of the norm, we have

\[
\left\| \sum_{j=1}^{n} a_j x_j \right\| \geq \left\| a_k \left( \sum_{j=1}^{n} x_j \right) - \sum_{j=1}^{n} (a_k - a_j) x_j \right\|
\]

\[
\geq \left\| a_k \left( \sum_{j=1}^{n} x_j \right) \right\| - \left\| \sum_{j=1}^{n} (a_k - a_j) x_j \right\|
\]

\[
\geq \left\| a_k \left( \sum_{j=1}^{n} x_j \right) \right\| - n \| (a_k - a_j) x_j \|
\]

\[
\geq \left\| a_k \left( \sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|a_k - a_j\| \|x_j\|
\]

for any \( k \in \{1, \ldots, n\} \), which implies the first part in (2.1).

**Remark 1.** If there exists \( r > 0 \) so that \( \|a_j - a_k\| \leq r \|a_k\| \) for any \( j, k \in \{1, \ldots, n\} \), then, by the second part of (2.1), we have

\[
\left\| \sum_{j=1}^{n} a_j x_j \right\| \leq \min_{k \in \{1, \ldots, n\}} \left\{ \|a_k\| \right\} \left\{ \left\| \sum_{j=1}^{n} x_j \right\| + r \sum_{j=1}^{n} \|x_j\| \right\}.
\]

**Corollary 1.** If \( x_j \in A, j \in \{1, \ldots, n\} \), then

\[
\max_{k \in \{1, \ldots, n\}} \left\{ \left\| x_k \left( \sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|x_j - x_k\| \|x_j\| \right\}
\]

\[
\leq \max_{k \in \{1, \ldots, n\}} \left\{ \left\| x_k \left( \sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \| (x_j - x_k) x_j \| \right\} \leq \sum_{j=1}^{n} x_j^2
\]

\[
\leq \min_{k \in \{1, \ldots, n\}} \left\{ \left\| x_k \left( \sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \| (x_j - x_k) x_j \| \right\}
\]

**Corollary 2.** Assume that \( A \) is a unital normed algebra. If \( x_j \in A \) are invertible for any \( j \in \{1, \ldots, n\} \), then

\[
\min_{k \in \{1, \ldots, n\}} \|x_k^{-1}\| \left\| \sum_{j=1}^{n} x_j \right\| - \left\| \sum_{j=1}^{n} x_j \right\|
\]

\[
\leq \sum_{j=1}^{n} \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|
\]

\[
\leq \max_{k \in \{1, \ldots, n\}} \|x_k^{-1}\| \left\| \sum_{j=1}^{n} x_j \right\| - \left\| \sum_{j=1}^{n} x_j \right\|.
\]
Lemma 1. An equivalent form of (2.4) is:

\[
\begin{align*}
\sum_{j=1}^{n} \|x_{j}\| \|x_j\| & - \sum_{j=1}^{n} \|x_{j}^{-1}\| \|x_j\| \\
\max_{k \in \{1, \ldots, n\}} \|x_k^{-1}\| & \leq \sum_{j=1}^{n} \|x_{j}\| \|x_j\| - \sum_{j=1}^{n} \|x_{j}^{-1}\| \|x_j\| \\
\min_{k \in \{1, \ldots, n\}} \|x_k^{-1}\| & \\
\end{align*}
\]

which provides both a refinement and a reverse inequality for the generalised triangle inequality.
or, equivalently,
\[
\frac{1}{2} \left\{ \|a(x+y)\| + \|b(x+y)\| - \|[b-a]y\| + \|(b-a)x\| \right\}
\]
\[
+ \frac{1}{2} \|a(x+y)\| - \|b(x+y)\| + \|[b-a]y\| - \|(b-a)x\|
\]
\[
\leq \|ax \pm by\|
\]
\[
\leq \frac{1}{2} \left\{ \|a(x+y)\| + \|b(x+y)\| + \|[b-a]y\| + \|(b-a)x\| \right\}
\]
\[
- \frac{1}{2} \|a(x+y)\| + \|b(x+y)\| - \|(b-a)y\| - \|(b-a)x\|.
\]

**Proof.** The inequality (3.1) follows from Theorem 1 for \( n = 2, a_1 = a, a_2 = b, x_1 = x \) and \( x_2 = y \).

Utilising the properties of real numbers,
\[
\min \{\alpha, \beta\} = \frac{1}{2} [\alpha + \beta - |\alpha - \beta|], \quad \max \{\alpha, \beta\} = \frac{1}{2} [\alpha + \beta + |\alpha - \beta|]; \quad \alpha, \beta \in \mathbb{R};
\]
the inequality (3.1) is clearly equivalent with (3.2).

The following result contains some upper bounds for \( \|ax \pm by\| \) that are perhaps more useful for applications.

**Theorem 2.** If \( (a, b), (x, y) \in A^2 \), then
\[
\|ax \pm by\| \leq \min \{\|a(x \pm y)\|, \|b(x \pm y)\|\} + \|b-a\| \max \{\|x\|, \|y\|\}
\]
and
\[
\|ax \pm by\| \leq \|x \pm y\| \max \{\|a\|, \|b\|\} + \min \{\|b-a\| x\|, \|b-a\| y\|\}
\]

**Proof.** Observe that \( \|(b-a)x\| \leq \|b-a\| \|x\| \) and \( \|(b-a)y\| \leq \|b-a\| \|y\| \), and then
\[
\|(b-a)x\|, \|(b-a)y\| \leq \|b-a\| \max \{\|x\|, \|y\|\},
\]
which implies that
\[
\min \{\|a(x \pm y)\| + \|(b-a)y\|, \|b(x \pm y)\| + \|(b-a)x\|\}
\]
\[
\leq \min \{\|a(x \pm y)\|, \|b(x \pm y)\|\} + \|b-a\| \max \{\|x\|, \|y\|\}
\]
\[
\leq \|x \pm y\| \min \{\|a\|, \|b\|\} + \|b-a\| \max \{\|x\|, \|y\|\}.
\]

Utilising the second inequality in (3.1), we deduce (3.3).

Also, since \( \|a(x \pm y)\| \leq \|a\| \|x \pm y\| \) and \( \|b(x \pm y)\| \leq \|b\| \|x \pm y\| \), hence
\[
\|a(x \pm y)\|, \|b(x \pm y)\| \leq \|x \pm y\| \max \{\|a\|, \|b\|\},
\]
which implies that
\[
\min \{\|a(x \pm y)\| + \|(b-a)y\|, \|b(x \pm y)\| + \|(b-a)x\|\}
\]
\[
\leq \|x \pm y\| \max \{\|a\|, \|b\|\} + \min \{\|b-a\| x\|, \|b-a\| y\|\}
\]
\[
\leq \|x \pm y\| \max \{\|a\|, \|b\|\} + \|b-a\| \min \{\|x\|, \|y\|\},
\]
and the inequality (3.4) is also proved.

The following corollary may be more useful for applications.
Corollary 3. If \((a, b), (x, y) \in A^2\), then
\[
\|ax \pm by\| \leq \|x \pm y\| \cdot \frac{\|a\| + \|b\|}{2} + \|b - a\| \cdot \frac{\|x\| + \|y\|}{2}.
\]

Proof. Follows from Theorem 2 by adding the last inequality in (3.3) to the last inequality (3.4) and utilising the property that \(\min \{\alpha, \beta\} + \max \{\alpha, \beta\} = \alpha + \beta\), \(\alpha, \beta \in \mathbb{R}\).

The following lower bounds for \(\|ax \pm by\|\) can be stated as well:

Theorem 3. For any \((a, b)\) and \((x, y) \in A^2\), we have:
\[
\max \{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \max \{\|x\|, \|y\|\}
\leq \max \{\|a(x \pm y)\|, \|b(x \pm y)\|\} - \|b - a\| \max \{\|x\|, \|y\|\}
\leq \|ax \pm by\|
\]
and
\[
\min \{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \min \{\|x\|, \|y\|\}
\leq \min \{\|a(x \pm y)\|, \|b(x \pm y)\|\} - \|b - a\| \min \{\|x\|, \|y\|\}
\leq \|ax \pm by\|.
\]

Proof. Observe that, by (3.5) we have that
\[
\max \{\|a(x \pm y)\| - \|(b - a)y\|, \|b(x \pm y)\| - \|(b - a)x\|\}
\geq \max \{\|ax \pm ay\|, \|bx \pm by\|\} - \|b - a\| \max \{\|x\|, \|y\|\}
\geq \max \{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \max \{\|x\|, \|y\|\}
\]
and on utilising the first inequality in (3.1), the inequality (3.7) is proved.

Observe also that, since
\[
\|a(x \pm y)\|, \|b(x \pm y)\| \geq \min \{\|ax\| - \|ay\|, \|bx\| - \|by\|\},
\]
then
\[
\max \{\|a(x \pm y)\| - \|(b - a)y\|, \|b(x \pm y)\| - \|(b - a)x\|\}
\geq \min \{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \min \{\|(b - a)x\|, \|(b - a)y\|\}
\geq \min \{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \min \{\|x\|, \|y\|\}.
\]

Then, by the first inequality in (3.1), we deduce (3.8).

Corollary 4. For any \((a, b), (x, y) \in A^2\), we have
\[
\frac{1}{2} \cdot \|\|ax\| - \|ay\| + \|bx\| - \|by\|\| - \|b - a\| \cdot \|x\| + \|y\| \leq \|ax \pm by\|.
\]

The proof follows from Theorem 3 by adding (3.7) to (3.8). The details are omitted.

4. Applications for Two Invertible Elements

In this section we assume that \(A\) is a unital algebra with the unity \(1\). The following results provide some upper bounds for the quantity \(\||x^{-1}|| x \pm ||y^{-1}|| y\|\), where \(x\) and \(y\) are invertible in \(A\).
Proposition 1. If \((x, y) \in A^2\) are invertible, then

\[
(4.1) \quad \|x^{-1}\| \|x \pm y\| \leq \|x \pm y\| \min \{\|x^{-1}\|, \|y^{-1}\|\} + \|x^{-1}\| - \|y^{-1}\| \max \{\|x\|, \|y\|\}
\]

and

\[
(4.2) \quad \|x^{-1}\| \|x \pm y\| \leq \|x \pm y\| \max \{\|x^{-1}\|, \|y^{-1}\|\} + \|x^{-1}\| - \|y^{-1}\| \min \{\|x\|, \|y\|\}.
\]

Proof. Follows by Theorem 2 on choosing \(a = \|x^{-1}\| \cdot 1\) and \(b = \|y^{-1}\| \cdot 1\).

Corollary 5. With the above assumption for \(x\) and \(y\), we have

\[
(4.3) \quad \|x^{-1}\| \|x \pm y\| \leq \|x \pm y\| \cdot \left(\frac{\|x^{-1}\| + \|y^{-1}\|}{2} + \|x^{-1}\| - \|y^{-1}\| \cdot \frac{\|x\| + \|y\|}{2}\right).
\]

Lower bounds for \(\|x^{-1}\| \|x \pm y\|\) are provided below:

Proposition 2. If \((x, y) \in A^2\) are invertible, then

\[
(4.4) \quad \|x \pm y\| \min \{\|x^{-1}\|, \|y^{-1}\|\} - \|x^{-1}\| - \|y^{-1}\| \max \{\|x\|, \|y\|\} \leq \|x^{-1}\| \|x \pm y\|
\]

and

\[
(4.5) \quad \|x \pm y\| \min \{\|x^{-1}\|, \|y^{-1}\|\} - \|x^{-1}\| - \|y^{-1}\| \min \{\|x\|, \|y\|\} \leq \|x^{-1}\| \|x \pm y\|.
\]

Proof. The first inequality in (4.4) follows from the second inequality in (3.7) on choosing \(a = \|x^{-1}\| \cdot 1\) and \(b = \|y^{-1}\| \cdot 1\).

We know from the proof of Theorem 3 that

\[
(4.6) \quad \max \{\|a(x \pm y)\| - \|(b - a) y\|, \|b(x \pm y)\| - \|(b - a) x\|\} \leq \|ax \pm by\|.
\]

If in this inequality we choose \(a = \|x^{-1}\| \cdot 1\) and \(b = \|y^{-1}\| \cdot 1\), then we get

\[
\|x^{-1}\| \|x \pm y\| \geq \min \{\|x^{-1}\| \|x \pm y\| - \|x^{-1}\| - \|y^{-1}\| \|y\|, \|y^{-1}\| \|x \pm y\| - \|x^{-1}\| - \|y^{-1}\| \|x\}\}
\]

and the inequality (4.5) is obtained.

Corollary 6. If \((x, y) \in A^2\) are invertible, then

\[
(4.7) \quad \|x \pm y\| \cdot \left(\frac{\|x^{-1}\| + \|y^{-1}\|}{2} - \|x^{-1}\| - \|y^{-1}\| \cdot \frac{\|x\| + \|y\|}{2}\right) \leq \|x^{-1}\| \|x \pm y\|.
\]
Remark 3. We observe that the inequalities (4.3) and (4.7) are in fact equivalent with:

\[
\left\| \|x^{-1}\| x \pm \|y^{-1}\| y - \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} \right\| \\
\leq \left( \|x^{-1}\| - \|y^{-1}\| \right) \cdot \frac{\|x\| + \|y\|}{2}.
\]

Now we consider the dual expansion \( \|\|y^{-1}\| x \pm \|x^{-1}\| y \| \), for which the following upper bounds can be stated.

Proposition 3. If \((x, y)\) are invertible in \(A\), then

\[
\|\|y^{-1}\| x \pm \|x^{-1}\| y \| \\
\leq \| x \pm y \| \min \left\{ \|x^{-1}\|, \|y^{-1}\| \right\} + \|x^{-1}\| - \|y^{-1}\| \max \{\|x\|, \|y\|\}
\]

and

\[
\|\|y^{-1}\| x \pm \|x^{-1}\| y \| \\
\leq \| x \pm y \| \max \left\{ \|x^{-1}\|, \|y^{-1}\| \right\} + \|x^{-1}\| - \|y^{-1}\| \min \{\|x\|, \|y\|\}.
\]

In particular,

\[
\|\|y^{-1}\| x \pm \|x^{-1}\| y \| \\
\leq \| x \pm y \| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} + \|x^{-1}\| - \|y^{-1}\| \cdot \frac{\|x\| + \|y\|}{2}.
\]

The proof follows from Theorem 2 on choosing \(a = \|y^{-1}\| \cdot 1\) and \(b = \|x^{-1}\| \cdot 1\).

The lower bounds for the quantity \(\|\|y^{-1}\| x \pm \|x^{-1}\| y \|\) are incorporated in:

Proposition 4. If \((x, y)\) are invertible in \(A\), then

\[
\| x \pm y \| \max \left\{ \|x^{-1}\|, \|y^{-1}\| \right\} - \|x^{-1}\| - \|y^{-1}\| \max \{\|x\|, \|y\|\}
\leq \|\|y^{-1}\| x \pm \|x^{-1}\| y \|
\]

and

\[
\| x \pm y \| \min \left\{ \|x^{-1}\|, \|y^{-1}\| \right\} - \|x^{-1}\| - \|y^{-1}\| \min \{\|x\|, \|y\|\}
\leq \|\|y^{-1}\| x \pm \|x^{-1}\| y \|.
\]

In particular,

\[
\| x \pm y \| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} - \|x^{-1}\| - \|y^{-1}\| \cdot \frac{\|x\| + \|y\|}{2}
\leq \|\|y^{-1}\| x \pm \|x^{-1}\| y \|.
\]

Remark 4. We observe that the inequalities (4.11) and (4.14) are equivalent with

\[
\|\|y^{-1}\| x \pm \|x^{-1}\| y \| - \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2}
\leq \|\|x^{-1}\| - \|y^{-1}\| \cdot \frac{\|x\| + \|y\|}{2}.
\]
References


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