NOTE ON A CLASS OF COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE POLYGAMMA FUNCTIONS

FENG QI, SENLIN GUO, AND BAI-NI GUO

Abstract. In this article, some monotonicity of the function \( x^\alpha \psi^{(i)}(x + \beta) \) and the complete monotonicity of the functions \( \frac{1}{k!} \psi^{(i)}(x + \beta) - \psi^{(i+1)}(x + \beta) \) and \( \alpha \psi^{(i)}(x + \beta) - x \psi^{(i+1)}(x + \beta) \) in \((0, \infty)\) for \( i \in \mathbb{N}, \alpha > 0 \) and \( \beta \geq 0 \) are investigated, where \( \psi^{(i)}(x) \) is the well known polygamma functions. Moreover, lower and upper bounds for infinite series whose coefficients involves the Bernoulli numbers are established.

1. Introduction

Recall \([7, 11, 14]\) that a function \( f \) is called completely monotonic on an interval \( I \) if \( f \) has derivatives of all orders on \( I \) and \( 0 \leq (-1)^k f^{(k)}(x) < \infty \) for all \( k \geq 0 \) on \( I \). The well known Bernstein’s Theorem \([14, \text{p. 161}]\) states that \( f \in \mathcal{C}([0, \infty]) \) if and only if \( f(x) = \int_0^\infty e^{-xs} d\mu(s) \), where \( \mu \) is a nonnegative measure on \([0, \infty)\) such that the integral converges for all \( x > 0 \). The class of completely monotonic functions on \( I \) is denoted by \( \mathcal{C}[I] \). For more information on \( \mathcal{C}[I] \), please refer to \([5, 6, 7, 8, 9, 10, 11, 14]\) and the references therein.

By using the convolution theorem of Laplace transforms, the increasingly monotonicity of \( x^\alpha \psi^{(i)}(x + 1) \) is presented \([9, 10]\): The function \( x^\alpha \psi^{(i)}(x + 1) \) is strictly increasing in \((0, \infty)\) if and only if \( \alpha \geq i \), where \( \psi(x) \), the logarithmic derivative of the classical Euler’s gamma function \( \Gamma(x) \), is called psi function and \( \psi^{(i)}(x) \) for \( i \in \mathbb{N} \) are called polygamma functions. In \([3]\), in order to show the subadditive property of the function \( \psi^{(i)}(a + e^t) \), it proved that the function \( x\psi^{(i)}(x + a) \) is strictly increasing on \([0, \infty)\) for \( a \geq 1 \). In \([2]\), it was also showed, using the convolution theorem of Laplace transforms, that the function \( x^\alpha \psi^{(k)}(x) \) for \( k \geq 1 \) is strictly decreasing in \((0, \infty)\) if and only if \( k \leq 2k \) and is strictly increasing in \([0, \infty)\) if and only if \( e \geq k + 1 \). In \([4]\), the monotonicity of the more general function \( x^\alpha \psi^{(i)}(x + \beta) \) was studied without using the convolution theorem of Laplace transforms and, except the above results, the following conclusions are obtained:

For \( i \in \mathbb{N}, \alpha > 0 \) and \( \beta \geq 0 \),

1. the function \( x^\alpha \psi^{(i)}(x + \beta) \) is strictly increasing in \((0, \infty)\) if \( (\alpha, \beta) \in \{ \alpha \geq i, 1 \leq \beta < 1 \} \cup \{ \alpha \geq i, \beta \geq \frac{\alpha - i + 1}{2} \} \cup \{ \alpha \geq i + 1, \beta \leq \frac{\alpha - i + 1}{2} \} \) and only if \( \alpha \geq i \);

2. \( \frac{1}{2} \psi^{(i)}(x) - |\psi^{(i+1)}(x)| \in \mathcal{C}([0, \infty]) \) if and only if \( \alpha \geq i + 1 \);

3. \( |\psi^{(i+1)}(x)| - \frac{1}{2} \psi^{(i)}(x) \in \mathcal{C}([0, \infty]) \) if and only if \( 0 < \alpha \leq i \);

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Theorem 1. The function \( x^\alpha |\psi^{(i)}(x + \beta) | \) in \((0, \infty)\) is strictly increasing if and only if \( \alpha \geq i + 1 \) and strictly decreasing if and only if \( 0 \leq \alpha \leq i \).

(2) For \( \beta \geq \frac{1}{2} \), the function \( x^\alpha |\psi^{(i)}(x + \beta) | \) is strictly increasing in \([0, \infty)\) if and only if \( \alpha \geq i \).

Let \( \delta : (0, \infty) \to (0, \frac{1}{2}) \) be defined by
\[
\delta(t) = \frac{e^t(t - 1) + 1}{(e^t - 1)^2} \tag{1}
\]
for \( t \in (0, \infty) \) and \( \delta^{-1} : (0, \frac{1}{2}) \to (0, \infty) \) stand for the inverse function of \( \delta \). If \( 0 < \beta < \frac{1}{2} \) and
\[
\alpha \geq i + 1 - \left[ \frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} + \beta - 1 \right] \delta^{-1}(\beta), \tag{2}
\]
then the function \( x^\alpha |\psi^{(i)}(x + \beta) | \) is strictly increasing in \((0, \infty)\).

Remark 1. It is noted that
\[
0 < \left[ \frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} + \beta - 1 \right] \delta^{-1}(\beta) < 1
\]
for \( \beta \in (0, 1) \), since \( \lim_{\beta \to 0^+} \beta \delta^{-1}(\beta) = 0 \).

Theorem 2. Let \( i \in \mathbb{N} \), \( \alpha \geq 0 \) and \( \beta \geq 0 \).

(1) \( \alpha |\psi^{(i)}(x) | - x|\psi^{(i+1)}(x) | \in \mathcal{C}([0, \infty]) \) if and only if \( \alpha \geq i + 1 \).

(2) \( x|\psi^{(i+1)}(x) | - \alpha |\psi^{(i)}(x) | \in \mathcal{C}([0, \infty]) \) if and only if \( 0 \leq \alpha \leq i \).

(3) If \( \beta \geq \frac{1}{2} \), then \( \alpha |\psi^{(i)}(x + \beta) | - x|\psi^{(i+1)}(x + \beta) | \in \mathcal{C}([0, \infty]) \) if and only if \( \alpha \geq i \).

(4) If \( 0 < \beta < \frac{1}{2} \) and inequality (2) holds true, then \( \alpha |\psi^{(i)}(x + \beta) | - x|\psi^{(i+1)}(x + \beta) | \in \mathcal{C}([0, \infty]) \).

Theorem 3. Let \( i \in \mathbb{N} \), \( \alpha \geq 0 \) and \( \beta \geq 0 \).

(1) \( \alpha |\psi^{(i)}(x) | - |\psi^{(i+1)}(x) | \in \mathcal{C}([0, \infty]) \) if and only if \( \alpha \geq i + 1 \).

(2) \( |\psi^{(i+1)}(x) | - \alpha |\psi^{(i)}(x) | \in \mathcal{C}([0, \infty]) \) if and only if \( 0 \leq \alpha \leq i \).

(3) If \( \beta \geq \frac{1}{2} \), then \( \frac{1}{2} |\psi^{(i)}(x + \beta) | - |\psi^{(i+1)}(x + \beta) | \in \mathcal{C}([0, \infty]) \) if and only if \( \alpha \geq i \).

(4) If \( 0 < \beta < \frac{1}{2} \) and inequality (2) holds true, then \( \frac{1}{2} |\psi^{(i)}(x + \beta) | - |\psi^{(i+1)}(x + \beta) | \in \mathcal{C}([0, \infty]) \).
Theorem 4. Let \( 0 < \beta < \frac{1}{2} \) and \( \delta^{-1} \) be the inverse function of \( \delta \) defined by (1). Then the following inequalities holds for \( t \in (0, \infty) \):

\[
\frac{1}{2} > \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k-1}}{(2k-1)!} > 0, \\
\frac{t}{2} > \sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!} > \max \left\{ 0, \frac{t}{2} - 1 \right\}, \\
\sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!} > \left( \frac{1}{2} - \beta \right) t + \left[ \frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} - \beta + 1 \right] \delta^{-1}(\beta) - 1,
\]

where \( B_k \) stands for the Bernoulli numbers defined by

\[
x e^x - 1 = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.
\]

2. Lemmas

In order to prove our main results, the following lemmas are necessary.

Lemma 1 ([1, 12, 13]). The polygamma functions \( \psi^{(k)}(x) \) are expressed for \( x > 0 \) and \( k \in \mathbb{N} \) as

\[
\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \, dt.
\]

For \( x > 0 \) and \( r > 0 \),

\[
\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} \, dt.
\]

For \( i \in \mathbb{N} \) and \( x > 0 \),

\[
\psi^{(i-1)}(x+1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i}.
\]

Lemma 2 ([5, 6]). Let \( f(x) \) be defined in an infinite interval \( I \). If \( \lim_{x \to \infty} f(x) = 0 \) and \( f(x) - f(x + \varepsilon) \geq 0 \) for any given \( \varepsilon > 0 \), then \( f(x) \geq 0 \) in \( I \).

3. Proofs of Theorems

Proof of Theorem 1. Direct calculation and rearrangement yields

\[
\frac{g'_{i, \alpha, \beta}(x)}{x^{\alpha-1}} = \alpha |\psi^{(i)}(x + \beta)| - x |\psi^{(i+1)}(x + \beta)| \\
= (-1)^{i+1} \left[ \alpha \psi^{(i)}(x + \beta) + x \psi^{(i+1)}(x + \beta) \right]
\]

and

\[
\lim_{x \to \infty} \frac{g'_{i, \alpha, \beta}(x)}{x^{\alpha-1}} = 0.
\]

Straightforwardly computing in virtue of formulas (9), (8) and (7) gives
\[
\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} - \frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)^{\alpha-1}} = (-1)^i \{ \alpha \psi^{(i)}(x + \beta) - \psi^{(i)}(x + \beta + 1) \} \\
+ x \{ \psi^{(i+1)}(x + \beta) - \psi^{(i+1)}(x + \beta + 1) \} - \psi^{(i+1)}(x + \beta + 1) \}
\]
\[
= \frac{d\alpha}{(x + \beta)^{i+1}} - \frac{(i+1)!x}{(x + \beta)^{i+2}} + (-1)^{i+2} \psi^{(i+1)}(x + \beta)
\]
\[
= (-1)^{i+2} \psi^{(i+1)}(x + \beta) + \frac{d\alpha - i - 1}{(x + \beta)^{i+1}} + \frac{(i+1)(\beta + 1)}{(x + \beta)^{i+2}}
\]
\[
= \int_0^\infty \left[ \frac{t}{1-e^{-t}} + (\beta - 1)t + \alpha - i - 1 \right] t e^{-(x+\beta)t} \, dt 
\]
\[
= \int_0^\infty h_{i,\alpha,\beta}(t) t e^{-(x+\beta)t} \, dt.
\]

If \( \beta = 0 \), the function \( h'_{i,\alpha,0}(t) = \frac{\eta^i}{(t-1)^{i+1}} < 0 \) and \( h_{i,\alpha,0}(t) \) is decreasing in \((0, \infty)\) with \( \lim_{t \to 0^+} h_{i,\alpha,0}(t) = \alpha - i \) and \( \lim_{t \to \infty} h_{i,\alpha,0}(t) = \alpha - i - 1 \). For \( \alpha \geq i + 1 \), the functions \( h_{i,\alpha,0}(t) \) and \( g_{i,\alpha,0}(s) - g_{i,\alpha,0}(x+1) \) are positive in \((0, \infty)\). Combining this with (11) and considering Lemma 2, it is obtained that the functions \( g_{i,\alpha,0}(x) \) and \( g_{i,\alpha,0}(x) \) are positive in \((0, \infty)\), which means that the function \( g_{i,\alpha,0}(x) \) is strictly increasing in \((0, \infty)\) for \( \alpha \geq i + 1 \). Similarly, for \( \alpha \leq i \), the function \( g_{i,\alpha,0}(x) \) is strictly decreasing in \((0, \infty)\).

If \( \beta > 0 \), then the function \( h'_{i,\alpha,\beta}(t) = \frac{\eta^i(t-1)^{i+2}}{(t-1)^{i+1}} \leq \frac{\lambda_1(t)}{(t-1)^{i+1}} \) and \( \lambda_1(t) = 1 + (t - 1)e^t > 0 \) in \((0, \infty)\), and the function \( \lambda_1(t) \) is increasing with \( \lambda_1(0) = 0 \), thus \( \lambda_1(t) > 0 \) and \( \lambda'(t) > 0 \). Hence, the functions \( \lambda(t) \) and \( h'_{i,\alpha,\beta}(t) \) are strictly increasing in \((0, \infty)\) with \( \lim_{t \to 0^+} h'_{i,\alpha,\beta}(t) = \beta - \frac{1}{2} \) and \( \lim_{t \to \infty} h'_{i,\alpha,\beta}(t) = \beta \). Thus, if \( \beta \geq \frac{1}{2} \), the function \( h'_{i,\alpha,\beta}(t) \) is positive and the function \( h_{i,\alpha,\beta}(t) \) is strictly decreasing in \((0, \infty)\) with \( \lim_{t \to 0^+} h_{i,\alpha,\beta}(t) = \alpha - i \) and \( \lim_{t \to \infty} h_{i,\alpha,\beta}(t) = \alpha - i \). Accordingly, for \( \alpha \geq i \) and \( \beta \geq \frac{1}{2} \), the function \( h_{i,\alpha,\beta}(t) \) is decreasing in \((0, \infty)\). Therefore, for \( \alpha > i \) and \( \beta > \frac{1}{2} \), by the same argument as above, it is deduced that the function \( g_{i,\alpha,\beta}(x) \) is strictly decreasing in \((0, \infty)\).

If \( 0 < \beta < \frac{1}{2} \), since the function \( h'_{i,\alpha,\beta}(t) \) is strictly increasing in \((0, \infty)\) with \( \lim_{t \to 0^+} h'_{i,\alpha,\beta}(t) = \beta - \frac{1}{2} < 0 \) and \( \lim_{t \to \infty} h'_{i,\alpha,\beta}(t) = \beta > 0 \), then the function \( h_{i,\alpha,\beta}(t) \) attains its unique minimum at some point \( t_0 \in (0, \infty) \). It is easy to see that the function \( \delta(t) \) defined by (1) satisfies \( \delta(t_0) = \beta \) for \( 0 < \beta < \frac{1}{2} \), equals \( -[\lambda(t + 1)] \) and is positive and strictly decreasing with \( \lim_{t \to 0^+} \delta(t) = \frac{1}{2} \) and \( \lim_{t \to \infty} \delta(t) = 0 \). Therefore, the unique minimum of \( h_{i,\alpha,\beta}(t) \) equals
\[
\frac{\delta^{-1}(\beta)e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} + (\beta - 1)\delta^{-1}(\beta) + \alpha - i - 1,
\]
where \( \delta^{-1} \) is the inverse function of \( \delta \) defined by (1) and is strictly decreasing in \((0, \frac{1}{2})\) with \( \lim_{t \to 0^+} \delta^{-1}(s) = \infty \) and \( \lim_{t \to \frac{1}{2}^-} \delta^{-1}(s) = 0 \). As a result, while inequality (2) holds for \( 0 < \beta < \frac{1}{2} \), the function \( h_{i,\alpha,\beta}(t) \) is positive in \((0, \infty)\). Consequently, if \( 0 < \beta < \frac{1}{2} \) and inequality (2) is valid, then the function \( g_{i,\alpha,\beta}(x) \) is strictly increasing in \((0, \infty)\). The sufficiency is proved.
Now we are in a position to prove the necessity. In [8], it was proved that $$\psi(x) - \ln x + \frac{a}{2} \in C[[0, \infty)]$$ if and only if $$\alpha \geq 1$$ and $$\ln x - \frac{a}{2} - \psi(x) \in C[[0, \infty)]$$ if and only if $$\alpha \leq \frac{1}{2}$$. From this it is deduced that inequality
\[
\frac{(k - 1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) = \left|\psi^{(k)}(x)\right| < \frac{(k - 1)!}{x^k} + \frac{k!}{x^{k+1}}
\] holds in $$(0, \infty)$$ for $$k \in \mathbb{N}$$.

If $$g_{i, \alpha, 0}(x)$$ is strictly decreasing in $$(0, \infty)$$, then
\[
0 \geq \lim_{x \to \infty} x^{i+1-\alpha}g_{i, \alpha, 0}(x)
\]
Applying (13) into (14) leads to
\[
0 \geq \lim_{x \to \infty} x^{i+1-\alpha}g_{i, \alpha, 0}(x) = \alpha x^i \left|\psi^{(i)}(x)\right| - x^{i+1} \left|\psi^{(i+1)}(x)\right| < 0
\]
and, applying (9) into (15) and using (13),
\[
0 \leq \lim_{x \to 0^+} x^{i+2-\alpha}g_{i, \alpha, 0}(x)
\]
which means $$\alpha \leq i$$.

If $$g_{i, \alpha, 0}(x)$$ is strictly increasing in $$(0, \infty)$$, then
\[
x^{i+2-\alpha}g_{i, \alpha, 0}(x) = \alpha x^i \left|\psi^{(i)}(x)\right| - x^{i+2} \left|\psi^{(i+1)}(x)\right| > 0
\]
and, applying (9) into (15) and using (13),
\[
0 \leq \lim_{x \to 0^+} x^{i+2-\alpha}g_{i, \alpha, 0}(x)
\]
which means $$\alpha \geq i + 1$$.

If the function $$g_{i, \alpha, \beta}(x)$$ is strictly increasing in $$(0, \infty)$$ for $$\beta > 0$$, then
\[
x^{i+1-\alpha}g_{i, \alpha, \beta}(x) = \alpha x^i \left|\psi^{(i)}(x + \beta)\right| - x^{i+1} \left|\psi^{(i+1)}(x + \beta)\right| > 0
\]
Applying (13) in (16) and taking limit leads to
\[
0 \leq \lim_{x \to \infty} x^{i+1-\alpha}g_{i, \alpha, \beta}(x)
\]
which means $$\alpha \geq i$$. The proof of Theorem 1 is complete. \qed
Proof of Theorem 2. If \( h_{i,\alpha,\beta}(t) \geq 0 \) in \((0, \infty)\), then
\[
\pm \int_0^\infty h_{i,\alpha,\beta}(t) t e^{-(x+\beta)t} \, dt \in \mathbb{C}([-\beta, \infty]),
\]
which is equivalent to
\[
\pm \left[ \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} - \frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)\alpha-1} \right] \in \mathbb{C}([0, \infty]) \text{ by (12)},
\]
and then, by definition,
\[
(-1)^j \left[ \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} - \frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)\alpha-1} \right]^{(j)}
= (-1)^j \left[ \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} \right]^{(j)} - (-1)^j \left[ \frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)\alpha-1} \right]^{(j)} \geq 0
\]
in \((0, \infty)\) for \( j \geq 0 \). Further, formulas (7) and (10) imply
\[
\lim_{x \to -\infty} \left[ \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} \right]^{(j)} = \lim_{x \to -\infty} (-1)^j \left[ \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} \right] = 0. \tag{17}
\]
By (17) and Lemma 2, it is concluded that
\[
\pm \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} = \pm [\alpha|\psi(i)(x+\beta)| - x|\psi(i+1)(x+\beta)|] \in \mathbb{C}([0, \infty])
\]
if \( h_{i,\alpha,\beta}(t) \geq 0 \) in \((0, \infty)\). The proof of Theorem 1 tells us that the function \( h_{i,\alpha,\beta}(t) \) is positive in \((0, \infty)\) if either \( \beta = 0 \) and \( \alpha \geq i+1 \), or \( \beta \geq \frac{1}{2} \) and \( \alpha \geq i \), or \( 0 < \beta < \frac{1}{2} \) and inequality (2) validating, and that \( h_{i,\alpha,\beta}(t) \) is negative in \((0, \infty)\) if \( \beta = 0 \) and \( \alpha \leq i \). As a result, the function \( \alpha|\psi(i)(x+\beta)\| - x|\psi(i+1)(x+\beta)| \) is completely monotonic in \((0, \infty)\) for either \( \beta = 0 \) and \( \alpha \geq i+1 \), or \( \beta \geq \frac{1}{2} \) and \( \alpha \geq i \), or \( 0 < \beta < \frac{1}{2} \) and inequality (2) being true, and \( x|\psi(i+1)(x+\beta)| - \alpha|\psi(i)(x+\beta)| \in \mathbb{C}([0, \infty]) \) for \( \beta = 0 \) and \( \alpha \leq i \).

The proofs of necessities are the same as those in Theorem 1. The proof of Theorem 2 is complete. \(\square\)

Proof of Theorem 3. This follows from Theorem 2 and the following facts that
\[
\pm \left[ \frac{\alpha}{x}|\psi(i)(x+\beta)| - |\psi(i+1)(x+\beta)| \right] = \pm \frac{1}{x} \left[ \alpha|\psi(i)(x+\beta)| - x|\psi(i+1)(x+\beta)| \right],
\]
\( \frac{1}{2} \in \mathbb{C}([0, \infty]) \), and that the product of two completely monotonic functions is also completely monotonic on the union of their domains. \(\square\)

Proof of Theorem 4. Let \( B_k(x) \) be the Bernoulli polynomials defined \([1, 12, 13]\) by
\[
\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^\infty B_k(x) \frac{t^k}{k!} \tag{18}
\]
It is well known that the Bernoulli numbers \( B_k \) and \( B_k(x) \) are connected by \( B_k(1) = (-1)^k B_k(0) = (-1)^k B_k \) and \( B_{2k+1}(0) = B_{2k+1} = 0 \) for \( k \geq 1 \), and that the first few Bernoulli numbers and polynomials are
\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30},
\]
\[
B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.
\]
Using these notations, the functions \( h_{i,\alpha,\beta}(t) \) and \( h'_{i,\alpha,\beta}(t) \) can be rewritten as
\[
h_{i,\alpha,\beta}(t) = \frac{te^t}{e^t - 1} + (\beta - 1)t + \alpha - i - 1
\]
and...
\[
\begin{align*}
\alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=2}^{\infty} B_k (1 + \frac{t}{k}) \\
= \alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=2}^{\infty} (-1)^k B_k \frac{t^k}{k!} \\
= \alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=1}^{\infty} (-1)^{k+1} B_{k+1} \frac{t^{k+1}}{(k+1)!} \\
= \alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!} \\
\end{align*}
\]

\[
h'_{i,\alpha,\beta}(t) = \beta - \frac{1}{2} + \sum_{k=1}^{\infty} B_{2k+1} \frac{t^{2k+1}}{(2k+1)!}.
\]

The proof of Theorem 1 states that

(1) \(h'_{i,0,0}(t) < 0\) in \((0, \infty)\);

(2) if \(\alpha \geq i + 1\), then \(h_{i,\alpha,0}(t) > 0\) in \((0, \infty)\);

(3) if \(0 < \alpha \leq i\), then \(h_{i,\alpha,0}(t) < 0\) in \((0, \infty)\);

(4) if \(\beta \geq \frac{1}{2}\), then \(h'_{i,\alpha,\beta}(t) > 0\) in \((0, \infty)\);

(5) if \(\alpha \geq i\) and \(\beta \geq \frac{1}{2}\), then \(h_{i,\alpha,\beta}(t) > 0\) in \((0, \infty)\);

(6) if \(0 < \beta < \frac{1}{2}\) and inequality (2) holds true, then \(h_{i,\alpha,\beta}(t) > 0\) in \((0, \infty)\).

From these and standard argument, Theorem 4 is proved. \(\square\)

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