A GENERALISATION OF THE PEČARIĆ-RAJIĆ INEQUALITY IN NORMED LINEAR SPACES

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Abstract. In this paper we establish a generalisation of the recent *Pečarić-Rajić inequality* by providing upper and lower bounds for the norm of the linear combination $\sum_{j=1}^{n}\alpha_j x_j$ where $\alpha_j \in \mathbb{K}$ and $x_j \in X$ for $j \in \{1,\dots,n\}$ with $n \geq 2$. Applications for two vectors that are related to the *Massera-Schäffer, Dunkl-Williams* and *Maligranda-Mercer inequalities* are given. Some bounds for the quantity $\|x/\|y\| - y/\|x\|\|$ with $x, y \in X \setminus \{0\}$, are also provided.

1. Introduction

In the recent paper [13], J. Pečarić and R. Rajić proved the following inequality for n nonzero vectors $x_k, k \in \{1,\dots,n\}$ in the real or complex normed linear space $(X, \|\cdot\|)$:

$$\max_{k\in\{1,\dots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\left|\|x_j\| - \|x_k\|\right|\right]\right\} \leq \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \leq \min_{k\in\{1,\dots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\left|\|x_j\| - \|x_k\|\right|\right]\right\} \tag{1.1}$$

and showed that this inequality implies the following *refinement of the generalised triangle inequality* obtained by M. Kato et al. in [8]:

$$\min_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left[n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right] \leq \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \leq \max_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left[n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]. \tag{1.2}$$

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The inequality (1.2) can be also obtained as a particular case of the author's result established in [1]

$$\max_{1\le j\le n}\{\|x_j\|\}\left[\sum_{j=1}^{n}\|x_j\|^{p-1}-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^{p}\right] \tag{1.3}$$
$$\ge\sum_{j=1}^{n}\|x_j\|^{p}-n^{1-p}\left\|\sum_{j=1}^{n}x_j\right\|^{p}\ge\min_{1\le j\le n}\{\|x_j\|\}\left[\sum_{j=1}^{n}\|x_j\|^{p-1}-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^{p}\right],$$

where $p\ge 1$ and $n\ge 2$.

Notice that, in [1], a more general inequality for convex functions has been obtained as well.

In [13], Pečarić and Rajić also observed that, for $n=2, x_1=x$ and $x_2=-y$ their result reduces to

$$\frac{\|x-y\|-|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}\le\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\le\frac{\|x-y\|+|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}} \tag{1.4}$$

which holds for each nonzero vector $x,y\in X$.

The second inequality in (1.4) has been obtained by L. Maligranda in [9]. It provides a refinement of the *Massera-Schäffer inequality* [10]

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\le\frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}},$$

which, in its turn, is a refinement of the *Dunkl-Williams inequality* [7]

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\le\frac{4\|x-y\|}{\|x\|+\|y\|}.$$

The first inequality in (1.4) was obtained by P.R. Mercer in [11].

The main aim of this paper is to establish a generalisation of the *Pečarić-Rajić inequality* (1.1) by providing upper and lower bounds for the norm of the linear combination $\sum_{j=1}^{n}\alpha_j x_j$ where $\alpha_j\in\mathbb{K}$ and $x_j\in X$ for $j\in\{1,\dots,n\}$ with $n\ge 2$. Applications for two vectors that are related to the *Maligranda-Mercer inequalities* (1.4) are given. Some bounds for the dual quantity $\|x/\|y\|-y/\|x\|\|$ with $x,y\in X\setminus\{0\}$, are also provided.

2. A General Norm Inequality for n Vectors

We can state the following result

Theorem 1. *Let $(X,\|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. If $\alpha_j\in\mathbb{K}$ and $x_j\in X$ for $j\in\{1,\dots,n\}$ with $n\ge 2$, then*

$$\max_{k\in\{1,\dots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\alpha_j-\alpha_k|\|x_j\|\right\} \tag{2.1}$$
$$\le\left\|\sum_{j=1}^{n}\alpha_j x_j\right\|\le\min_{k\in\{1,\dots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}|\alpha_j-\alpha_k|\|x_j\|\right\}$$

Proof. Observe that, for any fixed $k \in \{1, \ldots, n\}$ we have

$$\sum_{j=1}^{n} \alpha_j x_j = \alpha_k \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} (\alpha_j - \alpha_k) x_j. \tag{2.2}$$

Taking the norm in (2.2) and utilising the triangle inequality we have successively

$$\begin{aligned} \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| &\leq \left\| \alpha_k \sum_{j=1}^{n} x_j \right\| + \left\| \sum_{j=1}^{n} (\alpha_j - \alpha_k) x_j \right\| \\ &\leq |\alpha_k| \left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\|, \end{aligned}$$

which, on taking the minimum over $k \in \{1, \ldots, n\}$, produces the second inequality in (2.1).

Since, obviously, by (2.2) we have

$$\sum_{j=1}^{n} \alpha_j x_j = \alpha_k \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} (\alpha_k - \alpha_j) x_j$$

then on utilising the continuity property of the norm we also have

$$\begin{aligned} \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| &\geq \left| \left\| \alpha_k \sum_{j=1}^{n} x_j \right\| - \left\| \sum_{j=1}^{n} (\alpha_j - \alpha_k) x_j \right\| \right| \\ &\geq \left\| \alpha_k \sum_{j=1}^{n} x_j \right\| - \left\| \sum_{j=1}^{n} (\alpha_j - \alpha_k) x_j \right\| \\ &\geq |\alpha_k| \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\| \end{aligned}$$

which, on taking the maximum over $k \in \{1, \ldots, n\}$ produces the first part of (2.1) and the theorem is completely proved. ∎

Remark 1. *If some information is available about the location of the scalars* $\alpha_j \neq 0, j \in \{1, \ldots, n\}$ *namely, if*

$$\left| \frac{\alpha_j}{\alpha_k} - 1 \right| \leq \rho \quad \text{for each } j, k \in \{1, \ldots, n\}$$

for a given $\rho > 0$, *then we get from the second part of (2.1) that*

$$\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \leq \min_{k \in \{1, \ldots, n\}} \{|\alpha_k|\} \left[\left\| \sum_{j=1}^{n} x_j \right\| + \rho \sum_{j=1}^{n} \|x_j\| \right].$$

If $x_j \in X$ *for* $j \in \{1, \ldots, n\}$ *are such that*

$$\left\| \sum_{j=1}^{n} x_j \right\| - \rho \sum_{j=1}^{n} \|x_j\| \geq 0$$

then the following nontrivial lower bound can be stated as well

$$\max_{k\in\{1,\dots,n\}}\{|\alpha_k|\}\left[\left\|\sum_{j=1}^n x_j\right\|-\rho\sum_{j=1}^n\|x_j\|\right]\leq\left\|\sum_{j=1}^n\alpha_j x_j\right\|.$$

Corollary 1. *Let $(X,\|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. If $x_j\in X$ for $j\in\{1,\dots,n\}$ with $n\geq 2$, then*

$$\max_{k\in\{1,\dots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^n x_j\right\|-\sum_{j=1}^n\left|\|x_j\|-\|x_k\|\right|\|x_j\|\right\}\tag{2.3}$$
$$\leq\left\|\sum_{j=1}^n\|x_j\|x_j\right\|\leq\min_{k\in\{1,\dots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^n x_j\right\|+\sum_{j=1}^n\left|\|x_j\|-\|x_k\|\right|\|x_j\|\right\}.$$

The proof is obvious by Theorem 1 on choosing $\alpha_k=\|x_k\|$, $k\in\{1,\dots,n\}$.

From (2.3) we can deduce some upper and lower bounds for the nonnegative quantity $\sum_{j=1}^n\|x_j\|^2-\left\|\sum_{j=1}^n\|x_j\|x_j\right\|$ as follows:

Corollary 2. *If $x_j\in X$ for $j\in\{1,\dots,n\}$ with $n\geq 2$, then*

$$(0\leq)\min_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left(\sum_{j=1}^n\|x_j\|-\left\|\sum_{j=1}^n x_j\right\|\right)\tag{2.4}$$
$$\leq\sum_{j=1}^n\|x_j\|^2-\left\|\sum_{j=1}^n\|x_j\|x_j\right\|$$
$$\leq\max_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left(\sum_{j=1}^n\|x_j\|-\left\|\sum_{j=1}^n x_j\right\|\right).$$

Proof. Assume that $\min_{k\in\{1,\dots,n\}}\{\|x_k\|\}=\|x_{k_0}\|$ with $k_0\in\{1,\dots,n\}$. Then, on utilising the second inequality in (2.3) we have

$$\left\|\sum_{j=1}^n\|x_j\|x_j\right\|\leq\|x_{k_0}\|\left\|\sum_{j=1}^n x_j\right\|+\sum_{j=1}^n\left|\|x_j\|-\|x_{k_0}\|\right|\|x_j\|$$
$$=\|x_{k_0}\|\left\|\sum_{j=1}^n x_j\right\|+\sum_{j=1}^n\|x_j\|^2-\|x_{k_0}\|\sum_{j=1}^n\|x_j\|$$

which is clearly equivalent to the first inequality in (2.4).

The second part follows likewise and the details are omitted. ∎

Remark 2. *If $x_j\in X\setminus\{0\}$ for $j\in\{1,\dots,n\}$ with $n\geq 2$, then from (2.1) for $\alpha_k=1/\|x_k\|$, $k\in\{1,\dots,n\}$ we deduce the Pečarić-Rajić inequality (1.1).*

3. Inequalities for Two Vectors

The case of two vectors may be of interest for applications in the Geometry of Banach Spaces.

We start with the following result:

Proposition 1. *For any two vectors $x, y \in X$ and two scalars $\alpha, \beta \in \mathbb{K}$ we have the double inequality*

$$\begin{aligned} &\frac{1}{2}\left[\left(|\alpha|+|\beta|\right)\|x+y\|-|\alpha-\beta|\left(\|x\|+\|y\|\right)\right] \\ &\qquad+\frac{1}{2}\left|\left(|\alpha|-|\beta|\right)\|x+y\|+|\alpha-\beta|\left(\|x\|-\|y\|\right)\right| \\ &\leq\|\alpha x+\beta y\| \\ &\leq \frac{1}{2}\left[\left(|\alpha|+|\beta|\right)\|x+y\|+|\alpha-\beta|\left(\|x\|+\|y\|\right)\right] \\ &\qquad-\frac{1}{2}\left|\left(|\alpha|-|\beta|\right)\|x+y\|-|\alpha-\beta|\left(\|x\|-\|y\|\right)\right| . \end{aligned} \tag{3.1}$$

Proof. If we apply Theorem 1 for $n=2,\ \alpha_1=\alpha,\ \alpha_2=\beta,\ x_1=x$ and $x_2=y$ we have

$$\begin{aligned} &\max\left\{|\alpha|\|x+y\|-|\alpha-\beta|\|y\|,|\beta|\|x+y\|-|\alpha-\beta|\|x\|\right\} \\ &\leq\|\alpha x+\beta y\| \\ &\leq \min\left\{|\alpha|\|x+y\|+|\alpha-\beta|\|y\|,|\beta|\|x+y\|+|\alpha-\beta|\|x\|\right\} . \end{aligned} \tag{3.2}$$

We utilize the properties that

$$\max\{a,b\}=\frac{1}{2}\left(a+b+|a-b|\right), \min\{a,b\}=\frac{1}{2}\left(a+b-|a-b|\right),$$

for any $a, b \in \mathbb{R}$ and since

$$\begin{aligned} &\max\left\{|\alpha|\|x+y\|-|\alpha-\beta|\|y\|,|\beta|\|x+y\|-|\alpha-\beta|\|x\|\right\} \\ &\qquad=\frac{1}{2}\left[\left(|\alpha|+|\beta|\right)\|x+y\|-|\alpha-\beta|\left(\|x\|+\|y\|\right)\right] \\ &\qquad\qquad+\frac{1}{2}\left|\left(|\alpha|-|\beta|\right)\|x+y\|+|\alpha-\beta|\left(\|x\|-\|y\|\right)\right| \end{aligned}$$

and

$$\begin{aligned} &\min\left\{|\alpha|\|x+y\|+|\alpha-\beta|\|y\|,|\beta|\|x+y\|+|\alpha-\beta|\|x\|\right\} \\ &\qquad\leq\frac{1}{2}\left[\left(|\alpha|+|\beta|\right)\|x+y\|+|\alpha-\beta|\left(\|x\|+\|y\|\right)\right] \\ &\qquad\qquad-\frac{1}{2}\left|\left(|\alpha|-|\beta|\right)\|x+y\|-|\alpha-\beta|\left(\|x\|-\|y\|\right)\right| , \end{aligned}$$

hence by (3.2) we deduce the desired result (3.1). ∎

The following particular cases are of interest.

Corollary 3. *Under the assumptions of Proposition 1 and if $|\alpha|=|\beta|=1$, then*

$$\left|\|\alpha x+\beta y\|-\|x+y\|\right|\leq|\alpha-\beta|\min\left\{\|x\|,\|y\|\right\}, \tag{3.3}$$

for any $x, y \in X$.

Corollary 4. *Under the assumptions of Proposition 1 and if $\|x\|=\|y\|=1$, then*

$$\left|\|\alpha x+\beta y\|-\left(|\alpha|+|\beta|\right)\cdot\left\|\frac{x+y}{2}\right\|\right|\leq|\alpha-\beta|-\left||\alpha|-|\beta|\right|\cdot\left\|\frac{x+y}{2}\right\|, \tag{3.4}$$

for any $\alpha, \beta \in \mathbb{K}$.

4. Dual Versions of the Maligranda-Mercer Inequality

In this section we provide two dual versions of the *Maligranda-Mercer inequality:*

$$\frac{\|x-y\|-|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}\leq\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leq\frac{\|x-y\|+|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}} \tag{4.1}$$

namely, we obtain upper and lower bounds for the quantity

$$\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|$$

in the case when the vectors x and y are nonzero in the normed linear space $(X,\|\cdot\|)$.

Theorem 2. *For any $x,y\in X\diagdown\{0\}$ we have*

$$\begin{aligned}0&\leq\frac{\|x-y\|}{\min\{\|x\|,\|y\|\}}-\frac{|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}} \\ &\leq\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|\leq\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}+\frac{|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}.\end{aligned} \tag{4.2}$$

Proof. We use the inequality (3.1) for $\alpha=1/\|y\|$ and $\beta=1/\|x\|$.

Firstly, we observe that

$$\begin{aligned}I&:=\frac{1}{2}\left[(|\alpha|+|\beta|)\|x+y\|-|\alpha-\beta|(\|x\|+\|y\|)\right] \\ &\qquad+\frac{1}{2}\left|(|\alpha|-|\beta|)\|x+y\|+|\alpha-\beta|(\|x\|-\|y\|)\right| \\ &=\frac{1}{2}\left[\left(\frac{\|x\|+\|y\|}{\|x\|\|y\|}\right)\|x+y\|-\frac{|\|x\|-\|y\||}{\|x\|\|y\|}(\|x\|+\|y\|)\right] \\ &\qquad+\frac{1}{2}\left|\left(\frac{\|x\|-\|y\|}{\|x\|\|y\|}\right)\|x+y\|+\frac{|\|x\|-\|y\||}{\|x\|\|y\|}(\|x\|-\|y\|)\right| \\ &=\frac{1}{2}\left(\frac{\|x\|+\|y\|}{\|x\|\|y\|}\right)(\|x+y\|-|\|x\|-\|y\||) \\ &\qquad+\frac{1}{2}\frac{|\|x\|-\|y\||}{\|x\|\|y\|}\left|\|x+y\|+|\|x\|-\|y\||\right|\end{aligned} \tag{4.3}$$

and since

$$\left|\|x+y\|+|\|x\|-\|y\||\right|=\|x+y\|+|\|x\|-\|y\||,$$

we get from (4.3) that

$$\begin{aligned}I=\frac{1}{2}\|x+y\|&\left[\frac{\|x\|+\|y\|}{\|x\|\|y\|}+\frac{|\|x\|-\|y\||}{\|x\|\|y\|}\right] \\ &-\frac{1}{2}|\|x\|-\|y\||\left[\frac{\|x\|+\|y\|}{\|x\|\|y\|}-\frac{|\|x\|-\|y\||}{\|x\|\|y\|}\right].\end{aligned} \tag{4.4}$$

Moreover, it is clear that

$$\frac{1}{2}\left[\frac{\|x\|+\|y\|}{\|x\|\|y\|}+\frac{|\|x\|-\|y\||}{\|x\|\|y\|}\right]=\max\left\{\frac{1}{\|x\|},\frac{1}{\|y\|}\right\}=\frac{1}{\min\{\|x\|,\|y\|\}}$$

and

$$\frac{1}{2}\left[\frac{\|x\|+\|y\|}{\|x\|\|y\|}-\frac{|\|x\|-\|y\||}{\|x\|\|y\|}\right]=\min\left\{\frac{1}{\|x\|},\frac{1}{\|y\|}\right\}=\frac{1}{\max\{\|x\|,\|y\|\}}$$

and then, by (4.4) we deduce

$$I=\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}-\frac{|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}. \tag{4.5}$$

Secondly, if we define J by

$$J:=\frac{1}{2}\left[(|\alpha|+|\beta|)\|x+y\|+|\alpha-\beta|(\|x\|+\|y\|)\right] \\ -\frac{1}{2}\left|(|\alpha|-|\beta|)\|x+y\|-|\alpha-\beta|(\|x\|-\|y\|)\right|$$

then for $\alpha=1/\|y\|$ and $\beta=1/\|x\|$ we get in a similar manner the equality

$$J=\frac{\|x+y\|}{\max\{\|x\|,\|y\|\}}+\frac{|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}. \tag{4.6}$$

Finally, by making use of the representations (4.5) and (4.6) we deduce from the inequality (3.1) that

$$\begin{aligned} 0 &\le \frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}-\frac{|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}} \\ &\le \left\|\frac{x}{\|y\|}+\frac{y}{\|x\|}\right\| \le \frac{\|x+y\|}{\max\{\|x\|,\|y\|\}}+\frac{|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}. \end{aligned} \tag{4.7}$$

which is clearly equivalent with (4.2). ∎

The second results looks slightly different:

Theorem 3. *For any two nonzero vectors $x,y\in X$ we have*

$$\left|\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|-\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}\right|\le\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}\,(\le 2)\,. \tag{4.8}$$

Proof. For $\alpha=\frac{1}{\|y\|}$ and $\beta=\frac{1}{\|x\|}$ in the left side of (3.2), we have

$$\begin{aligned} &\max\left\{\frac{1}{\|y\|}\|x+y\|-\frac{\|x\|+\|y\|}{\|x\|\|y\|}\cdot\|y\|,\frac{1}{\|x\|}\|x+y\|-\frac{\|x\|+\|y\|}{\|x\|\|y\|}\cdot\|x\|\right\} \\ &=\frac{1}{2}\left[\frac{\|x+y\|(\|x\|+\|y\|)}{\|x\|\|y\|}-\frac{(\|x\|+\|y\|)^2}{\|x\|\|y\|}\right] \\ &\qquad+\frac{1}{2}\left|\frac{\|x+y\|(\|x\|-\|y\|)}{\|x\|\|y\|}+\frac{(\|x\|+\|y\|)(\|x\|-\|y\|)}{\|x\|\|y\|}\right| \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\cdot\frac{\|x+y\|\left(\|x\|+\|y\|\right)}{\|x\|\,\|y\|}-\frac{1}{2}\cdot\frac{\left(\|x\|+\|y\|\right)^{2}}{\|x\|\,\|y\|}\\
&\quad+\frac{1}{2}\cdot\frac{\left|\|x\|-\|y\|\right|\|x+y\|}{\|x\|\,\|y\|}+\frac{1}{2}\cdot\frac{\left|\|x\|-\|y\|\right|\left(\|x\|+\|y\|\right)}{\|x\|\,\|y\|}\\
&= \frac{1}{2}\cdot\frac{\|x+y\|}{\|x\|\,\|y\|}\left[\|x\|+\|y\|+\left|\|x\|-\|y\|\right|\right]\\
&\quad-\frac{1}{2}\cdot\frac{\|x\|+\|y\|}{\|x\|\,\|y\|}\left[\|x\|+\|y\|-\left|\|x\|-\|y\|\right|\right]\\
&= \|x+y\|\max\left\{\frac{1}{\|x\|},\frac{1}{\|y\|}\right\}-\left(\|x\|+\|y\|\right)\min\left\{\frac{1}{\|x\|},\frac{1}{\|y\|}\right\}\\
&= \frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}-\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}.
\end{aligned}$$

On utilising the first inequality in (3.2) we then conclude that

$$\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}-\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}\le\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|,\qquad x,y\in X\backslash\{0\}. \tag{4.9}$$

We also have

$$\begin{aligned}
&\min\left\{\frac{\|x+y\|}{\|y\|}-\frac{\|x\|+\|y\|}{\|x\|\,\|y\|}\cdot\|y\|,\frac{\|x+y\|}{\|x\|}-\frac{\|x\|+\|y\|}{\|x\|\,\|y\|}\cdot\|x\|\right\}\\
&=\frac{1}{2}\left[\frac{\|x+y\|\left(\|x\|+\|y\|\right)}{\|x\|\,\|y\|}+\frac{\left(\|x\|+\|y\|\right)^{2}}{\|x\|\,\|y\|}\right]\\
&\qquad-\frac{1}{2}\left|\frac{\|x+y\|\left(\|x\|-\|y\|\right)}{\|x\|\,\|y\|}-\frac{\left(\|x\|+\|y\|\right)\left(\|x\|-\|y\|\right)}{\|x\|\,\|y\|}\right|\\
&=\frac{1}{2}\cdot\frac{\|x+y\|\left(\|x\|+\|y\|\right)}{\|x\|\,\|y\|}+\frac{1}{2}\cdot\frac{\left(\|x\|+\|y\|\right)^{2}}{\|x\|\,\|y\|}\\
&\qquad-\frac{1}{2}\cdot\frac{\left|\|x\|-\|y\|\right|}{\|x\|\,\|y\|}\left[\|x\|+\|y\|-\|x+y\|\right]\\
&= \frac{1}{2}\cdot\frac{\|x+y\|}{\|x\|\,\|y\|}\left[\|x\|+\|y\|+\left|\|x\|-\|y\|\right|\right]\\
&\qquad+\frac{1}{2}\cdot\frac{\|x\|+\|y\|}{\|x\|\,\|y\|}\left[\|x\|+\|y\|-\left|\|x\|-\|y\|\right|\right]\\
&= \|x+y\|\max\left\{\frac{1}{\|x\|},\frac{1}{\|y\|}\right\}+\left(\|x\|+\|y\|\right)\min\left\{\frac{1}{\|x\|},\frac{1}{\|y\|}\right\}\\
&= \frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}+\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}.
\end{aligned}$$

On utilising the second inequality in (3.2) we deduce

$$\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|\le\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}+\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}. \tag{4.10}$$

The desired result (4.8) is clearly equivalent with (4.9) and (4.10) and the proof is complete. ∎

5. Bounds for the Čebyšev Functional

For $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_n) \in \mathbb{K}^n$ and $\mathbf{y} := (y_1, \ldots, y_n) \in X^n$ we consider the *unweighted Čebyšev functional* defined by

$$C_n(\boldsymbol{\beta}, \mathbf{y}) := \frac{1}{n}\sum_{j=1}^{n} \beta_j y_j - \frac{1}{n}\sum_{j=1}^{n} \beta_j \cdot \frac{1}{n}\sum_{j=1}^{n} y_j.$$

This functional plays an important role in providing error bounds for approximating $\frac{1}{n}\sum_{j=1}^{n} \beta_j y_j$ by the simpler quantities $\frac{1}{n}\sum_{j=1}^{n} \beta_j$ and $\frac{1}{n}\sum_{j=1}^{n} y_j$.

We remark that, this functional has been considered previously by the author and some bounds have been established. We recall here some simple results.

With the above assumptions for $X, \boldsymbol{\alpha}$ and $\mathbf{y}$, we have

$$\|C_n(\boldsymbol{\alpha}, \mathbf{y})\| \leq \begin{cases} \frac{1}{12}\left(n^2-1\right) \max\limits_{j\in\{1,\ldots,n-1\}} |\Delta\alpha_j| \max\limits_{j\in\{1,\ldots,n-1\}} \|\Delta y_j\|, & [6]; \\ \frac{1}{2}\cdot\left(1-\frac{1}{n}\right) \sum\limits_{j=1}^{n-1} |\Delta\alpha_j| \sum\limits_{j=1}^{n-1} \|\Delta y_j\|, & [3]; \\ \frac{1}{6}\frac{n^2-1}{n}\left(\sum\limits_{j=1}^{n-1} |\Delta\alpha_j|^p\right)^{\frac{1}{p}} \left(\sum\limits_{j=1}^{n-1} \|\Delta y_j\|^q\right)^{\frac{1}{q}}, & \\ p>1, \frac{1}{p}+\frac{1}{q}=1, & [2], \end{cases} \tag{5.1}$$

where $\Delta z_j = z_{j+1} - z_j$ is the forward difference. Here the constants $\frac{1}{12}, \frac{1}{2}$ and $\frac{1}{6}$ are best possible in the sense that they cannot be replaced by smaller quantities.

In [5] we also have established that

$$\begin{aligned} &\|C_n(\boldsymbol{\alpha}, \mathbf{y})\| \\ &\leq \frac{1}{n^2} \times \begin{cases} \max\limits_{i\in\{1,\ldots,n-1\}} \left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}\alpha_k & \sum_{k=1}^{n}\alpha_k \end{pmatrix}\right| \cdot \sum_{j=1}^{n-1} \|\Delta y_j\|; \\ \left(\sum\limits_{i=1}^{n-1} \left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}\alpha_k & \sum_{k=1}^{n}\alpha_k \end{pmatrix}\right|^q\right)^{\frac{1}{q}} \cdot \left(\sum\limits_{j=1}^{n-1} \|\Delta y_j\|^p\right)^{\frac{1}{p}} \\ \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{i=1}^{n-1} \left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}\alpha_k & \sum_{k=1}^{n}\alpha_k \end{pmatrix}\right| \cdot \max\limits_{j\in\{1,\ldots,n-1\}} \|\Delta y_j\|. \end{cases} \end{aligned} \tag{5.2}$$

and

$$\|C_n(\boldsymbol{\alpha},\mathbf{y})\| \leq \frac{1}{n}\times\begin{cases} \max\limits_{i\in\{1,\dots,n-1\}}\left|\frac{1}{n}\sum\limits_{k=1}^{n}\alpha_k-\frac{1}{i}\sum\limits_{k=1}^{i}\alpha_k\right|\cdot\sum\limits_{i=1}^{n-1}i\|\Delta y_i\|; \\ \left(\sum\limits_{i=1}^{n-1}i\left|\frac{1}{n}\sum\limits_{k=1}^{n}\alpha_k-\frac{1}{i}\sum\limits_{k=1}^{i}\alpha_k\right|^q\right)^{\frac{1}{q}}\cdot\left(\sum\limits_{i=1}^{n-1}i\|\Delta y_i\|^p\right)^{\frac{1}{p}} \\ \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{i=1}^{n-1}i\left|\frac{1}{n}\sum\limits_{k=1}^{n}\alpha_k-\frac{1}{i}\sum\limits_{k=1}^{i}\alpha_k\right|\cdot\max\limits_{i\in\{1,\dots,n-1\}}\|\Delta y_i\|. \end{cases} \tag{5.3}$$

Finally, we recall the following result from [4]:

If there exists the complex numbers $a, A\in\mathbb{C}$ such that

$$\operatorname{Re}\left[(A-\alpha_i)\left(\overline{\alpha_i}-\overline{a}\right)\right]\geq 0 \quad \text{for each } \ i\in\{1,\dots,n\}$$

or, equivalently,

$$\left|\alpha_i-\frac{a+A}{2}\right|\leq\frac{1}{2}|A-a| \quad \text{for each } \ i\in\{1,\dots,n\},$$

then one has the inequality:

$$\|C_n(\boldsymbol{\beta},\mathbf{y})\|\leq\frac{1}{2}|A-a|\cdot\frac{1}{n}\sum_{i=1}^{n}\left\|y_i-\frac{1}{n}\sum_{j=1}^{n}y_j\right\|. \tag{5.4}$$

The constant $\frac{1}{2}$ in the right hand side of the inequality (5.4) is best possible in the sense that it cannot be replaced by a smaller quantity.

For many other results that hold for n-tuples $\boldsymbol{\beta}$ and $\mathbf{y}$ of real numbers we recommend the chapters devoted to Grüss and Čebyšev inequalities from the books [12] and [14].

In the following we provide other upper and lower bounds for $\|C_n(\boldsymbol{\beta},\mathbf{y})\|$.

Proposition 2. *For any* $\boldsymbol{\beta}:=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$ *and* $\mathbf{y}:=(y_1,\dots,y_n)\in X^n$ *we have*

$$\begin{aligned}\|C_n(\boldsymbol{\beta},\mathbf{y})\| &\leq \min_{k\in\{1,\dots,n\}}\left\{\frac{1}{n}\sum_{j=1}^{n}|\beta_j-\beta_k|\left\|y_j-\frac{1}{n}\sum_{\ell=1}^{n}y_\ell\right\|\right\} \\ &\leq\begin{cases}\min\limits_{k\in\{1,\dots,n\}}\left\{\max\limits_{j\in\{1,\dots,n\}}\{|\beta_j-\beta_k|\}\right\}\frac{1}{n}\sum_{j=1}^{n}\left\|y_j-\frac{1}{n}\sum_{\ell=1}^{n}y_\ell\right\|; \\ \min\limits_{k\in\{1,\dots,n\}}\left\{\left[\frac{1}{n}\sum_{j=1}^{n}|\beta_j-\beta_k|^p\right]^{\frac{1}{p}}\right\}\left[\frac{1}{n}\sum_{j=1}^{n}\left\|y_j-\frac{1}{n}\sum_{\ell=1}^{n}y_\ell\right\|^q\right]^{\frac{1}{q}} \\ \qquad\qquad \textit{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \min\limits_{k\in\{1,\dots,n\}}\left\{\frac{1}{n}\sum_{j=1}^{n}|\beta_j-\beta_k|\right\}\max\limits_{j\in\{1,\dots,n\}}\left\{\left\|y_j-\frac{1}{n}\sum_{\ell=1}^{n}y_\ell\right\|\right\}.\end{cases}\end{aligned} \tag{5.5}$$

Proof. We observe that

$$C_n(\boldsymbol{\beta},\mathbf{y})=\frac{1}{n}\sum_{j=1}^{n}\beta_j\left(y_j-\frac{1}{n}\sum_{\ell=1}^{n}y_\ell\right).$$

Now, on applying the second inequality in Theorem 1 for $\alpha_j=\beta_j$ and $x_j=y_j-\frac{1}{n}\sum_{\ell=1}^{n}y_\ell$ we deduce the first part of (5.5). The second part is obvious by the Hölder inequality. ∎

The following results can be stated as well:

Proposition 3. *For any* $\boldsymbol{\beta}:=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$ *and* $\mathbf{y}:=(y_1,\dots,y_n)\in X^n$ *we have*

$$\max_{k\in\{1,\dots,n\}}\left\{\left|\beta_k-\frac{1}{n}\sum_{\ell=1}^{n}\beta_\ell\right|\left\|\frac{1}{n}\sum_{j=1}^{n}y_j-z\right\|-\frac{1}{n}\sum_{j=1}^{n}\left|\beta_j-\beta_k\right|\left\|y_j-z\right\|\right\}\tag{5.6}$$
$$\le\left\|C_n(\boldsymbol{\beta},\mathbf{y})\right\|$$
$$\le\min_{k\in\{1,\dots,n\}}\left\{\left|\beta_k-\frac{1}{n}\sum_{\ell=1}^{n}\beta_\ell\right|\left\|\frac{1}{n}\sum_{j=1}^{n}y_j-w\right\|+\frac{1}{n}\sum_{j=1}^{n}\left|\beta_j-\beta_k\right|\left\|y_j-w\right\|\right\}$$

for any $z,w\in X$.

Proof. Follows from Theorem 1 on noticing that

$$C_n(\boldsymbol{\beta},\mathbf{y})=\frac{1}{n}\sum_{j=1}^{n}\left(\beta_j-\frac{1}{n}\sum_{\ell=1}^{n}\beta_\ell\right)(y_j-t)$$

for any $t\in X$. ∎

Remark 3. *As a particular case, one can state the following inequality*

$$\max_{k\in\{1,\dots,n\}}\left\{\left|\beta_k-\frac{1}{n}\sum_{\ell=1}^{n}\beta_\ell\right|\left\|\frac{1}{n}\sum_{j=1}^{n}y_j\right\|-\frac{1}{n}\sum_{j=1}^{n}\left|\beta_j-\beta_k\right|\left\|y_j\right\|\right\}\tag{5.7}$$
$$\le\left\|C_n(\boldsymbol{\beta},\mathbf{y})\right\|$$
$$\le\min_{k\in\{1,\dots,n\}}\left\{\left|\beta_k-\frac{1}{n}\sum_{\ell=1}^{n}\beta_\ell\right|\left\|\frac{1}{n}\sum_{j=1}^{n}y_j\right\|+\frac{1}{n}\sum_{j=1}^{n}\left|\beta_j-\beta_k\right|\left\|y_j\right\|\right\}$$

that provides simpler upper and lower bounds for the norm of the unweighted Čebyšev functional $C_n(\boldsymbol{\beta},\mathbf{y})$.

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