BOUNDS FOR THE DEVIATION OF A FUNCTION FROM A GENERALISED CHORD GENERATED BY ITS EXTREMITIES WITH APPLICATIONS

SEVER S. DRAGOMIR

Abstract. Bounds for the deviation of a real-valued function \( f \) defined on a compact interval \([a, b]\) to the generalised chord

\[
\frac{v(b) - v(t)}{v(b) - v(a)} \cdot f(a) + \frac{v(t) - v(a)}{v(b) - v(a)} \cdot f(b),
\]

where \( v : [a, b] \to \mathbb{R} \) and \( v(a) \neq v(b) \), that connects its end points \((a, f(a))\) and \((b, f(b))\) are given. Applications for normalised positive linear functionals are provided as well.

1. Introduction

Consider a function \( f : [a, b] \to \mathbb{R} \) and assume that it is bounded on \([a, b]\). Denote by \( \Phi_f(t) \) the error in approximating the function \( f \) by its (straight line) chord \( d_f \) which connects the points \((a, f(a))\) and \((b, f(b))\), i.e.,

\[
\Phi_f(t) := \frac{b-t}{b-a} \cdot f(a) + \frac{t-a}{b-a} f(b) - f(t), \quad t \in [a, b].
\]

In the recent paper [3], sharp error estimates for \( \Phi_f(t) \) under various assumptions on the function \( f \) have been derived. We recall here some of them.

If there exist the constants \(-\infty < m < M < \infty\) such that \( m \leq f(t) \leq M \) for each \( t \in [a, b] \), then \( |\Phi_f(t)| \leq M - m \). The multiplication constant 1 in front of \((M - m)\) cannot be replaced by a smaller quantity. If \( f : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\), then

\[
0 \leq \Phi_f(t) \leq \frac{1}{b-a} (t - a) (b - t) \left[ f'_-(b) - f'_+(a) \right]
\]

\[
\leq \frac{1}{4} (b - a) \left[ f'_-(b) - f'_+(a) \right],
\]

for any \( t \in [a, b] \). In the case where the lateral derivatives \( f'_-(b) \) and \( f'_+(a) \) are finite, then the second inequality and the constant \( \frac{1}{4} \) are sharp.

Date: November 22, 2007.
1991 Mathematics Subject Classification. Primary 26D15.
Key words and phrases. Bounded variation, Stieltjes integral, Monotonicity, Normalised positive functionals.
If \( f : [a, b] \to \mathbb{R} \) is a function of bounded variation, then

\[
|\Phi_f(t)| \leq \frac{b-t}{b-a} \cdot \frac{t}{a} \int_a^b |f(t)| + \frac{t-a}{b-a} \int_t^b |f(t)| \\
\begin{cases}
\left[ \frac{1}{2} + \left| \frac{t-a}{b-a} \right| \right] \frac{b}{a} |f(t)|; \\
\left[ \left( \frac{b-t}{b-a} \right)^p + \left( \frac{t-a}{b-a} \right)^q \right]^{\frac{1}{p}} \left( \frac{t}{a} \int_a^b |f(t)| \right)^{\frac{1}{q}} & \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\
\frac{1}{2} \frac{b}{a} |f(t)| + \frac{1}{2} \left| \frac{t}{a} \int_a^b |f(t)| - \frac{b}{a} \right|.
\end{cases}
\]

The first inequality in (1.3) is sharp. The constant \( \frac{1}{2} \) is best possible in the first and third branches.

In particular, if \( f \) is \( L \)-Lipschitzian on \( [a, b] \), i.e., \( |f(t) - f(s)| \leq L|t-s| \) for any \( t, s \in [a, b] \), then

\[
|\Phi_f(t)| \leq \frac{2(b-t)(t-a)}{b-a} L \leq \frac{1}{2} (b-a) L,
\]

for any \( t \in [a, b] \). The constants 2 and \( \frac{1}{2} \) are best possible.

For extensions to \( n \)-time differentiable functions see [4].

In this paper we consider a natural generalisation of the above problem by introducing the error function for the approximation of \( f(t) \) with \( \frac{v(b) - v(t)}{v(b) - v(a)} \cdot f(a) + \frac{v(t) - v(a)}{v(b) - v(a)} \cdot f(b) \), where \( v : [a, b] \to \mathbb{R} \) is another function with the property that \( v(a) \neq v(b) \). Error bounds for different pairs of functions \( f, v \) are derived. Applications in obtaining error bounds in approximating the quantity \( A(f \circ u) \) by the generalised trapezoid formula

\[
\frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(a) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(b),
\]

where \( A \) is a normalised linear functional are also given.

2. Bounds for \( \Phi_{f,v} \) when \( f, v \) are of Bounded Variation

For a function \( p : [a, b] \to \mathbb{R} \) we define the kernel \( Q_p : [a, b]^2 \to \mathbb{R} \) by

\[
Q_p(t, s) := \begin{cases}
p(t) - p(b) & \text{if } a \leq s \leq t \leq b, \\
p(t) - p(a) & \text{if } a \leq t < s \leq b.
\end{cases}
\]

With this notation we have the following representation of the function \( \Phi_{f,v} \), where

\[
\Phi_{f,v}(t) = \frac{v(t) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - v(t)}{v(b) - v(a)} \cdot f(a) - f(t)
\]

with \( t \in [a, b] \).
Lemma 1. If $f, v : [a, b] \to \mathbb{R}$ are bounded functions on $[a, b]$, then

\begin{equation}
\Phi_{f,v} (t) = \frac{1}{v(b) - v(a)} \int_a^b Q_v (t, s) \, df (s) = \frac{1}{v(b) - v(a)} \int_a^b Q_f (t, s) \, dv (s)
\end{equation}

provided $v(b) \neq v(a)$, where the integrals are taken in the Riemann-Stieltjes sense.

Proof. We have

\begin{equation}
\Phi_{f,v} (t) = \frac{[v(t) - v(b)] [f(t) - f(a)] + [v(t) - v(a)] [f(b) - f(t)]}{v(b) - v(a)} = \frac{[v(t) - v(b)] \int_t^a df (s) + [v(t) - v(a)] \int_t^b df (s)}{v(b) - v(a)} = \frac{1}{v(b) - v(a)} \int_a^b Q_v (t, s) \, df (s).
\end{equation}

Also, by rearranging the terms in the first equality, we also have

\begin{equation}
\Phi_{f,v} (t) = \frac{[f(a) - f(t)] \int_t^b dv (s) + [f(b) - f(t)] \int_t^a df (s)}{v(b) - v(a)} = \frac{1}{v(b) - v(a)} \int_a^b Q_f (t, s) \, dv (s)
\end{equation}

and the representation (2.2) is proved. \(\blacksquare\)

The following estimation result can be stated.

Theorem 1. Assume that $f, v : [a, b] \to \mathbb{R}$ are bounded and $v(a) \neq v(b)$.

(i) If $f$ is of bounded variation on $[a, b]$, then

\begin{equation}
|\Phi_{f,v} (t)| \leq \frac{v(b) - v(t)}{v(b) - v(a)} \left[ \frac{1}{t} \int_a^t (f) + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \int_a^b (f) \right] = \max \left\{ \frac{v(b) - v(t)}{v(b) - v(a)}, \frac{v(t) - v(a)}{v(b) - v(a)} \right\} \left[ \frac{1}{t} \int_a^t (f) + \left| \frac{b}{t} \right| \int_a^b (f) \right]^{\frac{1}{p}}
\end{equation}

where

\begin{align*}
&\frac{1}{t} \int_a^t (f) + \left| \frac{b}{t} \right| \int_a^b (f) \\
&\quad \leq \frac{v(b) - v(t)}{v(b) - v(a)} + \frac{v(t) - v(a)}{v(b) - v(a)} = \max \left\{ \frac{v(b) - v(t)}{v(b) - v(a)}, \frac{v(t) - v(a)}{v(b) - v(a)} \right\} \left[ \frac{1}{t} \int_a^t (f) + \left| \frac{b}{t} \right| \int_a^b (f) \right]^{\frac{1}{p}}
\end{align*}

and

\begin{align*}
&\frac{v(b) - v(t)}{v(b) - v(a)} + \frac{v(t) - v(a)}{v(b) - v(a)} = \max \left\{ \frac{v(b) - v(t)}{v(b) - v(a)}, \frac{v(t) - v(a)}{v(b) - v(a)} \right\} \left[ \frac{1}{t} \int_a^t (f) + \left| \frac{b}{t} \right| \int_a^b (f) \right]^{\frac{1}{p}}
\end{align*}
Proof. Utilising the equality (2.3) and taking the modulus, we have successively:

\[ |\Phi_{f,v}(t)| \leq \frac{f(b) - f(t)}{v(b) - v(a)} \left( \int_a^t (v) + \frac{f(t) - f(a)}{v(b) - v(a)} \right) \cdot \frac{b}{t} (v) \]

\[
\leq \begin{cases} 
\max \left\{ \left| \frac{f(b) - f(t)}{v(b) - v(a)} \right|, \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right| \right\} \left( \int_a^t (v) \right) \\
\left[ \left| \frac{f(b) - f(t)}{v(b) - v(a)} \right| + \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right| \right]^{\frac{1}{2}} \left\{ \left[ \int_a^t (f) \right]^q + \left[ \frac{b}{t} (f) \right]^q \right\}^{\frac{1}{q}}, \\
\left[ \frac{\left| f(b) - f(t) \right| + \left| f(t) - f(a) \right|}{|v(b) - v(a)|} \right] \left\{ \frac{b}{2} \left[ \int_a^t (f) \right] + \frac{1}{2} \left[ \int_a^t (v) \right] - \frac{b}{2} \left[ \int_a^t (f) \right] \right\}, 
\end{cases}
\]

if \( p > 1, \frac{1}{p} + \frac{1}{q} = 1; \)

where for the last inequality we have used the Hölder inequality.

The inequality (2.6) goes likewise by utilising the equality (2.4).

Remark 1. Since \( v(a) \neq v(b) \), we can assume without loss the generality that \( v(a) < v(b) \). Now, if we assume that

\[ v(a) \leq v(t) \leq v(b) \quad \text{for any} \quad t \in (a,b), \]

then from the first branch of (2.5) we get the inequality

\[ |\Phi_{f,v}(t)| \leq \left[ \frac{1}{2} + \frac{v(t) - v(a) + v(b)}{2v(b) - v(a)} \right] \frac{b}{t} (f), \quad t \in [a,b]. \]

The constant \( \frac{1}{2} \) is sharp in (2.8).

To prove the sharpness of the constant we take in (2.8) \( v(t) = t \) and then choose \( t = \frac{a + b}{2} \). This produces the result:

\[ \left| f \left( \frac{a + b}{2} \right) - \frac{f(a) + f(b)}{2} \right| \leq \frac{b}{2} (f), \]

which is sharp since for \( f(t) = |t - \frac{a + b}{2}|, t \in [a,b] \) we obtain in both sides of (2.9) the same quantity \( \frac{b - a}{2} \).
Remark 2. We also remark that, if \( v \) satisfies (2.7), then from the last inequality in (2.5) we get

\[
|\Phi_{f,v}(t)| \leq \frac{1}{2} \sqrt{f(a)} + \frac{1}{2} \sqrt{f(b) - \sqrt{f(t)}}, \quad t \in [a,b]
\]

for which the first constant \( \frac{1}{2} \) is also best possible.

Remark 3. If \( f \) satisfies the property that \( f(a) \leq f(t) \leq f(b) \) for any \( t \in [a,b] \), then from the first inequality in (2.6) we get

\[
(2.10) \quad |\Phi_{f,v}(t)| \leq \frac{1}{2} \sqrt{f(a)} + \frac{1}{2} \sqrt{f(b) - \sqrt{f(t)}}, \quad t \in [a,b].
\]

The first constant \( \frac{1}{2} \) in (2.12) is best possible.

With the same assumptions for \( f \) we have from the second inequality in (2.6) that

\[
(2.11) \quad |\Phi_{f,v}(t)| \leq \frac{1}{2} \sqrt{f(a)} + \frac{1}{2} \sqrt{f(b) - \sqrt{f(t)}}, \quad t \in [a,b].
\]

The first constant \( \frac{1}{2} \) in (2.12) is best possible.

Indeed, if we choose \( v(t) = t \) and then \( t = \frac{a+b}{2} \) in (2.12), we have

\[
(2.13) \quad \left| \frac{f(a) + f(b)}{2} - f \left( \frac{a+b}{2} \right) \right| \leq \frac{1}{2} \left| f(b) - f(a) \right|.
\]

Now, for \( f : [a,b] \to \mathbb{R}, f(t) = 0 \) if \( t \in [a,b] \) and \( f(b) = k > 0 \), we obtain on both sides the same quantity \( \frac{k}{2} \).

3. Bounds for \( \Phi_{f,v} \) when \( v(a) < v(t) < v(b) \) (\( f(a) < f(t) < f(b) \))

The following result may be stated as well.

Theorem 2. Assume that \( f, v : [a,b] \to \mathbb{R} \) are bounded and \( v(a) \neq v(b) \).

(i) If \( v(a) < v(t) < v(b) \) for any \( t \in (a,b) \), then

\[
(3.1) \quad |\Phi_{f,v}(t)| \leq \frac{1}{4} |v(b) - v(a)| \left[ \frac{f(t) - f(a)}{v(t) - v(a)} + \frac{f(b) - f(t)}{v(b) - v(t)} \right], \quad t \in [a,b].
\]

The constant \( \frac{1}{4} \) is best possible.

(ii) If \( f(a) < f(t) < f(b) \) for \( t \in (a,b) \), then

\[
(3.2) \quad |\Phi_{f,v}(t)| \leq \frac{1}{4} \left[ \frac{|v(b) - v(t)|}{|v(b) - v(a)|} \left[ \frac{f(t) - f(a)}{v(t) - v(a)} + \frac{f(b) - f(t)}{v(b) - v(t)} \right] + \frac{|v(t) - v(a)|}{|v(b) - v(t)|} \right], \quad t \in [a,b].
\]

Proof. (i) From the first equality in (2.3), we have:

\[
\Phi_{f,v}(t) \leq \frac{|v(b) - v(t)|}{|v(b) - v(a)|} \left[ \frac{f(t) - f(a)}{v(t) - v(a)} + \frac{f(b) - f(t)}{v(b) - v(t)} \right] = \frac{|v(b) - v(t)|}{|v(b) - v(a)|} \left[ \frac{f(t) - f(a)}{v(t) - v(a)} + \frac{f(b) - f(t)}{v(b) - v(t)} \right] \leq \frac{1}{4} |v(b) - v(a)| \left[ \frac{f(t) - f(a)}{v(t) - v(a)} + \frac{f(b) - f(t)}{v(b) - v(t)} \right]
\]

since, for any \( t \in (a,b) \),

\[
|v(b) - v(t)| |v(t) - v(a)| \leq \frac{1}{4} |v(b) - v(a)|^2.
\]
For the best constant, choose \( v(t) = t \) and then \( t = \frac{a+b}{2} \) in (3.1) to obtain

\[
(3.3) \quad \left| \frac{f(a) + f(b)}{2} - f\left( \frac{a+b}{2} \right) \right| \leq \frac{1}{2} \left[ \left| f\left( \frac{a+b}{2} \right) - f(a) \right| + \left| f(b) - f\left( \frac{a+b}{2} \right) \right| \right].
\]

If we consider the function \( f : [a,b] \to \mathbb{R} \),

\[
f(t) = \begin{cases} 
0 & \text{if } t \in [a,b) \\
\kappa & \text{if } t = b, \kappa > 0,
\end{cases}
\]

then (3.3) becomes an equality with both terms \( \frac{\kappa}{2} \).

(ii) The proof goes likewise and the details are omitted.

**Remark 4.**

(a) Under the assumptions of (i) of Theorem 2 and if there exist \( L_a > 0, L_b > 0, \alpha, \beta \geq 0 \) such that

\[
(3.4) \quad \left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| \leq L_a (t - a)^\alpha, \quad \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \leq L_b (b - t)^\beta, \quad t \in (a,b),
\]

then we have the inequality:

\[
(3.5) \quad |\Phi_{f,v}(t)| \leq \frac{1}{2} \left[ |v(b) - v(a)| \right] L_a (t - a)^\alpha + L_b (b - t)^\beta, \quad t \in (a,b).
\]

(\( \text{aa} \)) Under the assumptions of (ii) of Theorem 2 and if there exist the constants \( H_a, H_b > 0 \) and \( \gamma, \delta \geq 0 \) such that

\[
(3.6) \quad \left| \frac{v(t) - v(a)}{f(t) - f(a)} \right| \leq H_a (t - a)^\gamma, \quad \left| \frac{v(b) - v(t)}{f(b) - f(t)} \right| \leq H_b (b - t)^\delta, \quad t \in (a,b),
\]

then we have the inequality:

\[
(3.7) \quad |\Phi_{f,v}(t)| \leq \frac{1}{4} \left[ \frac{|f(b) - f(a)|^2}{|v(b) - v(a)|} \right] H_a (t - a)^\gamma + H_b (b - t)^\delta, \quad t \in (a,b).
\]

The following corollary provides some uniform bounds in the case where the functions are differentiable.

**Corollary 1.** Assume that \( f,v : [a,b] \to \mathbb{R} \) are continuous on \([a,b]\) and differentiable on \((a,b)\) with \( v(a) \neq v(b) \).

(i) If \( v(a) < v(t) < v(b) \) and \( v'(t) \neq 0 \) for \( t \in (a,b) \), then

\[
(3.8) \quad |\Phi_{f,v}(t)| \leq \frac{1}{2} \cdot |v(b) - v(a)| \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|, \quad t \in (a,b).
\]

(ii) If \( f(a) < f(t) < f(b) \) and \( f'(t) \neq 0 \) for \( t \in (a,b) \), then

\[
(3.9) \quad |\Phi_{f,v}(t)| \leq \frac{1}{2} \cdot \frac{|f(b) - f(a)|^2}{|v(b) - v(a)|} \sup_{s \in (a,b)} \left| \frac{v'(s)}{f'(s)} \right|, \quad t \in (a,b).
\]

**Proof.** (i) Applying Cauchy’s mean value theorem, we deduce that for any \( t \in (a,b) \) there exists an \( s \) between \( t \) and \( a \) such that

\[
\frac{f(t) - f(a)}{v(t) - v(a)} = \frac{f'(s)}{v'(s)}.
\]
Therefore,
\[
\left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| \leq \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|, \quad t \in (a,b)
\]
and in a similar manner,
\[
\left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \leq \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|, \quad t \in (a,b).
\]

Utilising the inequality (2.13) we deduce (3.8).

The proof of (ii) goes likewise and we omit the details. \[
\]

4. Bounds for \( \Phi_{f,v} \) when \( f, v \) are Lipschitzian

We can state the following result.

**Theorem 3.** Assume that \( f, v : [a,b] \to \mathbb{R} \) are bounded functions on \([a,b]\) and \( v(a) \neq v(b) \).

(i) If there exist constants \( M_a, M_b > 0 \) and \( \alpha, \beta > 0 \) such that \( |f(t) - f(a)| \leq M_a (t-a)^\alpha \), \( |f(b) - f(t)| \leq M_b (b-t)^\beta \) for any \( t \in [a,b] \) and \( v : [a,b] \to \mathbb{R} \) is Riemann integrable on \([a,b]\), then
\[
|\Phi_{f,v}(t)| \leq M_a \left| \frac{v(b) - v(t)}{f(b) - f(t)} \right| (t-a)^\alpha + M_b \left| \frac{v(t) - v(a)}{f(t) - f(a)} \right| (b-t)^\beta
\]
for any \( t \in [a,b] \).

(ii) If there exist constants \( N_a, N_b > 0, \gamma, \delta > 0 \) such that \( |v(t) - v(a)| \leq N_a (t-a)^\gamma \), \( |v(b) - v(t)| \leq N_b (b-t)^\delta \) for any \( t \in [a,b] \), then
\[
|\Phi_{f,v}(t)| \leq N_b \left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| (b-t)^\delta + N_a \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| (t-a)^\gamma
\]
for any \( t \in [a,b] \).

**Proof.** Utilising the representation (2.3) we have:
\[
|\Phi_{f,v}(t)| \leq \frac{|f(t) - f(a)| |v(b) - v(t)| + |v(t) - v(a)| |f(b) - f(t)|}{|v(b) - v(a)|}
\]
for any \( t \in [a,b] \), which clearly produces the desired inequalities (4.1) and (4.2).

We notice that, if more information is provided for \( f \) and \( v \), then more specific bounds can be obtained. For instance, if \( f \) is as in (i) of Theorem 3 and \( v(a) < v(t) < v(b) \) for each \( t \in (a,b) \), then we get from (4.1) the following inequality:
\[
|\Phi_{f,v}(t)| \leq \left[ \frac{1}{2} + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \right] \left[ M_a (t-a)^\alpha + M_b (b-t)^\beta \right]
\]
for any \( t \in [a,b] \).

Similarly, if \( v \) satisfies condition (ii) of Theorem 3 and \( f(a) < f(t) < f(b) \) for each \( t \in (a,b) \), then
\[
|\Phi_{f,v}(t)| \leq \left[ \frac{1}{2} \cdot \left| \frac{f(b) - f(a)}{v(b) - v(a)} \right| + \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right| \right]
\times \left[ N_b (b-t)^\delta + N_a (t-a)^\gamma \right]
\]
for any \( t \in [a,b] \).
If $f$ is $M$–Lipschitzian, then from (4.1) we get
\begin{equation}
|\Phi_{f,v}(t)| \leq M \left[ \frac{v(b) - v(t)}{v(b) - v(a)} (t-a) + \frac{|v(t) - v(a)|}{v(b) - v(a)} (b-t) \right]
\end{equation}
(4.5) \quad \leq M \left[ \frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right] \left[ \frac{v(b) - v(t)}{v(b) - v(a)} + \frac{|v(t) - v(a)|}{v(b) - v(a)} \right],
for any $t \in [a,b]$.

Also, if $v$ is $N$–Lipschitzian, then from (4.1) we get
\begin{equation}
|\Phi_{f,v}(t)| \leq N \left[ \frac{f(t) - f(a)}{v(b) - v(a)} (b-t) + \frac{|f(b) - f(t)|}{v(b) - v(a)} (t-a) \right]
\end{equation}
(4.6) \quad \leq N \left[ \frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right] \left[ \frac{f(t) - f(a)}{v(b) - v(a)} + \frac{|f(b) - f(t)|}{v(b) - v(a)} \right],
for any $t \in [a,b]$.

Moreover, if $f$ is $M$–Lipschitzian and $v(a) < v(t) < v(b)$ for any $t \in [a,b]$, then from (4.5) we get the simpler inequality:
\begin{equation}
|\Phi_{f,v}(t)| \leq M \left[ \frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right]
\end{equation}
(4.7) \quad \text{for any } t \in [a,b].

If $v$ is $N$–Lipschitzian and $f(a) < f(t) < f(b)$, $v(a) < v(b)$, then from (4.6) we also have:
\begin{equation}
|\Phi_{f,v}(t)| \leq N \left[ \frac{f(b) - f(a)}{v(b) - v(a)} \left( \frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right) \right],
\end{equation}
(4.8) \quad \text{for each } t \in [a,b].

5. Applications for Positive Linear Functionals

Let $L$ be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties
\begin{enumerate}
\item[(L1)] $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
\item[(L2)] $1 \in L$, i.e., if $f_0(t) = 1, t \in E$, then $f_0 \in L$.
\end{enumerate}

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying
\begin{enumerate}
\item[(A1)] $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;
\item[(A2)] If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$;
\item[(A3)] The mapping $A$ is normalised if $A(1) = 1$.
\end{enumerate}

For a function $u : E \to [a,b]$, we consider the function
\[ \Phi_{f,v}(u) := \frac{v \circ u - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - v \circ u}{v(b) - v(a)} \cdot f(a) - f \circ u \]
and assume throughout this section that $\Phi_{f,v}(u) \in L$.

It is obvious that for a normalised linear functional $A : L \to \mathbb{R}$ we have
\[ A(\Phi_{f,v}(u)) = \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \]
and the inequalities in the previous section can be utilised to provide various upper bounds for the quantity
\[ |A(\Phi_{f,v}(u))| \, . \]

For the sake of brevity we give here only some bounds that are simple and perhaps more useful for applications.
**Proposition 1.** Let \( f : [a,b] \to \mathbb{R} \) be of bounded variation on \([a,b]\) and \( v(a) < v(b) \), \( v(a) \leq v(t) \leq v(b) \) for each \( t \in [a,b] \). If \( u \in L \) so that \( \Phi_{f,v}(u) \in L \) and \( A : L \to \mathbb{R} \) is a normalised positive linear functional on \( L \), then:

\[
(5.1) \quad \left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \\
\leq \left[ \frac{1}{2} + \frac{1}{v(b) - v(a)} A \left( \left| \frac{v \circ u - \frac{v(a) + v(b)}{2}}{v(b) - v(a)} \cdot 1 \right| \right) \right]^b_a \bigg( f \bigg) .
\]

**Proof.** Utilising the inequality (2.8) and the properties of the functional \( A \), we have

\[
|A(\Phi_{f,v}(u))| \leq A(\Phi_{f,v}(u)) \\
\leq A \left( \left[ \frac{1}{2} + \frac{1}{v(b) - v(a)} A \left( \left| \frac{v \circ u - \frac{v(a) + v(b)}{2}}{v(b) - v(a)} \cdot 1 \right| \right) \right] \bigg( f \bigg) \right) \\
= \left[ \frac{1}{2} + \frac{1}{v(b) - v(a)} A \left( \left| \frac{v \circ u - \frac{v(a) + v(b)}{2}}{v(b) - v(a)} \cdot 1 \right| \right) \right] \bigg( f \bigg) \\
\]

and the inequality (5.1) is proved. \( \blacksquare \)

**Proposition 2.** Let \( f, v : [a,b] \to \mathbb{R} \) be bounded and \( v(a) \neq v(b) \). Also, assume that \( u \in L \) such that \( \Phi_{f,v}(u) \in L \) and \( A : L \to \mathbb{R} \) is a normalised positive linear functional on \( L \).

(i) If \( v(a) < v(t) < v(b) \) for each \( t \in [a,b] \), then

\[
(5.2) \quad \left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \\
\leq \frac{1}{4} \left[ v(b) - v(a) \right] \bigg[ A \left( \left| \frac{f - f(a) \cdot 1}{v - v(a) \cdot 1} \right| \right) + A \left( \left| \frac{f(b) \cdot 1 - f}{v(b) \cdot 1 - v} \right| \right) \bigg] ,
\]

provided \( \frac{f(a) \cdot 1 - f}{v(a) \cdot 1 - v} \in L \);

(ii) If \( f(0) < f(t) < f(b) \) for \( t \in (a,b) \), then

\[
(5.3) \quad \left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \\
\leq \frac{1}{4} \left[ v(b) - v(a) \right] \bigg[ A \left( \left| \frac{v - v(a) \cdot 1}{f - f(a) \cdot 1} \right| \right) + A \left( \left| \frac{v(b) \cdot 1 - v}{f(b) \cdot 1 - f} \right| \right) \bigg] ,
\]

provided \( \frac{v(a) \cdot 1 - v}{f(a) \cdot 1 - f} \in L \).

Utilising Corollary 1 we can state the following result that can be utilised for applications.

**Proposition 3.** Let \( f, v : [a,b] \to \mathbb{R} \) be continuous on \([a,b]\) and differentiable on \((a,b)\). Also, assume that \( u \in L \) such that \( \Phi_{f,v}(u) \in L \) and \( A : L \to \mathbb{R} \) is a normalised positive functional on \( L \).
If \( v \) is strictly monotonic on \([a, b]\), then

\[
\left| A(v \circ u) - v(a) \right| \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u)
\]
\[
\leq \frac{1}{2} \left| v(b) - v(a) \right| \sup_{s \in (a, b)} \left| f''(s) \right| \cdot \left| v'(s) \right|.
\]

If \( f \) is strictly monotonic on \([a, b]\), then

\[
\left| A(v \circ u) - v(a) \right| \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u)
\]
\[
\leq \frac{1}{2} \cdot \left[ f(b) - f(a) \right]^{2} \sup_{s \in (a, b)} \left| \frac{v'(s)}{f'(s)} \right|,
\]

provided \( v(a) \neq v(b) \).

For other inequalities for isotonic linear functionals, see the papers [1], [2], [6] and the books [5] and [7].

References


School of Computer Science & Mathematics, Victoria University, PO Box 14428, Melbourne, VIC 8001, Australia.

E-mail address: Sever.Dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir