SHARP ERROR BOUNDS IN APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL BY A GENERALISED TRAPEZOID FORMULA AND APPLICATIONS

P. CERONE AND S.S. DRAGOMIR

Abstract. Sharp error bounds in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the generalised trapezoid formula $f(b)\left[u(b)-\frac{1}{b-a}\int_a^b u(s)\,ds\right]+f(a)\left[\frac{1}{b-a}\int_a^b u(s)\,ds-u(a)\right]$ for various pairs (f,u) of functions are given. Applications for weighted integrals are also provided.

1. Introduction

In [8], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the *generalised trapezoid formula*

$$[u(b)-u(x)]\,f(b)+[u(x)-u(a)]\,f(a),\qquad x\in[a,b] \tag{1.1}$$

the authors considered the error functional

$$T(f,u;a,b;x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]\,f(b)-[u(x)-u(a)]\,f(a) \tag{1.2}$$

and proved that

$$|T(f,u;a,b;x)|\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f),\qquad x\in[a,b], \tag{1.3}$$

provided that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and u is of $r-H-$Hölder type, that is, $u:[a,b]\to\mathbb{R}$ satisfies the condition $|u(t)-u(s)|\le H|t-s|^r$ for any $t,s\in[a,b]$, where $r\in(0,1]$ and $H>0$ are given.

The dual case, namely, when f is of $q-K-$Hölder type and u is of bounded variation has been considered by the authors in [2] in which they obtained the bound:

$$\begin{aligned}(1.4)\qquad &|T(f,u;a,b;x)|\\ &\le K\left[(x-a)^q\bigvee_a^x(u)+(b-x)^q\bigvee_x^b(u)\right]\end{aligned}$$

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$$\leq \begin{cases} K\left[(x-a)^{q}+(b-x)^{q}\right]\left[\frac{1}{2}\bigvee_{a}^{b}(u)+\frac{1}{2}\left|\bigvee_{a}^{x}(u)-\bigvee_{x}^{b}(u)\right|\right]; \\ K\left[(x-a)^{q\alpha}+(b-x)^{q\alpha}\right]^{\frac{1}{\alpha}}\left[\left[\bigvee_{a}^{x}(u)\right]^{\beta}-\left[\bigvee_{x}^{b}(u)\right]^{\beta}\right]^{\frac{1}{\beta}} \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{q}\bigvee_{a}^{b}(u), \end{cases}$$

for any $x\in[a,b]$.

The case where f is monotonic and u is of $r-H-$Hölder type, which provides a refinement for (1.3), respectively the case where u is monotonic and f of $q-K-$Hölder type were considered by Cheung and Dragomir in [5], while the case where one function was of Hölder type and the other was Lipschitzian were considered in [1]. For other recent results in estimating the error $T(f,u;a,b,x)$ for absolutely continuous integrands f and integrators u of bounded variation, see [3] and [4].

The main aim of the present paper is to investigate the error bounds in approximating the Stieltjes integral by a different generalised trapezoid rule than the one from (1.1) in which the value $u(x)$, $x\in[a,b]$ is replaced with the integral mean $\frac{1}{b-a}\int_a^b u(s)\,ds$. Applications in approximating the weighted integrals $\int_a^b h(t)f(t)\,dt$ are also provided.

2. Representation Results

We consider the following error functional $T_g(f;u)$ in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the generalised trapezoid formula

$$f(b)\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right]+f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right],$$

$$\text{(2.1)}\quad T_g(f;u):=f(b)\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right] \\ +f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right]-\int_a^b f(t)\,du(t).$$

If we consider the associated functions Φ_f,Γ_f and Δ_f defined by

$$\Phi_f(t):=\frac{(t-a)f(b)+(b-t)f(a)}{b-a}-f(t),\quad t\in[a,b],$$
$$\Gamma_f(t):=(t-a)\left[f(b)-f(t)\right]-(b-t)\left[f(t)-f(a)\right],\quad t\in[a,b]$$

and

$$\Delta_f(t):=\frac{f(b)-f(t)}{b-t}-\frac{f(t)-f(a)}{t-a},\quad t\in(a,b),$$

then we observe that

$$\text{(2.2)}\qquad \Phi_f(t)=\frac{1}{b-a}\Gamma_f(t)=\frac{(b-t)(t-a)}{b-a}\Delta_f(t),\quad \text{for any } t\in(a,b).$$

The following representation result can be stated.

Theorem 1. *let $f, u : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and such that the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ and the Riemann integral $\int_a^b u(t)\, dt$ exist. Then we have the identities:*

$$T_g(f; u) = \int_a^b \Phi_f(t)\, du(t) = \frac{1}{b-a}\int_a^b \Gamma_f(t)\, du(t) \tag{2.3}$$
$$= \frac{1}{b-a}\int_a^b (b-t)(t-a)\Delta_f(t)\, du(t) = D(u; f),$$

where

$$D(u; f) = \int_a^b u(t)\, df(t) - [f(b) - f(a)] \cdot \frac{1}{b-a}\int_a^b u(t)\, dt. \tag{2.4}$$

Proof. Integrating the Riemann-Stieltjes integral by parts, we have

$$\begin{aligned}
&\int_a^b \Phi_f(t)\, du(t) \\
&= \int_a^b \left[\frac{f(a)(b-t) + f(b)(t-a)}{b-a} - f(t)\right] du(t) \\
&= \frac{1}{b-a}\left\{[f(a)(b-t) + f(b)(t-a)]\, u(t)\Big|_a^b \right. \\
&\qquad \left. - \int_a^b u(t)\, d[f(a)(b-t) + f(b)(t-a)]\right\} - \int_a^b f(t)\, du(t) \\
&= \frac{1}{b-a}\left\{[f(b)u(b) - f(a)u(a)](b-a) - [f(b) - f(a)]\int_a^b u(t)\, dt\right\} - \int_a^b f(t)\, du(t) \\
&= f(b)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\, dt\right] + f(a)\left[\frac{1}{b-a}\int_a^b u(t)\, dt - u(a)\right] \\
&= T_g(f; u),
\end{aligned}$$

and the first equality in (2.3) is proved.

The second and third identity is obvious by the relation (2.2).

For the last equality, we use the fact that for any $g, h : [a, b] \to \mathbb{R}$ bounded functions for which the Riemann-Stieltjes integral $\int_a^b h(t)\, dg(t)$ and the Riemann integral $\int_a^b g(t)\, dt$ exist, we have the representation (see for instance [6])

$$D(g; h) = \int_a^b \Phi_h(t)\, dg(t). \tag{2.5}$$

The proof is now complete. ∎

In the case where u is an integral, the following identity can be stated.

Corollary 1. *Let* $p, h : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *be Riemann integrable. Then we have the identity:*

$$\begin{aligned} T_g\left(f; \int_a ph\right) &= \frac{1}{b-a}\left[f(b)\cdot\int_a^b (t-a)\,p(t)\,h(t)\,dt + f(a)\cdot\int_a^b (b-t)\,p(t)\,h(t)\,dt\right] \\ &\qquad - \int_a^b p(t)\,f(t)\,h(t)\,dt \\ &= \int_a^b \Phi_f(t)\,p(t)\,h(t)\,dt. \end{aligned} \tag{2.6}$$

Proof. Since p and h are continuous, the function $u(t) = \int_a^t p(s)\,h(s)\,ds$ is differentiable and $u'(t) = p(t)\,h(t)$ for each $t \in (a,b)$.

Integrating by parts, we have

$$\begin{aligned} \int_a^b u(t)\,dt &= \left(\int_a^t p(s)\,h(s)\,ds\right)\cdot t\Big|_a^b - \int_a^b tp(t)\,h(t)\,dt \\ &= b\int_a^b p(s)\,h(s)\,ds - \int_a^b tp(t)\,h(t)\,dt \\ &= \int_a^b (b-t)\,p(t)\,h(t)\,dt. \end{aligned}$$

Since

$$\begin{aligned} u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt &= \int_a^b p(t)\,h(t)\,dt - \frac{1}{b-a}\int_a^b (b-t)\,p(t)\,h(t)\,dt \\ &= \frac{1}{b-a}\int_a^b (t-a)\,p(t)\,h(t)\,dt, \end{aligned}$$

then, by the definition of T_g in (2.1), we deduce the first part of (2.6).

The second part of (2.6) follows by (2.3). ∎

Remark 1. *In the particular case* $p(t) = 1$, $t \in [a,b]$, *we have the equality:*

$$\begin{aligned} T_g\left(f; \int_a h\right) &= \frac{1}{b-a}\left[f(b)\cdot\int_a^b (t-a)\,h(t)\,dt + f(a)\cdot\int_a^b (b-t)\,h(t)\,dt\right] \\ &\qquad - \int_a^b f(t)\,h(t)\,dt \\ &= \int_a^b \Phi_f(t)\,h(t)\,dt = \frac{1}{b-a}\int_a^b \Gamma_f(t)\,h(t)\,dt. \end{aligned} \tag{2.7}$$

3. Some Inequalities for f−Convex

The following result concerning the nonnegativity of the error functional $T_g(\cdot;\cdot)$ can be stated.

Theorem 2. *If u is monotonic nonincreasing and $f:[a,b]\to\mathbb{R}$ is such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\frac{f(b)-f(t)}{b-t}\geq\frac{f(t)-f(a)}{t-a},\quad \text{for any } t\in(a,b), \tag{3.1}$$

then $T_g(f;u)\geq 0$, or, equivalently

$$f(b)\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right]+f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right]\geq\int_a^b f(t)\,du(t). \tag{3.2}$$

A sufficient condition for (3.1) to hold is that f is convex on $[a,b]$.

Proof. The condition (3.1) is equivalent with the fact that $\Delta_f(t)\geq 0$ for any $t\in(a,b)$ and then, by the equality

$$T_g(f;u)=\frac{1}{b-a}\int_a^b (b-t)(t-a)\,\Delta_f(t)\,du(t)$$

we deduce that $T_g(f;u)\geq 0$.

If f is convex, then

$$\frac{t-a}{b-a}f(b)+\frac{b-t}{b-a}f(a)\geq f\left[\left(\frac{t-a}{b-a}\right)b+\left(\frac{b-t}{b-a}\right)a\right]=f(t)$$

which shows that $\Phi_f(t)\geq 0$, namely, the condition (3.1) is satisfied. ∎

Corollary 2. *Let $p,h:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ Riemann integrable. If $p(t)h(t)\geq 0$ for any $t\in[a,b]$ and f satisfies (3.1) or, sufficiently, f is convex on $[a,b]$, then*

$$\frac{1}{b-a}\left[f(b)\cdot\int_a^b (t-a)\,p(t)\,h(t)\,dt+f(a)\cdot\int_a^b (b-t)\,p(t)\,h(t)\,dt\right]\geq\int_a^b p(t)\,f(t)\,h(t)\,dt. \tag{3.3}$$

We are now able to provide some new results.

Theorem 3. *Assume that p and h are continuous and synchronous(asynchronous) on (a,b), i.e.,*

$$(p(t)-p(s))(h(t)-h(s))\geq(\leq)\,0,\quad \text{for any } t,s\in[a,b]. \tag{3.4}$$

If f satisfies (3.1) and is Riemann integrable on $[a,b]$ (or sufficiently, f is convex on $[a,b]$), then

$$T_g\left(f;\int_a p\right)\cdot T_g\left(f;\int_a h\right)\leq(\geq)\,T_g\left(f;\int_a 1\right)\cdot T_g\left(f;\int_a ph\right) \tag{3.5}$$

where

$$T_g\left(f;\int_a 1\right)=\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt. \tag{3.6}$$

Proof. We use the Čebyšev inequality:

$$\int_a^b \alpha(t)\,dt \int_a^b \alpha(t)\,p(t)\,h(t)\,dt \geq (\leq) \int_a^b \alpha(t)\,p(t)\,dt \int_a^b \alpha(t)\,h(t)\,dt, \tag{3.7}$$

which holds for synchronous (asynchronous) functions p, h and nonnegative α for which the involved integrals exist.

Now, on applying the Čebyšev inequality (3.7) for $\alpha(t) = \Phi_f(t) \geq 0$ and utilising the representation result (2.6), we deduce the desired inequality (3.5). ∎

We also have:

Theorem 4. *Assume that $f : [a,b] \to \mathbb{R}$ is Riemann integrable and satisfies (3.1) (or sufficiently, f is concave on $[a,b]$). Then for $p, h : [a,b] \to \mathbb{R}$ continuous, we have*

$$\left| T_g\left(f; \int_a ph\right)\right| \leq \sup_{t\in[a,b]} |h(t)|\, T_g\left(f; \int_a |p|\right) \tag{3.8}$$

and

$$\left| T_g\left(f; \int_a ph\right)\right| \leq \left[T_g\left(f; \int_a |p|^\alpha\right)\right]^{\frac{1}{\alpha}} \left[T_g\left(f; \int_a |h|^\beta\right)\right]^{\frac{1}{\beta}} \tag{3.9}$$

where $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. In particular, we have

$$\left| T_g\left(f; \int_a ph\right)\right|^2 \leq T_g\left(f; \int_a |p|^2\right) T_g\left(f; \int_a |h|^2\right). \tag{3.10}$$

Proof. Observe that

$$\begin{aligned}
\left| T_g\left(f; \int_a ph\right)\right| &= \left|\int_a^b \Phi_f(t)\,p(t)\,h(t)\,dt\right| \\
&\leq \int_a^b |\Phi_f(t)\,p(t)\,h(t)|\,dt \\
&= \int_a^b \Phi_f(t)\,|p(t)|\,|h(t)|\,dt \\
&\leq \sup_{t\in[a,b]} |h(t)| \int_a^b \Phi_f(t)\,|p(t)|\,dt \\
&= \sup_{t\in[a,b]} |h(t)|\, T_g\left(f; \int_a |p|\right)
\end{aligned}$$

and the inequality (3.8) is proved.

Further, by the Hölder inequality, we also have

$$\begin{aligned}
\left| T_g\left(f; \int_a ph\right)\right| &\leq \int_a^b \Phi_f(t)\,|p(t)|\,|h(t)|\,dt \\
&\leq \left(\int_a^b \Phi_f(t)\,|p(t)|^\alpha\,dt\right)^{\frac{1}{\alpha}} \left(\int_a^b \Phi_f(t)\,|h(t)|^\beta\,dt\right)^{\frac{1}{\beta}} \\
&= \left[T_g\left(f; \int_a |p|^\alpha\right)\right]^{\frac{1}{\alpha}} \left[T_g\left(f; \int_a |h|^\beta\right)\right]^{\frac{1}{\beta}}
\end{aligned}$$

for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and the theorem is proved. ∎

Remark 2. *The above result can be useful for providing some error estimates in approximating the weighted integral* $\int_a^b h(t) f(t) \, dt$ *by the generalised trapezoid rule*

$$\frac{1}{b-a}\left[f(b) \cdot \int_a^b (t-a) h(t) \, dt + f(a) \cdot \int_a^b (b-t) h(t) \, dt\right]$$

as follows:

$$\left|\frac{1}{b-a}\left[f(b) \cdot \int_a^b (t-a) h(t) \, dt + f(a) \cdot \int_a^b (b-t) h(t) \, dt\right] - \int_a^b h(t) f(t) \, dt\right| \leq \sup_{t \in [a,b]} |h(t)| \left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t) \, dt\right], \tag{3.11}$$

provided f *satisfies (3.1) and is Riemann integrable (or, sufficiently convex on* $[a,b]$*), which is continuous on* $[a,b]$.

If $h(t) = |w(t)|^{\frac{1}{\beta}}$, $t \in [a,b]$, *then for some* f *we also have*

$$\begin{aligned} &\left|\frac{1}{b-a}\left[f(b) \int_a^b (t-a) |w(t)|^{\frac{1}{\beta}} dt + f(a) \int_a^b (b-t) |w(t)|^{\frac{1}{\beta}} dt\right] - \int_a^b |w(t)|^{\frac{1}{\beta}} f(t) \, dt\right| \\ &\leq \left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t) \, dt\right]^{\frac{1}{\alpha}} \times \left\{\frac{1}{b-a}\left[f(b) \int_a^b (t-a) |w(t)| \, dt \right.\right. \\ &\qquad \left.\left. + f(a) \int_a^b (b-t) |w(t)| \, dt\right] - \int_a^b |w(t)| f(t) \, dt\right\}^{\frac{1}{\beta}}, \end{aligned} \tag{3.12}$$

with $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Finally, we can state the following Jensen type inequality for the error functional $T_g\left(f; \int_a^b h\right)$.

Theorem 5. *Assume* $f : [a,b] \to \mathbb{R}$ *is Riemann integrable and satisfies (3.1) (or sufficiently,* f *is convex on* $[a,b]$*), while* $h : [a,b] \to \mathbb{R}$ *is continuous. If* $F : \mathbb{R} \to \mathbb{R}$ *is convex(concave) then*

$$F\left(\frac{T_g\left(f; \int_a^b h\right)}{T_g\left(f; \int_a^b 1\right)}\right) \leq (\geq) \frac{T_g\left(f; \int_a^b F \circ h\right)}{T_g\left(f; \int_a^b 1\right)}. \tag{3.13}$$

Proof. By the use of Jensen's integral inequality, we have

$$F\left(\frac{\int_a^b \Phi_f(t) h(t) \, dt}{\int_a^b \Phi_f(t) \, dt}\right) \leq (\geq) \frac{\int_a^b \Phi_f(t) F(h(t)) \, dt}{\int_a^b \Phi_f(t) \, dt}. \tag{3.14}$$

Since, by the identity (2.6), we have

$$\int_a^b \Phi_f(t) F(h(t)) dt = T_g\left(f; \int_a^b F \circ h\right),$$

then (3.14) is equivalent with the desired result (3.13). ∎

4. Sharp Bounds Via Grüss Type Inequalities

Due to the identity (2.3) in which the error bound $T_g(f;u)$ can be represented as $D(u;f)$, where

$$D(u;f) = \int_a^b u(t) df(t) - [f(b) - f(a)] \cdot \frac{1}{b-a} \int_a^b u(t) dt,$$

is a Grüss type functional introduced in [9], any sharp bound for $D(u;f)$ will be a sharp bound for $T_g(f;u)$.

We can state the following result.

Theorem 6. *Let $f, u : [a,b] \to \mathbb{R}$ be bounded functions on $[a,b]$.*

(i) *If there exists the constants n, N such that $n \leq u(t) \leq N$ for any $t \in [a,b]$, u is Riemann integrable and f is $K-$Lipschitzian $(K > 0)$ then*

$$|T_g(f;u)| \leq \frac{1}{2} K (N-n)(b-a). \tag{4.1}$$

The constant $\frac{1}{2}$ is best possible in (4.1).

(ii) *If f is of bounded variation and u is $S-$Lipschitzian $(S > 0)$, then*

$$|T_g(f;u)| \leq \frac{1}{2} S (b-a) \bigvee_a^b (f). \tag{4.2}$$

The constant $\frac{1}{2}$ is best possible in (4.2)

(iii) *If f is monotonic nondecreasing and u is $S-$Lipschitzian, then*

$$\begin{aligned} |T_g(f;u)| &\leq \frac{1}{2} S (b-a) [f(b) - f(a) - P(f)] \\ &\leq \frac{1}{2} S (b-a) [f(b) - f(a)], \end{aligned} \tag{4.3}$$

where

$$P(f) = \frac{4}{(b-a)^2} \int_a^b \left(t - \frac{a+b}{2}\right) f(t) dt.$$

The constant $\frac{1}{2}$ is best possible in both inequalities.

(iv) *If f is monotonic nondecreasing and u is of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ exists, then*

$$|T_g(f;u)| \leq [f(b) - f(a) - Q(f)] \bigvee_a^b (u), \tag{4.4}$$

where

$$Q(f) := \frac{1}{b-a} \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f(t) dt.$$

The inequality (4.4) is sharp.

(v) *if f is continuous and convex on $[a,b]$ and u is of bounded variation on $[a,b]$, then*

$$|T_g(f;u)| \leq \frac{1}{4}\left[f'_-(b) - f'_+(a)\right]\bigvee_a^b(u). \tag{4.5}$$

The constant $\frac{1}{4}$ is sharp (if $f'_-(b)$ and $f'_+(a)$ are finite).

(vi) *If $f:[a,b]\to\mathbb{R}$ is continuous and convex on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned} 0 &\leq T_g(f;u) \\ &\leq 2\cdot\frac{f'_-(b)-f'_+(a)}{b-a}\cdot\int_a^b\left(t-\frac{a+b}{2}\right)u(t)\,dt \\ &\leq \begin{cases} \frac{1}{2}\left[f'_-(b)-f'_+(a)\right]\max\{|u(a)|,|u(b)|\}(b-a); \\ \frac{1}{(q+1)^{1/q}}\left[f'_-(b)-f'_+(a)\right]\|u\|_p(b-a)^{1/q} \quad \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[f'_-(b)-f'_+(a)\right]\|u\|_1. \end{cases} \end{aligned} \tag{4.6}$$

The constants 2 and $\frac{1}{2}$ are best possible in (4.6) (if $f'_-(b)$ and $f'_+(a)$ are finite).

Proof. The inequality (4.1) follows from the inequality (2.5) in [9] applied for $D(u;f)$, while (4.2) comes from (1.3) of [10]. The inequalities (4.3) and (4.4) follow from [6], while (4.5) and (4.6) are valid via the inequalities (2.8) and (2.1) from [7] applied for the functional $D(u;f)$. The details are omitted. ∎

If we consider the error functional in approximating the weighted integral $\int_a^b h(t)f(t)\,dt$ by the generalised trapezoid formula,

$$\frac{1}{b-a}\left[f(b)\cdot\int_a^b(t-a)h(t)\,dt + f(a)\cdot\int_a^b(b-t)h(t)\,dt\right],$$

namely, (see also (2.7))

$$\begin{aligned} E(f;h) &:= T_g\left(f;\int_a^b h\right) \\ &= \frac{1}{b-a}\left[f(b)\cdot\int_a^b(t-a)h(t)\,dt + f(a)\cdot\int_a^b(b-t)h(t)\,dt\right] \\ &\qquad -\int_a^b h(t)f(t)\,dt, \end{aligned} \tag{4.7}$$

then the following corollary provides various sharp bounds for the absolute value of $E(f;h)$.

Corollary 3. *Assume that f and u are Riemann integrable on $[a,b]$.*

(i) *If there exists the constants γ,Γ such that $\gamma\leq\int_a^t h(s)\,ds\leq\Gamma$ for each $t\in[a,b]$, and f is $K-$Lipschitzian on $[a,b]$, then*

$$|E(f;h)| \leq \frac{1}{2}K(\Gamma-\gamma)(b-a). \tag{4.8}$$

The constant $\frac{1}{2}$ is best possible in (4.8).

(ii) *If f is of bounded variation and $|h(t)| \leq M$ for each $t \in [a,b]$, then*

$$|E(f;h)| \leq \frac{1}{2} M (b-a) \bigvee_a^b (f). \tag{4.9}$$

The constant $\frac{1}{2}$ is best possible in (4.9).

(iii) *If f is monotonic nondecreasing and $|h(t)| \leq M$, $t \in [a,b]$, then*

$$\begin{aligned} |E(f;h)| &\leq \frac{1}{2} M (b-a) [f(b) - f(a) - P(f)] \\ &\leq \frac{1}{2} M (b-a) [f(b) - f(a)], \end{aligned} \tag{4.10}$$

where $P(f)$ is defined in Theorem 6. The constant $\frac{1}{2}$ is sharp in both inequalities.

(iv) *If f is monotonic nondecreasing and $\int_a^b |h(t)| dt < \infty$, then*

$$|E(f;h)| \leq [f(b) - f(a) - Q(f)] \int_a^b |h(t)| dt, \tag{4.11}$$

where $Q(f)$ is defined in Theorem 6. The inequality (4.11) is sharp.

(v) *If f is continuous and convex on $[a,b]$, and $\int_a^b |h(t)| dt < \infty$, then*

$$|E(f;h)| \leq \frac{1}{4} \left[f'_-(b) - f'_+(a) \right] \int_a^b |h(t)| dt. \tag{4.12}$$

The constant $\frac{1}{4}$ is sharp (if $f'_-(b)$ and $f'_+(a)$ are finite).

(vi) *If $f : [a,b] \to \mathbb{R}$ is continuous and convex on $[a,b]$ and $h(t) \geq 0$ for $t \in [a,b]$, then*

$$\begin{aligned} 0 &\leq E(f;h) \\ &\leq \frac{f'_-(b) - f'_+(a)}{b-a} \int_a^b (b-t)(t-a) h(t) dt \\ &\leq \begin{cases} \frac{1}{2} \left[f'_-(b) - f'_+(a) \right] (b-a) \int_a^b h(t) dt; \\ \frac{1}{(q+1)^{1/q}} \left[f'_-(b) - f'_+(a) \right] \left[\int_a^b \left(\int_a^t h(s) ds \right)^p dt \right]^{\frac{1}{p}} (b-a)^{1/q} \\ \qquad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[f'_-(b) - f'_+(a) \right] \int_a^b (b-t) h(t) dt. \end{cases} \end{aligned} \tag{4.13}$$

The first inequality in (4.13) is sharp (if $f'_-(b)$ and $f'_+(a)$ are finite).

Proof. We only prove the first inequality in (4.13).

Utilising the inequality (4.6) for $u(t) = \int_a^t h(s) ds$, we get

$$0 \leq E(f;h) \leq 2 \cdot \frac{f'_-(b) - f'_+(a)}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) \int_a^t h(s) dsdt. \tag{4.14}$$

However, on integrating by parts, we have

$$\begin{aligned}
&\int_a^b \left(t-\frac{a+b}{2}\right)\int_a^t h(s)\,dsdt \\
&= \int_a^b \left(\int_a^t h(s)\,ds\right) d\left[\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2\right] \\
&= \frac{1}{2}\left(t-\frac{a+b}{2}\right)^2 \int_a^t h(s)\,ds\Big|_a^b - \frac{1}{2}\int_a^b \left(t-\frac{a+b}{2}\right)^2 h(t)\,dt \\
&= \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 \int_a^b h(t)\,dt - \int_a^b \left(t-\frac{a+b}{2}\right)^2 h(t)\,dt\right] \\
&= \frac{1}{2}\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(t-\frac{a+b}{2}\right)^2\right] h(t)\,dt \\
&= \frac{1}{2}\int_a^b (b-t)(t-a)\,h(t)\,dt.
\end{aligned}$$

The rest of the inequality is obvious. ∎

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School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, 8001, VIC, Australia.

E-mail address: pietro.cerone@vu.edu.au
URL: http://rgmia.vu.edu.au/cerone

E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir