A NOTE ON AN OPEN PROBLEM

MIAO-QING AN

Abstract. The function $\frac{\Gamma(x+1)^{\frac{1}{x}}}{(x+\beta)^{\alpha}}$ is logarithmically completely monotonic on $(0,\infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$, and is logarithmically completely monotonic in $(-1,0)$ for $0 < \alpha \leq \frac{2\beta}{1+2\beta}$ and $\beta > 1$. This gives an answer to an open problem proposed by Feng Qi.

The classical gamma function

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \quad (x>0) \tag{1}$$

is one of the most important functions in analysis and its applications. The history and development of this function are described in detail [2]. The psi or digamma function $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed[6, p.16] as

$$\psi(x)=-\gamma+\int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \tag{2}$$

$$\psi^{(k)}(x)=(-1)^{k+1}\int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}}\,\mathrm{d}t \tag{3}$$

for $x>0$ and $k=1,2,\dots$, where $\gamma=0.57721566490153286\dots$ is the Euler-Mascheroni constant.

We recall that a function $f:(0,\infty)\longrightarrow \mathbf{R}$ is said to be completely monotonic if f has derivatives of all orders and

$$(-1)^n f^{(n)}(x)\geq 0 \tag{4}$$

for $x>0$ and n = 0, 1, 2, ... If f is nonconstant and completely monotonic, then the inequality (4) is strict, see [3]. Let $\mathcal{C}$ denote the set of completely monotonic functions.

A function f is said to be logarithmically completely monotonic on $(0,\infty)$ if f is positive and, for all $n\in\mathbb{N}$,

$$0\leq (-1)^n[\log f(x)]^{(n)}<\infty, \tag{5}$$

see[1, 7]. If inequality (5) is strict for all $x\in(0,\infty)$ and for all $n\geq 1$, then f is said to be strictly logarithmically completely monotonic. Let $\mathcal{L}$ on $(0,\infty)$ stand for the set of logarithmically completely monotonic functions.

The notion that logarithmically completely monotonic function was posed explicitly in [8] and published formally in [7] and a much useful and meaningful relation

2000 *Mathematics Subject Classification.* 33B15; 26A48.

Key words and phrases. Gamma function, psi function, polygamma function, complete monotonicity, logarithmically complete monotonicity.

$\mathcal{L} \subset \mathcal{C}$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in[7, 8].

In [5], H. Minc and L. Sathre proved that, if n is a positive integer and $\phi(n) = (n!)^{\frac{1}{n}}$, then

$$1 < \frac{\phi(n+1)}{\phi(n)} < \frac{n+1}{n}, \tag{6}$$

which can be rearranged as

$$[\Gamma(n+1)]^{1/n} < [\Gamma(n+2)]^{1/(n+1)}$$

and

$$\frac{[\Gamma(n+1)]^{1/n}}{n} > \frac{[\Gamma(n+2)]^{1/(n+1)}}{n+1},$$

since $\Gamma(n+1) = n!$.

In [4], the following monotonicity results for the Gamma function were established. The function $\left[\Gamma(1+\frac{1}{x})\right]^x$ decreases with $x > 0$ and $x\left[\Gamma(1+\frac{1}{x})\right]^x$ increases with $x > 0$, which recover the inequalities in (6) which refer to integer values of n. These are equivalent to the function $[\Gamma(1+x)]^{1/x}$ being increasing and $\frac{[\Gamma(1+x)]^{1/x}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}\left[[\Gamma(1+\frac{1}{x})^x\right]$ decreases for $0 < x < 1$, which is equivalent to $\frac{[\Gamma(1+x)]^{1/x}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In[9], Qi and Chen showed that the function $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing and strictly logarithmically convex in $(0,\infty)$, and the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing and strictly logarithmically concave in $(0,\infty)$. Using the monotonicity of above functions, Qi and Chen presented the following double inequality

$$\frac{x+1}{y+1} < \frac{[\Gamma(x+1)]^{1/x}}{[\Gamma(y+1)]^{1/y}} < \sqrt{\frac{x+1}{y+1}}$$

for $0 < x < y$, see Corollary 1 of [9].

In [8], Qi and Guo proposed an open problem

Open Problem 1. *Find conditions about α and β such that the ratio*

$$F(x) = \frac{[\Gamma(x+1)]^{\frac{1}{x}}}{(x+\beta)^\alpha} \tag{7}$$

is completely (absolutely, regularly) monotonic (convex) with $x > -1$.

In this paper , we give an answer to this problem and establish new inequalities.

Theorem 1. *The function $F(x)$ defined by (7) is strictly logarithmically completely monotonic in $(0,\infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$. Moreover, the function $F(x)$ is strictly completely monotonic in $(0,\infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$.*

*Proof.*Taking the logarithm of $F(x)$ defined by (7),

$$\begin{aligned} \log F(x) &= \frac{\log\Gamma(x+1)}{x} - \alpha\log(x+\beta) \\ &\triangleq g(x) - \alpha\log(x+\beta). \end{aligned} \tag{8}$$

Using Leibnitz' rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x) v^{(n-k)}(x), \tag{9}$$

we have

$$g^{(n)}(x) = \frac{1}{x^{n+1}} \sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!} \triangleq \frac{h_n(x)}{x^{n+1}}. \tag{10}$$

$$\begin{aligned} h_n^{'}(x) &= x^n \psi^{(n)}(x+1) \\ & \begin{cases} > 0, & \text{if n is odd and } x \in (0,\infty), \\ \leq 0, & \text{if in is odd and } x \in (-1,0) \text{ and n is even and } x \in (-1,\infty), \end{cases} \end{aligned} \tag{11}$$

where $\psi^{(-1)}(x+1) = \log \Gamma(x+1)$ and $\psi^{(0)}(x+1) = \psi(x+1)$.

$$\begin{aligned} (-1)^n \left(\log F(x)\right)^{(n)} &= \frac{1}{x^{n+1}} \left[(-1)^n h_n(x) + \frac{(n-1)! \alpha x^{n+1}}{(x+\beta)^n} \right] \\ &\triangleq \frac{v_{\alpha,\beta}(x)}{x^{n+1}} \end{aligned}$$

Using the representations

$$\frac{(n-1)!}{(x+1)^n} = \int_0^\infty t^{n-1} e^{-(x+1)t} \,\mathrm{d}t, x > 0, n = 1, 2, \ldots, \tag{12}$$

and (3), we conclude

$$\begin{aligned} v_{\alpha,\beta}^{'}(x) &= (-1)^n x^n \psi^{(n)}(x+1) + \frac{n! x^n \alpha\beta}{(x+\beta)^{n+1}} + \frac{(n-1)! x^n \alpha}{(x+\beta)^n} \\ &= x^n \int_0^\infty \left[\alpha(e^t - 1) + \alpha\beta t(e^t - 1) - t e^{\beta t} \right] \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} dt \\ &\triangleq x^n \int_0^\infty \phi(t) \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} dt \end{aligned} \tag{13}$$

where

$$\begin{aligned} \phi(t) &= \alpha\beta t(e^t - 1) - t e^{\beta t} + \alpha(e^t - 1) \\ &= (\alpha - 1)t + \sum_{m=2}^{\infty} [\alpha + m\beta(\alpha - \beta^{m-2})] \frac{t^m}{m!}. \end{aligned}$$

If $\alpha \geq 1$ and $0 \leq \beta \leq 1$, then $\phi(t) > 0$ and $v'_{\alpha,\beta}(x) > 0$. Hence, $v_{\alpha,\beta}(x) > v_{\alpha,\beta}(0) = 0$ and $(-1)^n \left(\log F(x)\right)^{(n)} > 0$, and thus, the function $F(x)$ is strictly logarithmically completely monotonic . The proof of Theorem 1 is complete.

Corollary 1. *For* $\alpha \geq 1$ *and* $0 \leq \beta \leq 1$,

$$\frac{\Gamma(x+1)^{\frac{1}{x}}}{\Gamma(y+1)^{\frac{1}{y}}} > \left(\frac{x+\beta}{y+\beta}\right)^{\alpha}, \tag{14}$$

in which $0 < x < y$.

Theorem 2. *The function $F(x)$ defined by (7) is strictly logarithmically completely monotonic in $(-1,0)$ for $0<\alpha\le\frac{2\beta}{1+2\beta}$ and $\beta>1$. Moreover, the function $F(x)$ is strictly completely monotonic in $(-1,0)$ for $0<\alpha\le\frac{2\beta}{1+2\beta}$ and $\beta>1$.*

Proof. By (13),

$$\begin{aligned}
\phi(t) &= \alpha\beta t(e^t-1)-te^{\beta t}+\alpha(e^t-1)\\
\phi(0) &= 0\\
\phi^{'}(t) &= e^t(\alpha+\alpha\beta+\alpha\beta t)-\alpha\beta-e^{\beta t}(1+\beta t)\\
\phi^{'}(0) &= \alpha-1\\
\phi^{''}(t) &= e^t\left[\alpha+2\alpha\beta+\alpha\beta t-\beta e^{(\beta-1)t}(2+\beta t)\right]\\
&\triangleq e^t u(t)
\end{aligned}$$

$$\begin{aligned}
u(0) &= \alpha+2\alpha\beta-2\beta\\
u'(t) &= \alpha\beta-\beta(\beta-1)e^{(\beta-1)t}(2+\beta t)-\beta^2e^{(\beta-1)t}\\
u^{'}(0) &= -3\beta^2+\alpha\beta+2\beta\\
u^{''}(t) &= e^{(\beta-1)t}\left[-\beta^2(\beta-1)^2t-2\beta(\beta-1)(2\beta-1)\right]
\end{aligned}$$

If $0<\alpha\le\frac{2\beta}{1+2\beta}$ and $\beta>1$, then $u^{''}(t)<0$ and $u^{'}(t)$ is strictly decreasing . So $u^{'}(t)<u^{'}(0)<0$ and $u(t)$ is strictly decreasing. Hence, $u(t)<u(0)<0$ and $\phi^{''}(t)<0$. Since $0<\alpha\le\frac{2\beta}{1+2\beta}$, we have $\phi^{'}(t)<\phi^{'}(0)<0$. So we conclude that $\phi(t)<\phi(0)=0$.

If n is odd, then $v^{'}_{\alpha,\beta}(x)>0$ on $(-1,0)$, and then, $v_{\alpha,\beta}(x)>v_{\alpha,\beta}(0)=0$ and $(-1)^n\left(\log F(x)\right)^{(n)}>0$. If n is even, then $v^{'}_{\alpha,\beta}(x)<0$ on $(-1,0)$, and then, $v_{\alpha,\beta}(x)<v_{\alpha,\beta}(0)=0$ and $(-1)^n\left(\log F(x)\right)^{(n)}>0$ on $(-1,0)$.

This means that the function $F(x)$ is strictly logarithmically completely monotonic on $(-1,0)$. The proof of Theorem 2 is complete.

Corollary 2. *For $0<\alpha\le\frac{2\beta}{1+2\beta}$ and $\beta>1$,*

$$\frac{\Gamma(x+1)^{\frac{1}{x}}}{\Gamma(y+1)^{\frac{1}{y}}}>\left(\frac{x+\beta}{y+\beta}\right)^{\alpha}, \tag{15}$$

in which $-1<x<y<0$.

Motivated by the open problem , we established a new function

$$G(x)=\frac{[\Gamma(x+\alpha)]^{\frac{1}{x}}}{(x+\beta)^{\gamma}} \tag{16}$$

in which α,β,γ are nonnegative. Our Theorem 3 consider its logarithmically completely monotonicity.

Theorem 3. *The function $G(x)$ defined by (16) is strictly logarithmically completely monotonic in $(0,\infty)$ for $\alpha \in (0,1]\cup[2,\infty)$, $\alpha-1 \le \beta \le \alpha$ and $\gamma \ge \max\left\{\frac{1}{\beta},1\right\}$. Moreover, the function $G(x)$ is strictly completely monotonic in $(0,\infty)$ for $\alpha \in (0,1]\cup[2,\infty)$, $\alpha-1\le\beta\le\alpha$ and $\gamma \ge \max\left\{\frac{1}{\beta},1\right\}$.*

Proof. Using (9), we obtain

$$\begin{aligned}(\log G(x))^{(n)} &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}[\log\Gamma(x+\alpha)]^{(k)} - \frac{(-1)^{n-1}\gamma(n-1)!}{(x+\beta)^n}\\ &= \left(\frac{1}{x}\right)^{(n)}\log\Gamma(x+\alpha) + \sum_{k=1}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}\psi^{(k-1)}(x+\alpha) + \frac{(-1)^n\gamma(n-1)!}{(x+\beta)^n}\\ &= \frac{(-1)^n n!}{x^{n+1}}\log\Gamma(x+\alpha) + \sum_{k=1}^{n}\frac{n!}{k!}\frac{(-1)^{n-k}}{x^{n-k+1}}\psi^{(k-1)}(x+\alpha) + \frac{(-1)^n\gamma(n-1)!}{(x+\beta)^n}\\ &\triangleq (-1)^n\frac{1}{x^{n+1}}\delta(x),\end{aligned}$$

and

$$\delta'(x) = x^n\left((-1)^n\psi^{(n)}(x+\alpha) + \frac{n!\beta\gamma}{(x+\beta)^{n+1}} + \frac{(n-1)!\gamma}{(x+\beta)^n}\right).$$

Using (3) and (12) for $x>0$ and $n\in\mathbb{N}$, we conclude

$$\begin{aligned}\frac{1}{x^n}\delta'(x) &= (-1)^n\psi^{(n)}(x+\alpha) + \frac{n!\beta\gamma}{(x+\beta)^{n+1}} + \frac{(n-1)!\gamma}{(x+\beta)^n}\\ &= \int_0^\infty\left[\gamma(e^t-1)+\beta\gamma t(e^t-1) - te^{(\beta-\alpha+1)t}\right]\frac{t^{n-1}e^{-(x+\beta)t}}{e^t-1}dt\\ &\triangleq \int_0^\infty u(t)\frac{t^{n-1}e^{-(x+\beta)t}}{e^t-1}dt,\end{aligned}$$

where

$$\begin{aligned}u(t) &= \beta\gamma t(e^t-1) - te^{(\beta-\alpha+1)t} + \gamma(e^t-1)\\ &= (\gamma-1)t + \sum_{m=2}^{\infty}\left\{\gamma + m\left[\beta\gamma - (\beta-\alpha+1)^{m-1}\right]\right\}\frac{t^m}{m!}.\end{aligned}$$

If $\alpha-1\le\beta\le\alpha$ and $\gamma\ge\max\left\{\frac{1}{\beta},1\right\}$, then $u(t)>0$ and $\delta'(x)>0$. Notice that $\Gamma(\alpha)\ge 1$ for $\alpha\in(0,1]\cup[2,\infty)$. Hence, $\delta(x)>\delta(0)=n!\log\Gamma(\alpha)\ge 0$ and $(-1)^n(\log G(x))^{(n)}>0$ in $(0,\infty)$, and thus, the function $G(x)$ is strictly logarithmically completely monotonic . The proof of Theorem 3 is complete.

Corollary 3. *For $\alpha\in(0,1]\cup[2,\infty)$, $\alpha-1\le\beta\le\alpha$ and $\gamma\ge\max\left\{\frac{1}{\beta},1\right\}$,*

$$\frac{\Gamma(x+\alpha)^{\frac{1}{x}}}{\Gamma(y+\alpha)^{\frac{1}{y}}} > \left(\frac{x+\beta}{y+\beta}\right)^{\gamma}, \tag{17}$$

in which $0<x<y$.

References

[1] R. D. Atanassov and U. V. Tsoukrovski, Some properties of a class of logarithmically completely monotonic functions, *C. R. Acad. Bulgare Sci.*, **41**(1988), 21-23.

[2] P. J. Davis, Leonhard Euler's integral:A historical profile of the gamma functions, *Amer. Math. Monthly*, **66**(1959), 848-869.

[3] J. Dubourdieu, Sur un théorème de M.s. Bernstein relatif á la transformation de Laplace-Stieltjes, *Compos. Math.* **7**(1939), 96-111.

[4] D. Kershaw and A. Laforgia, Monotonicity results for the gamma function, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **119** (1985), pp 127-133.

[5] H. Minc and L. Sathre, some inequalities involving $(r!)^{\frac{1}{r}}$, *Proc. Edinburgh Math. Soc.*, **14**(1964/65), 41-46.

[6] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin, 1966.

[7] F. Qi and C. P. Chen, A complete monotonicity Properties of the gamma function, *J. Math. Anal. Appl.*, **296** (2004), 603-607 .

[8] F. Qi and B. N. Guo, Complete monotonicities of functions involving the gamma and the digamma functions, *RGMIA Res. Rep. Coll.* **7**(2004), no 1, Art. 8. Available online at http://www.staff.vu.edu.au/rgmia/v7n1.asp.

[9] F. Qi and C. P. Chen, Monotonicity and convexity results for functions involving the gamma function, *RGMIA Res. Rep. Coll.* **6** (2003), no. 4, Art. 10.available online at:http://www.staff.vu.edu.au/rgmia/v6n4.asp

(miao-qing an) College of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan 454003, China

E-mail address: anxv1013@126.com, anxv1013@163.com