VICTORIA UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES

Binomial Type Sums

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TECHNICAL REPORT

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Abstract

Using ideas from residue theory, a method is developed which in turn allows a specific finite Binomial type sum to be expressed in closed form. The binomial type sum has applications in the areas of network reliability and discrete distributions.

AMS: 05A10, 40C99

Introduction

Some ideas of residue theory are used in expressing a finite sum in closed polynomial form. The sum to be considered is

$$\frac{(-1)^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^{r}\binom{n}{r}r^{n+m}$$

and it arises in the study of differential-difference equations in the work of Sofo and Cerone (to appear).

Binomial sums have been considered by Gould (1994) and the sum of the powers of natural numbers has been considered by de Bruyn (1995). Binomial type sums considered in this paper are related to Stirling numbers of the second kind, which in turn are related to Bernoulli, second order Eulerian and Bell numbers. However, the readers need not be acquainted with any of these special numbers as the authors develop a recurrence relation to determine the finite sum. The binomial type sum considered here has applications in the areas of network reliability, see for example the work of Prekopa et al. (1991) and in the representation of a discrete distribution, see the work of Brandt et al. (1990).

Firstly, in this paper, an infinite double sum is obtained from which a recurrence relation is developed. Then it is shown that the double sum may be expanded in a Maclaurin series. Secondly it is proved that the finite sum above can be written as a polynomial in n of degree m, and some examples are given. Finally a generalisation of the above finite sum is given.

Background and problem statement

Volterra integral equations of the form

$$\psi(t) = F(t) + \int_0^t \psi(t-x)\phi(x)dx \tag{1}$$

occur in a wide area of applications, (e.g. Tijms (1986)), as do differential-difference equations of the first order with a shift parameter, (e.g. Bellman and Cooke, (1963)).

Taking the Laplace transform of (1) and putting $F(t) = \delta(t)$, the Dirac delta impulse function, will produce an expression of the form

$$\Psi(p) = \frac{1}{1-\Phi(p)}. \tag{2}$$

Taking the Laplace transform of a differential-difference equation of the first order will produce a result like (2).

Now consider the rectangular wave $\phi(x) = H(a-x) = 1 - H(x-a)$, where $H(x)$ is the Heaviside function. The Laplace transform of $\phi(x)$ is:

$$\mathcal{L}\{\phi(x)\} = \Phi(p) = \frac{1-e^{-ap}}{p} \tag{3}$$

and the n-th moment of $\phi(x)$ is given by

$$M_n = \lim_{p\to 0}\left[(-1)^n \frac{d^n}{dp^n}\{\Phi(p)\}\right] = \frac{a^{n+1}}{n+1}. \tag{4}$$

Substituting (3) into (2) results in

$$\Psi(p) = \frac{p}{p-1+e^{-ap}}. \tag{5}$$

An expansion of (5) gives

$$\Psi(p) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\sum_{k=0}^{\infty}(-1)^k\frac{a^k r^k p^{k-n}}{k!}. \tag{6}$$

It may be noticed that $\Psi(p)$ can be written in the form $\Psi(p)=\sum_{m=0}^{\infty}\beta_m(a)p^m$

where

$$\beta_m(a)=\sum_{n=0}^{\infty}(-1)^{n+m}\frac{a^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m}. \tag{7}$$

In this paper the finite sum over r in equation (7) will be investigated.

A Recurrence Relation for $\beta_m(a)$

$\beta_m(a)$ may be given alternatively, from (5), as

$$\beta_m(a)=\lim_{p\to 0}\left[\frac{1}{m!}\frac{d^m}{dp^m}\left\{\frac{1}{1-\Phi(p)}\right\}\right]\ ;\ m=1,2,3...... \tag{8}$$

and in particular $\beta_0(a)=\frac{1}{1-\Phi(0)}=\frac{1}{1-a};a\neq 1.$

Furthermore, using the ideas developed by Cerone (1994),

$$m!\beta_m(a)=\lim_{p\to 0}\frac{d^m}{dp^m}\left[1+\frac{\Phi(p)}{1-\Phi(p)}\right]\ ;\ m=1,2,3.......$$

$$=\lim_{p\to 0}\frac{d^m}{dp^m}\left[\frac{\Phi(p)}{1-\Phi(p)}\right]$$

$$=\lim_{p\to 0}\sum_{k=0}^{m}\binom{m}{k}\frac{d^{m-k}}{dp^{m-k}}[\Phi(p)]\frac{d^k}{dp^k}\left[\frac{1}{1-\Phi(p)}\right]. \tag{9}$$

Hence, using (4) and (8) then, (9) may be written as

$$m!\beta_m(a)=\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{a^{m-k+1}k!}{(m-k+1)}.\beta_k(a) \tag{10}$$

that is, the $\beta_m(a)$ are given by the recurrence relation

$$(1-a)\beta_m(a)=\sum_{k=0}^{m-1}(-1)^{m-k}\frac{a^{m-(k-1)}}{(m-(k-1))!}.\beta_k(a)\ ;\ m=1,2,3.... \tag{11}$$

with $\quad \beta_0(a)=\frac{1}{1-a}.$

From (7), $\beta_m(a)=\sum_{n=0}^{\infty}(-1)^{n+m}\frac{a^{n+m}}{(n+m)!}.s(m,n)$

where $s(m,n)=\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m}$.

Putting $j=n+m$ gives $\beta_m(a)=\sum_{j=m}^{\infty}(-1)^j\frac{a^j}{j!}s(m,j-m)$

$$=(-1)^m\frac{a^m}{m!}s(m,0)+\sum_{j=m+1}^{\infty}(-1)^j\frac{a^j}{j!}s(m,j-m).$$

Since, $s(m,0)=\sum_{r=0}^{0}(-1)^r\binom{0}{r}r^{0+m}=0,\ m\neq 0$, then

$$\beta_m(a)=\sum_{j=m+1}^{\infty}(-1)^j\frac{a^j}{j!}s(m,j-m) \tag{12}$$

and so the problem lends itself in trying to express $\beta_m(a)$ as a Maclaurin series. From (7) and (12), $\beta_m(a)$ may be expanded in a Maclaurin series

$$\beta_m(a)=\sum_{q=m+1}^{\infty}\beta_m^{(q)}(0)\frac{a^q}{q!} \tag{13}$$

and the coefficients $\beta_m^{(q)}(0)$ can be calculated from the recurrence relation in (11), as follows.

From the left hand side of (11)

$$\frac{d^q}{da^q}\{(1-a)\beta_m(a)\}=\sum_{r=0}^{q}\binom{q}{r}\frac{d^{q-r}}{da^{q-r}}(1-a).\beta_m^{(r)}(a);q=0,1,2,\ldots..$$

and this term is zero only for $r=q$ and $r=q-1$, so that

$$\frac{d^q}{da^q}\{(1-a)\beta_m(a)\}=(1-a)\beta_m^{(q)}(a)+\binom{q}{q-1}(-1)\beta_m^{(q-1)}(a). \tag{14}$$

Further, from the right hand side of (11)

$$\frac{d^q}{da^q}\left[\sum_{k=0}^{m-1}(-1)^{m-k}\frac{a^{m-k+1}}{(m-k+1)!}\beta_k(a)\right]=\sum_{k=0}^{m-1}\frac{(-1)^{m-k}}{(m-k+1)!}\sum_{r=0}^{q}\binom{q}{r}\frac{d^{q-r}}{da^{q-r}}\{a^{m-k+1}\}.\beta_k^{(r)}(a)$$

$$=\sum_{k=0}^{m-1}(-1)^{m-k}\sum_{r=0}^{q}\binom{q}{r}\frac{a^{m-k+1-(q-r)}}{(m-k+1-(q-r))!}.\beta_k^{(r)}(a)$$

The last expression can be written as the sum of two terms corresponding to the a^0 term for $r = q-m+k-1$ and to the $a^j, j \geq 1$, term for $r > (q-m+k-1)$.

Hence,

$$\frac{d^q}{da^q}\left[\sum_{k=0}^{m-1}(-1)^{m-k}\frac{a^{m-k+1}}{(m-k+1)!}\beta_k(a)\right] = \sum_{k=0}^{m-1}(-1)^{m-k}\binom{q}{q-m+k-1}\beta_k^{(q-m+k-1)}(a)$$

$$+\sum_{k=0}^{m-1}(-1)^{m-k}\sum_{r=0}^{q}\binom{q}{r}\frac{a^{m-k+1-(q-r)}}{(m-k+1-(q-r))!}.\beta_k^{(r)}(a). \qquad (15)$$

Now, setting $a=0$ in (14) and (15) gives, after equating the right hand sides, the following recurrence relation for the coefficients of the Maclaurin series (13),

$$\beta_m^{(q)}(0) - q\beta_m^{(q-1)}(0) = \sum_{k=0}^{m-1}(-1)^{m-k}\binom{q}{q-m+k-1}.\beta_k^{(q-m+k-1)}(0)\ ,\ q = 0,1,2,\dots\ . (16)$$

These coefficients are demonstrated in Table 1.

q \ m	0	1	2	3	4	5	6	7	....
0	1	0	0	0	0	0	0	0	
1	1	0	0	0	0	0	0	0	
2	2!	-1	0	0	0	0	0	0	
3	3!	-6	1	0	0	0	0	0	
4	4!	-36	14	-1	0	0	0	0	
5	5!	-240	150	-30	1	0	0	0	
6	6!	-1800	1560	-540	62	-1	0	0	
7	7!	-15120	16800	-8400	1806	-126	1	0	
8	8!	-14120	191520	-126000	40824	-4914	254	-1	
:									

Table 1 : Coefficients of $\beta_m^{(q)}(0)$

Some observations that may be made from (16) and table 1 are as follows:

$\beta_0^{(0)}(0) = 1$; $\beta_0^{(q)}(0) = q!$; $\beta_m^{(0)}(0) = 0$ for $m \geq 1$;

$\beta_m^{(1)}(0) = \beta_m^{(0)}(0)$; $\beta_m^{(m+1)}(0) = (-1)^m$; and

$\beta_m^{(q)}(0) = 0$ for $q \leq m$.

The Polynomial

It will now be shown that the coefficients of the power series for $\beta_m(a)$ have the following property.

Theorem: The finite sum $P_m(n)=\frac{(-1)^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m}$ is a polynomial in n of degree m.

Proof: From (7),

$$\beta_m(a)=\sum_{n=1}^{m}\beta_m^{(m+n)}(0)\frac{a^{m+n}}{(m+n)!}+\sum_{n=m+1}^{\infty}(-1)^{n+m}\frac{a^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m}. \qquad (17)$$

From (11),

$$\beta_m(a)=\frac{(-1)^m}{(1-a)}\left[\frac{a^{m+1}}{(m+1)!}\beta_0(a)-\frac{a^m}{m!}\beta_1(a)+\frac{a^{m-1}}{(m-1)!}\beta_2(a)-\frac{a^{m-2}}{(m-2)!}\beta_3(a)+....+\frac{a^2}{2}\beta_{m-1}(a)\right];$$

$$m=1,2,3,....$$

$$=\frac{(-1)^m a^{m+1}}{(m+1)!(1-a)}\left[\frac{1}{1-a}+\frac{(m+1)a^2}{2a(1-a)^2}+\frac{(m+1)ma^3(a+2)}{12a^2(1-a)^3}+\frac{(m+1)m(m-1)a^4(1+2a)}{24a^3(1-a)^4}\right.$$

$$\left.+\frac{(m+1)m(m-1)(m-2)a^5\left(6+32a+8a^2-a^3\right)}{3!5!a^4(1-a)^5}+......+\frac{(m+1)!a^m}{2a^{m-1}(1-a)^m}X(a)\right]$$

where the function $X(a)$ is a polynomial in a, to be determined from the particular $\beta_{m-1}(a)$.

That is,

$$\beta_m(a)=\frac{(-1)^m a^{m+1}}{(m+1)!(1-a)^{m+1}}\left[(1-a)^{m-1}+\frac{(m+1)a(1-a)^{m-2}}{2}+\frac{(m+1)ma(a+2)(1-a)^{m-3}}{12}+\right.$$

$$\frac{(m+1)m(m-1)a(1+2a)(1-a)^{m-4}}{24}+\frac{(m+1)m(m-1)(m-2)a\left(6+32a+8a^2-a^3\right)(1-a)^{m-5}}{3!5!}$$

$$\left.+..........+\frac{(m+1)!a}{2}X(a)\right]$$ and so,

$$\beta_m(a)=(-1)^m \frac{a^{m+1}}{(m+1)!}\sum_{n=0}^{\infty}\binom{n+m}{n}a^n\left\{\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}a^k+\frac{(m+1)}{2}\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}a^{k+1}\right.$$

$$+\frac{m(m+1)}{12}\sum_{k=0}^{m-3}(-1)^k\binom{m-3}{k}\left[2a^{k+1}+a^{k+2}\right]+\frac{(m+1)m(m-1)}{24}\sum_{k=0}^{m-4}(-1)^k\binom{m-4}{k}\left[a^{k+1}+2a^{k+2}\right]$$

$$\left.+\frac{(m+1)m(m-1)(m-2)}{3!5!}\sum_{k=0}^{m-5}(-1)^k\binom{m-5}{k}\left[6a^{k+1}+32a^{k+2}+8a^{k+3}-a^{k+4}\right]+\ldots\ldots\frac{(m+1)!aX(a)}{2}\right\}$$

$$=\frac{(-1)^{2m}}{(m+1)!}\sum_{n=0}^{\infty}\binom{n+m}{n}a^{n+m+1}\left\{\ a^{m-1}\left[-1+\frac{m+1}{2}-\frac{(m+1)m}{12}+\frac{(m+1)m(m-1)(m-2)}{3!5!}+\ldots\right]\right.$$

$$+a^{m-2}\left[(m-1)-\frac{(m+1)(m-2)}{2}+\frac{(m+1)m(m-1)}{12}+\ldots\ldots\ldots\ldots\right]$$

$$\left.+\ldots\ldots\ldots a^0[\ldots\ldots\ldots\ldots\ldots\ldots\ldots]\right\}.$$

Thus, $\beta_m(a)$ may be expressed in the form

$$\beta_m(a)=\sum_{n=0}^{\infty}\binom{n+m}{n}\left\{a^{n+2m}F_1(m)+a^{n+2m-1}F_2(m)+\ldots\ldots+a^{n+m+1}F_m(m)\right\}$$

where the $F_j(m), j=1,2,\ldots,m,$ are functions dependent on the fixed parameter m only.

The summation indices are now adjusted to obtain coefficients of common powers of a in the following manner.

$$\beta_m(a)=\sum_{n=m-1}^{\infty}\binom{n+1}{n-m+1}a^{n+m+1}F_1(m)+\sum_{n=m-2}^{\infty}\binom{n+2}{n-m+2}a^{n+m+1}F_2(m)$$

$$+\ldots\ldots+\sum_{n=1}^{\infty}\binom{n+m-1}{n-1}a^{n+m+1}F_{m-1}(m)+\sum_{n=0}^{\infty}\binom{n+m}{n}a^{n+m+1}F_m(m),$$

and so,

$$\beta_m(a) = a^{2m}\binom{m}{0}F_1(m) + \sum_{n=m}^{\infty}\binom{n+1}{n-m+1}a^{n+m+1}F_1(m)$$

$$+\left[\binom{m+1}{1}a^{2m} + \binom{m}{0}a^{2m-1}\right]F_2(m) + \sum_{n=m}^{\infty}\binom{n+2}{n-m+2}a^{n+m+1}F_2(m)$$

$$+\ldots\ldots\ldots\ldots\ldots$$

$$+\left[\binom{2m-2}{m-2}a^{2m} + \binom{2m-3}{m-3}a^{2m-1} + \ldots\ldots + \binom{m+1}{1}a^{m+3} + \binom{m}{0}a^{m+2}\right]F_{m-1}(m) + \sum_{n=m}^{\infty}\binom{n+m-1}{n-1}a^{n+m+1}F_{m-1}(m)$$

$$+\left[\binom{2m-1}{m-1}a^{2m} + \binom{2m-2}{m-2}a^{2m-1} + \ldots\ldots + \binom{m+1}{1}a^{m+2} + \binom{m}{0}a^{m+1}\right]F_m(m) + \sum_{n=m}^{\infty}\binom{n+m}{n}a^{n+m+1}F_m(m) \quad (18)$$

Grouping of terms gives

$$\beta_m(a) = \sum_{n=m}^{\infty}a^{n+m+1}\left[\binom{n+1}{n-m+1}F_1(m) + \binom{n+2}{n-m+2}F_2(m) + \ldots\ldots + \binom{n+m-1}{n-1}F_{m-1}(m) + \binom{n+m}{n}F_m(m)\right]$$

$$+a^{2m}G_1(m) + a^{2m-1}G_2(m) + \ldots\ldots + a^{m+2}G_{m-1}(a) + a^{m+1}G_m(m),$$

and so,

$$\beta_m(a) = \sum_{n=m+1}^{\infty}a^{n+m}\left[\binom{n}{n-m}F_1(m) + \binom{n+1}{n-m+1}F_2(m) + \ldots\ldots + \binom{n+m-2}{n-2}F_{m-1}(m) + \binom{n+m-1}{n-1}F_m(m)\right]$$

$$+\sum_{n=1}^{m}a^{n+m}G_{m+1-n}(m) \quad (19)$$

where the functions $G_j(m)$, like $F_j(m)$, are dependent only on the fixed parameter m.

From the right hand side of (17) and (19) it may be seen that,

$$\sum_{n=1}^{m}\beta_m^{(m+n)}(0)\frac{a^{m+n}}{(m+n)!} = \sum_{k=1}^{m}a^{m+k}G_{m+1-k}(m) \quad (20)$$

and equating the powers of a^{m+j}, where $j = m+1, m+2, \ldots..$

$$\frac{(-1)^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m} = \sum_{k=0}^{m-1}\binom{n+k}{n-m+k}F_{k+1}(m). \quad (21)$$

Now since,

$$\binom{n+k}{n-m+k}=\frac{(n+k)(n+k-1)........(n+k-m+1)}{m!};k=0,1,2,.....,(m-1)$$

is a polynomial in n of degree m and the $F_{k+1}(m)$ functions depend on the fixed parameter m, then the right hand side of (21) is a polynomial in n of degree m.

Hence,

$$P_m(n)=\frac{(-1)^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m} \qquad (22)$$

is a polynomial in n of degree m, and the theorem is proved.

Finite differences may be used to determine a specific polynomial, however the following procedure establishes a recurrence relation to determine the values of $F_{k+1}(m)$ in (21).

Relations between $\beta_m^{(q)}(0), G_k(m)$ and $F_{k+1}(m)$

From (18) and (19) it can be seen that, on equating coefficients of $a^{m+j}, j=1,2,.....,m$ gives,

$$G_m(m)=\binom{m}{0}F_m(m)$$

$$G_{m-1}(m)=\binom{m}{0}F_{m-1}(m)+\binom{m+1}{1}F_m(m)$$

.

.

$$G_2(m)=\binom{m}{0}F_2(m)+\binom{m+1}{1}F_3(m)+......+\binom{2m-3}{m-3}F_{m-1}(m)+\binom{2m-2}{m-2}F(m)$$

$$G_1(m)=\binom{m}{0}F_1(m)+\binom{m+1}{1}F_2(m)+......+\binom{2m-2}{m-2}F_{m-1}(m)+\binom{2m-1}{m-1}F(m),$$

and therefore

$$G_k(m)=\sum_{j=0}^{m-k}\binom{m+j}{j}F_{j+k}(m)\ ,\ k=1,2,3,.....m.$$

The functions $F_j(m)$; $j=1,2,\ldots;m$, in (21) can now be recursively obtained from

$$\boldsymbol{M}\,\boldsymbol{F}=\boldsymbol{G} \tag{23}$$

where $\boldsymbol{M}$ is an $(m\,\mathrm{x}\,m)$ upper triangular matrix, $\boldsymbol{F}$ and $\boldsymbol{G}$ are $(m\,\mathrm{x}1)$ column vectors.

Similarly, from (20)

$$G_{m-k+1}(m)=\frac{\beta_m^{(m+k)}(0)}{(m+k)!}\ ;\ k=1,2,3,\ldots..m.$$

Now, putting $q=m+k$

$$G_{2m-q+1}(m)=\frac{\beta_m^{(q)}(0)}{q!}\quad ;\quad q=m+1,m+2,\ldots...2m,$$

and for the counter $j=2m-q+1$

$$G_j(m)=\frac{\beta_m^{(2m+1-j)}(0)}{(2m+1-j)!}\ ;\ j=m,m-1,m-2,\ldots...,3,2,1,$$

where the $\beta_m^{(q)}(0)$ values can be obtained from (16).

Therefore (23) may be written as

$$\boldsymbol{M}\,\boldsymbol{F}=\beta_m^{(2m+1-k)}(0)/(2m+1-k)!\ ;\ k=1,2,3,\ldots..m,\ \text{or}$$

$$\begin{bmatrix} \binom{m}{0} & \binom{m+1}{1} & \binom{m+2}{2} & \cdots\cdots & \binom{2m-1}{m-1} \\ & \binom{m}{0} & \binom{m+1}{1} & \cdots\cdots & \binom{2m-2}{m-2} \\ & & \ddots & & \\ & & \binom{m}{0} & \binom{m+1}{1} & \binom{m+2}{2} \\ & & & \binom{m}{0} & \binom{m+1}{1} \\ & & & & \binom{m}{0} \end{bmatrix} \begin{bmatrix} F_1(m) \\ F_2(m) \\ \vdots \\ F_{m-2}(m) \\ F_{m-1}(m) \\ F_m(m) \end{bmatrix} = \begin{bmatrix} \beta_m^{(2m)}(0)/(2m)! \\ \beta_m^{(2m+1)}(0)/(2m+1)! \\ \vdots \\ \beta_m^{(m+3)}(0)/(m+3)! \\ \beta_m^{(m+2)}(0)/(m+2)! \\ \beta_m^{(m+1)}(0)/(m+1)! \end{bmatrix}. \tag{24}$$

This matrix setup therefore allows a recursive evaluation of the functions $F_j(m)$; $j=1,2,3,....m$, in terms of the coefficients $\beta_m^{(q)}(0)$ in the Maclaurin series (13).

From the work of the previous section $F_1(m)$ takes the form

$$(m+1)!F_1(m)=-1+\frac{m+1}{2}-\frac{(m+1)m}{12}+0+\frac{(m+1)m(m-1)(m-2)}{6!}+....... \quad (25)$$

and for a particular value of m, that same number of terms are used on the right hand side of (25).

Therefore $F_1(1)=-\frac{1}{2}$, $F_1(2)=\frac{1}{12}$, $F_1(3)=0$, $F_1(4)=-\frac{1}{6!}$, $F_1(5)=0$,.... and because of the form of the polynomial in table 2, for every $m\geq 3$ and odd only, it is conjectured that $F_1(m)=0$. Of course all $F_1(m)$ values can be evaluated from (24).

Examples

(A) Let $m=3$ and from (21) and (22)

$$P_3(n)=\frac{(-1)^{n+3}}{(n+3)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+3}=\sum_{k=0}^{2}\binom{n+k}{n-3+k}F_{k+1}(3)$$

From (24) and table 1,

$$\binom{3}{0}F_3(3)=\frac{\beta_3^{(4)}(0)}{4!}=-\frac{1}{4!};\ \binom{3}{0}F_2(3)+\binom{4}{1}F_3(3)=\frac{\beta_3^{(5)}(0)}{5!}=-\frac{30}{5!};\ F_2(3)=-\frac{10}{5!}$$

$$\binom{3}{0}F_1(3)+\binom{4}{1}F_2(3)+\binom{5}{2}F_3(3)=\frac{\beta_3^{(6)}}{6!}(0)=-\frac{540}{6!};\ F_1(3)=0$$

and hence

$$P_3(n)=\binom{n}{n-3}(0)+\binom{n+1}{n-2}\left(-\frac{10}{5!}\right)+\binom{n+2}{n-1}\left(-\frac{1}{4!}\right)=-\frac{n^2(n+1)}{2.4!}$$

and $$\frac{(-1)^n}{(n+3)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+3}=\frac{n^2(n+1)}{2.4!}.$$

(B) The following example uses the idea of a multinominal distribution.
Consider n different coloured marbles in each of $(n+3)$ bags. In how many ways can one draw a marble from each bag such that all coloured marbles are represented in the 'hand' of $(n+3)$ marbles?

From the multinominal distribution, the total number of ways this can be done is:

$$\binom{n+3}{4}n(n-1)!+n(n-1)(n-2)!\binom{n+3}{3}\binom{n}{2}+\frac{n(n-1)(n-2)(n-3)!}{3!}\binom{n+3}{2}\binom{n+1}{2}\binom{n-1}{2} \quad (26)$$

$$=(n+3)!\left[\frac{n^2(n+1)}{2.4!}\right] \quad (27)$$

From the inclusion - exclusion principle, the total number of ways can be written down as

$$(-1)\binom{n}{1}+\binom{n}{2}2^{n+3}-\binom{n}{3}3^{n+3}+\ldots\ldots\ldots\ldots+(-1)^n\binom{n}{n}n^{n+3}=\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+3}.$$

However, from the work of the previous section

$$\frac{(-1)^n}{(n+3)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+3}=P_3(n)=\frac{n^2(n+1)}{2.4!}$$

so that

$$(-1)^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+3}=(n+3)!\frac{n^2(n+1)}{2.4!},$$

which is equivalent to (27).

For the general case of n coloured marbles and $(n+m)$ bags the expression at (26) becomes unwieldy and it is believed to be far better to use the methods of the previous sections.

Table 2, below lists the sum (22) in closed polynomial form for the values of $m = 0,1,2,3,4,5,6,7,8,9$.

m	$P_m(n)$
0	1
1	$n/2!$
2	$n(3n+1)/4!$
3	$n^2(n+1)/2.4!$
4	$n(15n^3+30n^2+5n-2)/2^3.6!$
5	$n^2(n+1)(3n^2+7n-2)/2^4.6!$
6	$n(63n^5+315n^4+315n^3-91n^2-42n+16)/2^3.9!$
7	$n^2(n+1)(9n^4+54n^3+51n^2-58n+16)/2^4.9!$
8	$n(135n^7+1260n^6+3150n^5+840n^4-2345n^3+540n^2-404n-144)/3.2^7.10!$
9	$n^2(n+1)(15n^6+165n^5+465n^4-17n^3-648n^2+548n-144)/3.2^8.10!$
.	
.	
.	

Table 2 : Some examples of $P_m(n)$ in closed form

These polynomials are related to the Stirling polynomials, see Graham et al. ((1989), by

$$P_m(n) = \frac{n!}{(n+m)!} S(n+m,n)$$

where $S(p,q)$ is the Stirling number of the second kind. From table 2, $P_3(4) = 5/3$. Using the table on page 835 of Abramowitz and Stegun (1965), $S(7,4) = 350$ and hence $P_3(4) = \frac{4!}{7!}.350 = \frac{5}{3}$.

In the next section a generalization of (22) is given whereby noninteger values may be summed.

A Generalization

Consider

$$Q_m(n,x) = \frac{(-1)^{n+m}}{(n+m)!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (x+r)^{n+m} \tag{28}$$

where m,n are natural numbers and x is a real number.

The following lemma is needed.

Lemma

$$R_i(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^i=\begin{cases}0 & ; \quad i=0,1,2,\ldots\ldots,(n-1)\\ (-1)^n n! & ; \quad i=n\end{cases}. \tag{29}$$

Proof

From the expression $(1-y)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k y^k$,

differentiating i times gives

$$\frac{d^i}{dy^i}\{(1-y)^n\}=(-1)^i n(n-1)\ldots\ldots(n-i+1)(1-y)^{n-i}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^i y^{k-i}\text{ ; for } i=0,1,2,\ldots.(n-1),$$

and so putting $y=1$, results in

$$0=\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^i.$$

Further for $i=n$, differentiating n times gives

$$(-1)^n n!=\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^n.$$

From (28)

$$Q_0(n,x)=\frac{(-1)^n}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(x+r)^n=\sum_{r=0}^{n}\sum_{k=0}^{n}\frac{(-1)^{r+n}}{n!}\binom{n}{r}\binom{n}{k}x^{n-k}r^k \tag{30}$$

Expanding and adding down columns results in

$$Q_0(n,x)=x^n\frac{(-1)^n}{n!}\binom{n}{0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^0+\frac{x^{n-1}(-1)^n}{n!}\binom{n}{1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^1+\frac{x^{n-2}(-1)^n}{n!}\binom{n}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^2$$

$$+\ldots\ldots.+\frac{x^0(-1)^n}{n!}\binom{n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^n.$$

Now, utilising the above lemma, from (29)

$$Q_0(n,x)=\frac{x^n(-1)^n}{n!}\binom{n}{0}.0+\ldots\ldots..+\frac{x^1(-1)^n}{n!}\binom{n}{n-1}.0+\frac{x^0(-1)^n}{n!}\binom{n}{n}(-1)^n n!$$

and hence, from (30)

$$Q_0(n,x)=\frac{(-1)^n}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(x+r)^n=1. \qquad (31)$$

The result in (31) can be integrated m times with respect to x to produce the result $Q_m(n,x)$ at (28).

The closed form of (28) can now be obtained as follows, integrating (31) and using the initial condition $Q_1(n,0)=P_1(n)=\frac{n}{2}$, from table 2 for $m=1$, results in

$$Q_1(n,x)=\frac{(-1)^n}{(n+1)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(x+r)^{n+1}=\frac{1}{2}(n+2x)$$

Using this procedure, table 3 below is given listing $Q_m(n,x)$ which gives (28) in closed form.

From (28)

$$Q_m(n,x)=\frac{(-1)^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\sum_{j=0}^{n+m}\binom{n+m}{j}r^{n+m-j}x^j=\sum_{j=0}^{n+m}\alpha_j x^j,$$

so that the coefficients of x^j can be expressed as

$$\alpha_j=\frac{(-1)^{n+m}}{(n+m)!}\binom{n+m}{j}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m-j};0\le j\le(n+m), \qquad (32)$$

and therefore $Q_m(n,x)$ appears to be a polynomial in x of degree $(n+m)$.

However, from (32), $\alpha_j = 0$ for $m+1 \le j \le n+m$ using the lemma (29), and hence $Q_m(n,x)$ is a polynomial in x of degree m. From the work of the previous section and above, $Q_m(n,x)$ is a polynomial in n and x of degree m for both n and x.

m	$Q_m(n,x)$
0	1
1	$(n+2x)/2$
2	$\left(3n^2+n(1+12x)+12x^2\right)/4!$
3	$\left(n^3+n^2(1+6x)+n\left(2x+12x^2\right)+8x^3\right)/2.4!$
4	$\left(15n^4+30n^3(1+4x)+5n^2(1+24x)(3x+1)\right)+2n\left(-1+60x^2(1+4x)\right)$
	$+240x^4)/2^3.6!$
.	
.	
.	
.	

Table 3 : Some examples of $Q_m(n,x)$ in closed form

Conclusion

A technique has been developed whereby Binomial type sums of powers can be expressed in closed form. An application to the multinominal distribution has been given.

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