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Integrals of logarithmic and hypergeometric functions

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Abstract. Integrals of logarithmic and hypergeometric functions are intrinsically connected with Euler sums. In this paper we explore many relations and explicitly derive closed form representations of integrals of logarithmic, hypergeometric functions and the Lerch phi transcendent in terms of zeta functions and sums of alternating harmonic numbers.

1 Introduction and Preliminaries

Let $\mathbb{N} := \{1, 2, 3, \cdots\}$ be the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, also let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the sets of real and complex numbers. In this paper we develop identities, new families of closed form representations of alternating harmonic numbers and reciprocal binomial coefficients, including integral representations, of the form:

$$\int_0^1 \frac{x \ln^{2m-1}(x)}{1-x}\, {}_2F_1\left[\begin{matrix} 1, 2; \\ 2+k; \end{matrix}\ -x\right] \mathrm{d}x, \qquad (k, m \in \mathbb{N}), \tag{1}$$

where ${}_2F_1\left[\begin{matrix} \cdot, \cdot; \\ \cdot; \end{matrix}\ z\right]$ is the classical generalized hypergeometric function. Some specific cases of similar integrals of the form (1) have been given by [11]. We also investigate integrals of the form

$$\int_0^1 \frac{x \ln^{2m-1}(x)}{1-x}\, \Phi(-x, 1, 1+r)\, \mathrm{d}x$$

and show that these integrals may be expressed in terms of a linear rational combination of zeta functions and known constants.

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Here $\Phi(z,t,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t}$ is the Lerch transcendent defined for $|z| < 1$ and $\Re(a) > 0$ and satisfies the recurrence

$$\Phi(z,t,a) = z\Phi(z,t,a+1) + a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function at $z = 1$,

$$\Phi(1,t,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}$$

and the Polylogarithm, or de Jonquière's function, when $a = 1$,

$$\mathrm{Li}_t(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^t}, t \in \mathbb{C} \text{ when } |z| < 1; \Re(t) > 1 \text{ when } |z| = 1.$$

Moreover

$$\int_0^1 \frac{\mathrm{Li}_t(px)}{x}\,\mathrm{d}x = \begin{cases} \zeta(1+t), \text{ for } p = 1 \\ (2^{-r}-1)\zeta(1+t), \text{ for } p = -1 \end{cases}.$$

A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda, \mu \in \mathbb{C})$ is defined, in terms of the familiar gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} \qquad (\lambda, \mu \in \mathbb{C}),$$

which, in the special case when $\mu = n, n \in \mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n \in \mathbb{N}),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is the Pochhammer symbol. Let, for $n \in \mathbb{N}$,

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1), \quad (H_0 := 0) \tag{2}$$

be the n-th harmonic number. Here γ denotes the Euler-Mascheroni constant and $\psi(z)$ is the Psi (or Digamma) function defined by

$$\psi(z) := \frac{\mathrm{d}}{\mathrm{d}z}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t.$$

An unusual, but intriguing representation, of the n-th harmonic number, has recently been given by [8], as

$$H_n = \pi \int_0^1 \left(x - \frac{1}{2}\right)\left(\frac{\cos\left(\frac{(4n+1)\pi x}{2}\right) - \cos\left(\frac{\pi x}{2}\right)}{\sin\left(\frac{\pi x}{2}\right)}\right)\mathrm{d}x.$$

Choi [3] has also given the definition, in terms of log-sine functions

$$\begin{aligned} H_n &= -4n \int_0^{\frac{\pi}{2}} \ln(\sin x) \sin x (\cos x)^{2n-1}\, \mathrm{d}x \\ &= -4n \int_0^{\frac{\pi}{2}} \ln(\cos x) \cos x (\sin x)^{2n-1}\, \mathrm{d}x. \end{aligned}$$

A *generalized harmonic number* $H_n^{(m)}$ of order m is defined, as follows:

$$H_n^{(m)} := \sum_{r=1}^{n} \frac{1}{r^m} \quad (m, n \in \mathbb{N}) \qquad \text{and} \qquad H_0^{(m)} := 0 \quad (m \in \mathbb{N}),$$

in terms of integral representations we have the result

$$H_n^{(m+1)} = \frac{(-1)^m}{m!} \int_0^1 \frac{(\ln x)^m (1 - x^n)}{1 - x}\, \mathrm{d}x. \tag{3}$$

In the case of *non-integer* values of n such as (for example) a value $\rho \in \mathbb{R}$, the generalized harmonic numbers $H_\rho^{(m+1)}$ may be defined, in terms of the Polygamma functions

$$\psi^{(n)}(z) := \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\psi(z)\} = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\{\log \Gamma(z)\} \qquad (n \in \mathbb{N}_0),$$

by

$$\begin{gathered} H_\rho^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\, \psi^{(m)}(\rho + 1) \\ (\rho \in \mathbb{R} \setminus \{-1, -2, -3, \cdots\}\,;\ m \in \mathbb{N}), \end{gathered} \tag{4}$$

where $\zeta(z)$ is the *Riemann zeta function*. Whenever we encounter harmonic numbers of the form $H_\rho^{(m)}$ at admissible real values of ρ, they may be evaluated by means of this known relation (4). In the exceptional case of (4) when $m = 0$, we may define $H_\rho^{(1)}$ by

$$H_\rho^{(1)} = H_\rho = \gamma + \psi(\rho + 1) \qquad \big(\rho \in \mathbb{R} \setminus \{-1, -2, -3, \dots\}\big).$$

In the case of non integer values of the argument $z = \frac{r}{q}$, we may write the generalized harmonic numbers, $H_z^{(\alpha+1)}$, in terms of polygamma functions

$$H_{\frac{r}{q}}^{(\alpha+1)} = \zeta(\alpha + 1) + \frac{(-1)^\alpha}{\alpha!} \psi^{(\alpha)}(\frac{r}{q} + 1), \quad \frac{r}{q} \neq \{-1, -2, -3, \dots\},$$

where $\zeta(z)$ is the zeta function. The evaluation of the polygamma function $\psi^{(\alpha)}(\dfrac{r}{a})$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [14], or Choi and Cvijovic [4] in terms of the Polylogarithmic or other special functions. Some specific values are listed in the books [18], [25] and [26]. Let us define the alternating zeta function

$$\bar{\zeta}(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = (1 - 2^{1-z})\zeta(z)$$

with $\bar{\zeta}(1) = \ln 2$, and also let

$$S^{+-}_{p,q} := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(p)}}{n^q}.$$

In the case where p and q are both positive integers and $p+q$ is an odd integer, Flajolet and Salvy [12] gave the identity:

$$\begin{aligned} 2S^{+-}_{p,q} &= \big(1-(-1)^p\big)\zeta(p)\bar{\zeta}(q) + 2\sum_{i+2k=q} \binom{p+i-1}{p-1} \zeta(p+i)\bar{\zeta}(2k) \\ &\quad + \bar{\zeta}(p+q) - 2\sum_{j+2k=p} \binom{q+j-1}{q-1} (-1)^j \bar{\zeta}(q+j)\bar{\zeta}(2k), \end{aligned} \tag{5}$$

where $\bar{\zeta}(0) = \frac{1}{2}, \bar{\zeta}(1) = \ln 2$, $\zeta(1) = 0$ and $\zeta(0) = -\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. Some results for sums of alternating harmonic numbers may be seen in the works [1], [2], [5], [6], [7], [9], [10], [12], [15], [16], [17], [20], [21], [19], [22], [23], [27], [28], [29] and [30] and references therein.

The following lemma will be useful in the development of the main theorems.

Lemma 1. *Let $r, p \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{r} \frac{(-1)^j}{j^p} = \frac{1}{2^p}\left(H^{(p)}_{[\frac{r}{2}]} + H^{(p)}_{[\frac{r-1}{2}]} \right) - H^{(p)}_{2[\frac{r+1}{2}]-1}, \tag{6}$$

where $[x]$ is the integer part of x. The following identities hold. For $0 < t \le 1$

$$t \ln^2\left(\frac{1+t}{t}\right) = 2\sum_{n=1}^{\infty} \frac{(-t)^{n+1} H_n}{n+1}$$

and when $t = 1$,

$$\ln^2 2 = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+1} = \zeta(2) - 2\mathrm{Li}_2\left(\frac{1}{2}\right). \tag{7}$$

Proof. The proof is given in the paper [24]. □

Lemma 2. *The following identity holds: for $m \in \mathbb{N}$,*

$$\begin{aligned} M(m,0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2m)}}{n} &= m\zeta(2m+1) - \bar{\zeta}(2m)\ln 2 \\ &\quad - \sum_{j=1}^{m-1} \bar{\zeta}(2j+1)\bar{\zeta}(2m-2j) \end{aligned} \tag{8}$$

$$= \frac{1}{(2m-1)!} \int_0^1 \frac{\ln^{2m-1} x}{1-x} \big(\ln(1+x) - \ln 2\big)\, \mathrm{d}x. \tag{9}$$

Proof. From (5) we choose $q = 1$ and $p = 2m$, $m \in \mathbb{N}$. Then (8) follows. Next from the definition (3),

$$\begin{aligned} M(m,0) &= -\frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1} x}{1-x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1-x^n)}{n}\,\mathrm{d}x \\ &= \frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1} x}{1-x}\big(\ln(1+x)-\ln 2\big)\,\mathrm{d}x. \end{aligned}$$

□

Lemma 3. *Let $m \in \mathbb{N}$. Then*

$$\begin{aligned} M(m,1) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n+1} &= \bar{\zeta}(2m+1) - m\zeta(2m+1) \\ &\quad + \bar{\zeta}(2m)\ln 2 + \sum_{j=1}^{m-1}\bar{\zeta}(2j+1)\bar{\zeta}(2m-2j) \qquad (10) \\ &= \frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1} x}{1-x}\left(\ln 2 - \frac{\ln(1+x)}{x}\right)\mathrm{d}x\,. \qquad (11) \end{aligned}$$

Proof. From the left hand side of (10) and by a change of counter

$$M(m,1) = \sum_{n=1}^{\infty}\frac{(-1)^n H_{n-1}^{(2m)}}{n} = -M(m,0) + \bar{\zeta}(2m+1)\,,$$

substituting for $M(m,0)$ we obtain the right hand side of (10). The integral (11) is obtained by the same method as that used in Lemma 2. □

Lemma 4. *Let $r \geq 2$ be a positive integer, $m \in \mathbb{N}$ and define*

$$M(m,r) := \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n+r}.$$

We have the recurrence relation

$$\begin{aligned} M(m,r) + M(m,r-1) &= \frac{(-1)^r}{(r-1)^{2m}}\Big(H_{r-1} - H_{[\frac{r-1}{2}]} - \big(1+(-1)^r\big)\ln 2\Big) \\ &\quad + \sum_{j=2}^{2m}\frac{(-1)^j}{(r-1)^{2m+1-j}}\bar{\zeta}(j) \end{aligned} \qquad (12)$$

with solution

$$\begin{aligned} M(m,r) &= (-1)^{r+1}M(m,1) - (-1)^r\ln 2\sum_{\rho=1}^{r-1}\frac{(-1)^{\rho+1}}{\rho^{2m}} \\ &\quad + (-1)^r\sum_{\rho=1}^{r-1}\frac{1}{\rho^{2m}}(H_\rho - H_{[\frac{\rho}{2}]} - \ln 2) + (-1)^r\sum_{\rho=1}^{r-1}\sum_{j=2}^{2m}\frac{(-1)^{\rho+1+j}}{\rho^{2m+1-j}}\bar{\zeta}(j), \end{aligned} \qquad (13)$$

with $M(m,0)$ and $M(m,1)$ given by (8) and (10) respectively.

Proof. By a change of counter

$$
\begin{aligned}
M(m,r) &= \sum_{n=1}^{\infty} \frac{(-1)^n H_{n-1}^{(2m)}}{n+r-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+r-1}\Big(H_n^{(2m)} - \frac{1}{n^{2m}}\Big) \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2m)}}{n+r-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2m}(n+r-1)} \\
&= -M(m,r-1) + \frac{1}{(r-1)^{2m}} \sum_{n=1}^{\infty} \frac{(-1)^{n+r}}{n} - \frac{1}{(r-1)^{2m}} \sum_{n=1}^{r-1} \frac{(-1)^{n+r}}{n} \\
&\quad - \frac{1}{(r-1)^{2m}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{j=2}^{2m} \frac{(-1)^j}{(r-1)^{2m+1-j}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^j}.
\end{aligned}
$$

From Lemma 1 and using the known results,

$$
\begin{aligned}
M(m,r) = -M(m,r-1) &- \frac{(-1)^r}{(r-1)^{2m}}\Big(\ln 2 + H_{\left[\frac{r-1}{2}\right]} - H_{r-1}\Big) \\
&- \frac{\ln 2}{(r-1)^{2m}} + \sum_{j=2}^{2m} \frac{(-1)^j}{(r-1)^{2m+1-j}} \bar{\zeta}(j).
\end{aligned}
\tag{14}
$$

From (14) we have the recurrence relation

$$
\begin{aligned}
M(m,r) + M(m,r-1) = &\frac{(-1)^r}{(r-1)^{2m}} (H_{r-1} - H_{\left[\frac{r-1}{2}\right]} - (1+(-1)^r) \ln 2) \\
&+ \sum_{j=2}^{2m} \frac{(-1)^j}{(r-1)^{2m+1-j}} \bar{\zeta}(j)\,,
\end{aligned}
$$

for $r \geq 2$, with $M(m,0)$ given by (8) and $M(m,1)$ given by (10). The recurrence relation is solved by the subsequent reduction of the

$$
M(m,r), M(m,r-1), M(m,r-2), \ldots, M(m,1)
$$

terms, finally arriving at the relation (13). □

It is of some interest to note that $M(m,r)$ may be expanded in a slightly different way so that it gives rise to another unexpected harmonic series identity. This is pursued in the next lemma.

Lemma 5. *For a positive integer $r \geq 2$, and $m \in \mathbb{N}$ we have the identity*

$$
\begin{aligned}
\Upsilon(m,r) &= \sum_{n=1}^{\infty} \frac{H_{2n}^{(2m)}}{(2n+r-1)(2n+r)} \\
&= M(m,r) - \frac{1}{2(r-1)^{2m}} H_{\frac{r-1}{2}} + \sum_{j=2}^{2m} \frac{(-1)^j}{2^j (r-1)^{2m+1-j}} \zeta(j)\,,
\end{aligned}
\tag{15}
$$

with $M(m,r)$ given by (13). For $r=1$,

$$\Upsilon(m,1)=\sum_{n=1}^{\infty}\frac{H_{2n}^{(2m)}}{2n(2n+1)}=\bar{\zeta}(2m+1)+(2^{-1-2m}-m)\zeta(2m+1)+\bar{\zeta}(2m)\ln 2+\sum_{j=1}^{m-1}\bar{\zeta}(2j+1)\bar{\zeta}(2m-2j)\,, \tag{16}$$

and for $r=0$

$$\Upsilon(m,0)=\sum_{n=1}^{\infty}\frac{H_{2n}^{(2m)}}{2n(2n-1)}=m\zeta(2m+1)-\bar{\zeta}(2m)\ln 2+\ln 2-\sum_{j=1}^{m-1}\bar{\zeta}(2j+1)\bar{\zeta}(2m-2j)-\sum_{j=2}^{2m}2^{-j}\zeta(j)\,. \tag{17}$$

Proof. From

$$M(m,r)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n+r}=\sum_{n=1}^{\infty}\frac{H_{2n}^{(2m)}}{(2n+r-1)(2n+r)}-\sum_{n=1}^{\infty}\frac{1}{(2n+r-1)(2n)^{2m}}.$$

Hence

$$\Upsilon(m,r):=\sum_{n=1}^{\infty}\frac{H_{2n}^{(2m)}}{(2n+r-1)(2n+r)}=M(m,r)+\sum_{n=1}^{\infty}\frac{1}{(2n+r-1)(2n)^{2m}},$$

then,

$$\begin{aligned}\Upsilon(m,r)&=M(m,r)+\frac{1}{(r-1)^{2m}}\sum_{n=1}^{\infty}\Big(\frac{1}{2n+r-1}-\frac{1}{2n}\Big)\\&\quad+\sum_{j=2}^{2m}\frac{(-1)^j}{2^j(r-1)^{2m+1-j}}\sum_{n=1}^{\infty}\frac{1}{n^j}\\&=M(m,r)-\frac{1}{2(r-1)^{2m}}H_{\frac{r-1}{2}}\\&\quad+\sum_{j=2}^{2m}\frac{(-1)^j}{2^j(r-1)^{2m+1-j}}\zeta(j)\,.\end{aligned}$$

For $r=1$

$$\Upsilon(m,1)=\sum_{n=1}^{\infty}\frac{H_{2n}^{(2m)}}{2n(2n+1)}=M(m,1)+\frac{1}{2^{2m+1}}\zeta(2m+1)$$

and substituting for $M(m,1)$, we obtain (16). For $r=0$

$$\Upsilon(m,0)=\sum_{n=1}^{\infty}\frac{H_{2n}^{(2m)}}{2n(2n-1)}=M(m,0)+\ln 2-\sum_{j=2}^{2m}2^{-j}\zeta(j)$$

and substituting for $M(m,0)$, we obtain (17). □

We develop some integral identities for Lemma 4 and Lemma 5, namely:

Lemma 6. *For* $r\in\mathbb{N}_0\setminus\{0,1\}$,

$$\frac{1}{(2m-1)!}\int_0^1\frac{x\ln^{2m-1}x}{1-x}\Phi(-x,1,1+r)\,\mathrm{d}x=M(m,r)-\zeta(2m)\Phi(-1,1,1+r) \quad (18)$$

where $M(m,r)$ *is given by* (13).

Proof. From $M(m,r):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n+r}$ and by the use of the integral representation (3)

$$\begin{aligned}M(m,r)&=\frac{1}{(2m-1)!}\int_0^1\frac{\ln^{2m-1}x}{1-x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1-x^n)}{n+r}\,\mathrm{d}x\\&=\frac{1}{(2m-1)!}\int_0^1\frac{\ln^{2m-1}x}{1-x}\big(x\Phi(-x,1,1+r)-\Phi(-1,1,1+r)\big)\,\mathrm{d}x\\&=\frac{1}{(2m-1)!}\int_0^1\frac{x\ln^{2m-1}x}{1-x}\Phi(-x,1,1+r)\,\mathrm{d}x+\zeta(2m)\Phi(-1,1,1+r)\end{aligned}$$

and by re-arrangement we obtain (18). □

Remark 1. For the case $r=0$, we have

$$\frac{1}{(2m-1)!}\int_0^1\frac{\ln^{2m-1}x\ln(1+x)}{1-x}\,\mathrm{d}x=M(m,0)-\zeta(2m)\ln 2.$$

This identity could also be obtained from (9), since it is known that

$$-\int_0^1\frac{\ln^{2m-1}x}{1-x}\,\mathrm{d}x=(2m-1)!\,\mathrm{Li}_{2m}(1)=(2m-1)!\,\zeta(2m).$$

For $r=1$ we obtain

$$\frac{1}{(2m-1)!}\int_0^1\frac{\ln^{2m-1}x\ln(1+x)}{x(1-x)}\,\mathrm{d}x=-M(m,1)-\zeta(2m)\ln 2,$$

for $m\geq 2$. An obvious result of the preceding two integrals yields the delightful identity

$$\begin{aligned}\frac{1}{(2m-1)!}\int_0^1\frac{\ln^{2m-1}x\ln(1+x)}{x}\,\mathrm{d}x&=-\bar{\zeta}(2m+1) \quad (19)\\&=-M(m,0)-M(m,1)=-\sum_{n=1}^{\infty}(-1)^{n+1}H_n^{(2m)}\Big(\frac{1}{n}+\frac{1}{n+1}\Big)\end{aligned}$$

The Wolfram on-line integrator yields no solution to these general integrals.

Lemma 7. *For $r \in \mathbb{N}_0 \setminus \{0,1\}$,*

$$\frac{1}{(2m-1)!}\int_0^1 \frac{x^2\ln^{2m-1}x}{2(1-x)}\left(\Phi\left(x^2,1,\frac{1+r}{2}\right)-\Phi\left(x^2,1,\frac{2+r}{2}\right)\right)\mathrm{d}x$$
$$=\Upsilon(m,r)+\frac{1}{2}\zeta(2m)\left(H_{\frac{r-1}{2}}-H_{\frac{r}{2}}\right) \quad (20)$$

where $\Upsilon(m,r)$ is given by (15) and $\Phi(x^2,1,\frac{1+r}{2})$ is the Lerch transcendent. For $r=0$

$$\frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1}x}{1-x}\left(x\tanh^{-1}x+\frac{1}{2}\ln(1-x^2)\right)\mathrm{d}x=\Upsilon(m,0)-\zeta(2m)\ln 2.$$

For $r=1$

$$\frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1}x}{1-x}\left(\frac{\tanh^{-1}x}{x}+\frac{1}{2}\ln(1-x^2)\right)\mathrm{d}x=-\Upsilon(m,1)-\zeta(2m)\ln 2.$$

By addition we have the result

$$\frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1}x}{1-x}\left((x+\frac{1}{x})\tanh^{-1}x+\ln(1-x^2)\right)\mathrm{d}x$$
$$=\Upsilon(m,0)-\Upsilon(m,1)-2\zeta(2m)\ln 2$$
$$=M(m,0)-M(m,1)-\left(\frac{1}{2^{2m+1}}+\ln 2\right)\zeta(2m)-\ln 2-\sum_{j=2}^{2m}2^{-j}\zeta(j)$$

and by further simplification

$$-\frac{1}{(2m-1)!}\int_0^1 \frac{(1+x)\ln^{2m-1}x}{x}\tanh^{-1}x\,\mathrm{d}x \quad (21)$$
$$=\Upsilon(m,1)+\Upsilon(m,0)$$
$$=\zeta(2m+1)+\ln 2-\sum_{j=2}^{2m+1}2^{-j}\zeta(j)$$
$$=\sum_{n=0}^{\infty}\frac{1}{(2\left[\frac{n}{2}\right]+1)(n+1)^{2m}}. \quad (22)$$

It may also be shown that the alternating companion to (22) is,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\left[\frac{n}{2}\right]+1)(n+1)^{2m}}=-\frac{1}{(2m-1)!}\int_0^1\frac{(1-x)\ln^{2m-1}x}{x}\tanh^{-1}x\,\mathrm{d}x$$
$$=\bar{\zeta}(2m+1)-\ln 2+\sum_{j=2}^{2m+1}2^{-j}\zeta(j),$$

where $[z]$ is the integer part of z.

Proof. Follow the same pattern as used in Lemma 6. The new sum (22) can be obtained from the Taylor series expansion of (21) in the following way:

$$\begin{aligned} -\frac{1}{(2m-1)!}\int_0^1 \frac{(1+x)\ln^{2m-1}x}{x}\tanh^{-1}x\,\mathrm{d}x \\ &= -\frac{1}{(2m-1)!}\sum_{n=0}^{\infty}\frac{1}{\left(2\left[\frac{n}{2}\right]+1\right)}\int_0^1 x^n\ln^{2m-1}x\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty}\frac{1}{\left(2\left[\frac{n}{2}\right]+1\right)(n+1)^{2m}} \\ &= \sum_{n=1}^{\infty}\frac{2H_{2n}^{(2m)}}{(2n-1)(2n+1)}. \end{aligned}$$

For $m=3$, we have

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{2H_{2n}^{(6)}}{(2n-1)(2n+1)} &= \sum_{n=0}^{\infty}\frac{1}{\left(2\left[\frac{n}{2}\right]+1\right)(n+1)^{6}} \\ &= \frac{\zeta(2)}{4}+\frac{\zeta(3)}{8}+\frac{\zeta(4)}{16}+\frac{\zeta(5)}{32}+\frac{\zeta(6)}{64}-\frac{127\zeta(7)}{128}-\ln 2. \end{aligned}$$

Mathematica cannot sum either of these two series. □

The next few theorems relate the main results of this investigation, namely the integral and closed form representation of integrals of the type (1).

2 Integral and Closed form identities

In this section we investigate integral identities in terms of closed form representations of infinite series of harmonic numbers of order $2m$ and inverse binomial coefficients. First we indicate the closed form representation of

$$X(m,k,p)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n^p\binom{n+k}{k}} \tag{23}$$

for the two cases $(k,p)=(k,1)$, $(k,0)$ and $k\in\mathbb{N}$.

Theorem 1. *Let $k\in\mathbb{N}$. Then from (23) with $p=1$ we have,*

$$X(m,k,1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n(\binom{n+k}{k}}=M(m,0)-\sum_{r=1}^{k}(-1)^{1+r}\binom{k}{r}M(m,r), \tag{24}$$

where $M(m,r)$ is given by (13).

Proof. Consider the expansion

$$\begin{aligned} X(m,k,1) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n\binom{n+k}{k}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}k!H_n^{(2m)}}{n(n+1)_k} \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}k!H_n^{(2m)}\left(\frac{\Omega}{n}+\sum_{r=1}^{k}\frac{\Lambda_r}{n+r}\right), \end{aligned}$$

where

$$\Lambda_r = \lim_{n\to -r}\left\{\frac{n+r}{n\prod_{r=1}^{k} n+r}\right\} = -\frac{(-1)^{r+1}}{k!}\binom{k}{r}, \quad \Omega = \frac{1}{k!}. \tag{25}$$

We can now express

$$\begin{aligned} X(m,k,1) &= \sum_{n=1}^{\infty}(-1)^{n+1}H_n^{(2m)}\Big(\sum_{r=1}^{k} -\frac{(-1)^{r+1}}{n+r}\binom{k}{r} + \frac{1}{n}\Big) \\ &= \sum_{n=1}^{\infty}\frac{H_n^{(2m)}}{n} - \sum_{r=1}^{k}(-1)^{1+r}\binom{k}{r}\sum_{n=1}^{\infty}\frac{H_n^{(2m)}}{n+r}\,. \end{aligned} \tag{26}$$

From (13) we have $M(m,r)$, hence substituting into (26), (24) follows. □

The other case of $X(m,k,0)$ can be evaluated in a similar fashion. We list the result in the next Theorem.

Theorem 2. *Under the assumptions of Theorem 1, we have for $p=0$,*

$$\begin{aligned} X(m,k,0) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{\binom{n+k}{k}} \\ &= \sum_{r=1}^{k}(-1)^{1+r}r\binom{k}{r}M(m,r)\,, \end{aligned} \tag{27}$$

and where $M(m,r)$ is given by (13).

Proof. Same procedure as that used in Theorem 1 can be used here. □

The following integral identities can be exactly evaluated by using the alternating harmonic number sums in Theorems 1 and 2. The following integral identities are the main results of this paper.

Theorem 3. *Let $m,k\in\mathbb{N}$. Then we have*

$$\begin{aligned} \frac{1}{(1+k)(2m-1)!}\int_0^1 \frac{x\ln^{2m-1}x}{1-x}\,{}_2F_1\left[\begin{matrix}1,1;\\2+k;\end{matrix}\; -x\right]\mathrm{d}x \\ = X(m,k,1) - \frac{\zeta(2m)}{1+k}\,{}_2F_1\left[\begin{matrix}1,1;\\2+k;\end{matrix}\; -1\right], \end{aligned} \tag{28}$$

where $X(m,k,1)$ is given by (24),

$$\begin{aligned} \frac{1}{(1+k)(2m-1)!}\int_0^1 \frac{x\ln^{2m-1}x}{1-x}\,{}_2F_1\left[\begin{matrix}1,2;\\2+k;\end{matrix}\; -x\right]\mathrm{d}x \\ = X(m,k,0) - \frac{\zeta(2m)}{1+k}\,{}_2F_1\left[\begin{matrix}1,2;\\2+k;\end{matrix}\; -1\right], \end{aligned} \tag{29}$$

where $X(m,k,0)$ is given by (27).

Proof. From the identity (3), we can write

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2m)}}{n\binom{n+k}{k}}=-\frac{1}{(2m-1)!}\int_0^1\frac{\ln^{2m-1}x}{1-x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1-x^n)}{n\binom{n+k}{k}}\,\mathrm{d}x$$
$$=\frac{1}{(1+k)(2m-1)!}\int_0^1\frac{\ln^{2m-1}x}{1-x}\left(\begin{array}{c} x\,{}_2F_1\left[\begin{array}{c}1,1;\\2+k;\end{array}-x\right]\\ \\ -{}_2F_1\left[\begin{array}{c}1,1;\\2+k;\end{array}-1\right]\end{array}\right)\mathrm{d}x\,.$$

where ${}_2F_1\left[\begin{array}{c}\cdot,\cdot;\\\cdot;\end{array}\cdot\right]$ is the generalized hypergeometric function. Since

$$\frac{1}{(2m-1)!}\int_0^1\frac{\ln^{2m-1}x}{1-x}\,{}_2F_1\left[\begin{array}{c}1,1;\\2+k;\end{array}-1\right]\mathrm{d}x=-{}_2F_1\left[\begin{array}{c}1,1;\\2+k;\end{array}-1\right]\zeta(2m),$$

then by re-arrangement we obtain the integral identity (28). The identity 29 follows in a similar way. □

Remark 2. From Theorem 3 we detail the following two examples. From (28) with $\{m,k\}=\{3,4\}$, we obtain

$$\frac{1}{720}\int_0^1\frac{x\ln^5 x}{1-x}\,{}_2F_1\left[\begin{array}{c}1,1;\\6;\end{array}-x\right]\mathrm{d}x=X(3,4,1)-\frac{\zeta(6)}{5}\,{}_2F_1\left[\begin{array}{c}1,1;\\6;\end{array}-1\right];$$

$$\frac{1}{720}\int_0^1\frac{\ln^5 x}{1-x}\Big(\frac{12(1+x)^4}{x^4}\ln(1+x)-\frac{12+42x+52x^2+25x^3}{x^3}\Big)\,\mathrm{d}x$$
$$=\frac{3090551}{279936}+\frac{84353\zeta(2)}{15552}+\frac{15743\zeta(4)}{128}+\frac{3739\zeta(6)}{192}-\frac{16040\ln 2}{729}$$
$$-\frac{63\ln 2\zeta(6)}{2}-\frac{13867\zeta(3)}{1728}-\frac{21\zeta(3)\zeta(4)}{4}-\frac{1775\zeta(5)}{192}$$
$$-\frac{15\zeta(2)\zeta(5)}{2}+\frac{2127\zeta(7)}{64}.$$

From (29) with $\{m,k\}=\{3,4\}$, we obtain

$$\frac{1}{720}\int_0^1\frac{x\ln^5 x}{1-x}\,{}_2F_1\left[\begin{array}{c}1,2;\\6;\end{array}-x\right]\mathrm{d}x=X(3,4,0)-\frac{\zeta(6)}{5}\,{}_2F_1\left[\begin{array}{c}1,2;\\6;\end{array}-1\right]$$
$$=-\frac{492155}{17496}-\frac{13373\zeta(2)}{972}-\frac{2471\zeta(4)}{108}-43\zeta(6)+\frac{40832\ln 2}{729}+63\ln 2\zeta(6)$$
$$+\frac{2191\zeta(3)}{108}+21\zeta(3)\zeta(4)+\frac{275\zeta(5)}{12}+15\zeta(2)\zeta(5)-\frac{129\zeta(7)}{2}.$$

The Wolfram on-line integrator, yields no solution to these general slow converging integrals.

Remark 3. It appears that, for $r \in \mathbb{N}_0$, $m \in \mathbb{N}$

$$K(m,r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2m)}}{(n+r)^2}$$

may not have a closed form solution, in terms of some common special functions. Remarkably, however, the sum of two consecutive terms of $K(m,r)$ does have a closed form solution, this result is pursued in the next Lemma.

Lemma 8. *For $r \in \mathbb{N}$, $m \in \mathbb{N}$*

$$\begin{aligned} K(m,r) + K(m,r+1) &= \sum_{n=1}^{\infty} (-1)^{n+1} H_n^{(2m)} \Big(\frac{1}{(n+r)^2} + \frac{1}{(n+r+1)^2} \Big) \\ &= \frac{1}{4r^{2m}} \Big(H_{\frac{r}{2}}^{(2)} - H_{\frac{r-1}{2}}^{(2)} \Big) \\ &\quad + \frac{2m(-1)^r}{r^{2m+1}} \Big(\big(1-(-1)^r\big) \ln 2 + H_{[\frac{r}{2}]} - H_r \Big) \\ &\quad + \sum_{j=2}^{2m} \frac{(-1)^j (2m+1-j)}{r^{2m+2-j}} \bar{\zeta}(j) \end{aligned} \tag{30}$$

and for $r = 0$,

$$\begin{aligned} K(m,0) + K(m,1) &= \sum_{n=1}^{\infty} (-1)^{n+1} H_n^{(2m)} \Big(\frac{1}{n^2} + \frac{1}{(n+1)^2} \Big) \\ &= \bar{\zeta}(2m+2) \qquad (31) \\ &= \frac{1}{(2m-1)!} \int_0^1 \frac{\ln^{2m-1} x \, \mathrm{Li}_2(-x)}{x} \, \mathrm{d}x \, . \qquad (32) \end{aligned}$$

Proof.

$$K(m,r) + K(m,r+1) = \sum_{n=1}^{\infty} (-1)^{n+1} H_n^{(2m)} \Big(\frac{1}{(n+r)^2} + \frac{1}{(n+r+1)^2} \Big)$$

and by a change of counter in the second sum, after simplification we obtain

$$\begin{aligned} K(m,r) + K(m,r+1) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2m}(n+r)^2} \\ &= \frac{1}{r^{2m}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+r)^2} + \frac{2m}{r^{2m+1}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+r} \\ &\quad + \sum_{j=1}^{2m} \frac{(-1)^j (2m+1-j)}{r^{2m+2-j}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^j} \\ &= \frac{1}{4r^{2m}} \Big(H_{\frac{r}{2}}^{(2)} - H_{\frac{r-1}{2}}^{(2)} \Big) + \frac{2m(-1)^r}{r^{2m+1}} \Big(\ln 2 + H_{[\frac{r}{2}]} - H_r \Big) - \frac{2m}{r^{2m+1}} \ln 2 \\ &\quad + \sum_{j=2}^{2m} \frac{(-1)^j (2m+1-j)}{r^{2m+2-j}} \bar{\zeta}(j) \end{aligned}$$

and upon simplification we obtain (30). For the case $r = 0$, (31) and (32) follow directly. □

Remark 4. The identity (32) with (31) is obtained using the definition (3). Also considering the Taylor expansion of (32) we notice

$$\begin{aligned}\frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1}x\,\mathrm{Li}_2(-x)}{x}\,\mathrm{d}x \\ &= \frac{1}{(2m-1)!}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{(j+1)^2}\int_0^1 x^j\ln^{2m-1}x\,\mathrm{d}x \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)^{2m+2}} = \bar{\zeta}(2m+2)\,.\end{aligned} \tag{33}$$

This gives the even zeta functions of alternating type and (19) gives the odd zeta functions of alternating type. Moreover

$$\lim_{m\to\infty}\left[\frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1}x\,\mathrm{Li}_2(-x)}{x}\,\mathrm{d}x\right] = 1$$

From (30), for $\{m,r\} = \{3,3\}$,

$$K(3,3)+K(3,4) = \frac{31\zeta(6)}{288} - \frac{5\zeta(5)}{72} + \frac{7\zeta(4)}{216} - \frac{\zeta(3)}{81} + \frac{2\zeta(2)}{729} - \frac{4\ln 2}{729} + \frac{91}{26244}.$$

It is possible to obtain a nice generalization of (33) as,

$$\frac{1}{(2m-1)!}\int_0^1 \frac{\ln^{2m-1}x\,\mathrm{Li}_p(-x)}{x}\,\mathrm{d}x = \bar{\zeta}(2m+p)$$

for $p \in \mathbb{N}$, and is analogous to the identity (see [13])

$$\int_0^1 \frac{\ln^m x\,\mathrm{Li}_p(x)}{x}\,\mathrm{d}x = (-1)^m m!\,\zeta(m+p+1)$$

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