Journal of King Saud University – Science (2017) 29, 263–273

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

Some q-analogues of Hermite–Hadamard inequality of functions of two variables on finite rectangles in the plane

M.A. Latif [a,*], **S.S. Dragomir** [b,a], **E. Momoniat** [a]

[a] School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
[b] College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

Received 26 April 2016; accepted 3 July 2016
Available online 3 August 2016

KEYWORDS
Quantum calculus;
Hermite–Hadamard inequality;
q_1q_2-Hölder inequality;
q_1q_2-Partial derivatives

Abstract Preliminaries of q-calculus for functions of two variables over finite rectangles in the plane are introduced. Some q-analogues of the famous Hermite–Hadamard inequality of functions of two variables defined on finite rectangles in the plane are presented. A q_1q_2-Hölder inequality for functions of two variables over finite rectangles is also established to provide some quantum estimates of trapezoidal type inequality of functions of two variables whose q_1q_2-partial derivatives in absolute value with certain powers satisfy the criteria of convexity on co-ordinates.

1. Introduction

Quantum calculus or q-calculus is the study of calculus without limits. In the eighteenth century, Euler initiated the study of q-calculus by introducing the number q in Newton's work of infinite series. Many remarkable results such as Jacobi's triple product identity and the theory of q-hypergeometric functions were obtained in the nineteenth century. In early twentieth century, Jackson (1910) had started a symmetric study of q-calculus and introduced q-definite integrals. The subject of quantum calculus has numerous applications in different areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, mechanics and in theory of relativity. This subject has received exceptional consideration by many researchers and hence it has appeared as an interdisciplinary subject between mathematics and physics. Interested readers are referred to Ernst (2012), Gauchman (2004) and Kac and Cheung (2002) for some recent developments in the theory of quantum calculus and theory of inequalities in quantum calculus.

Theory of inequalities and theory of convex functions have been observed to be profoundly dependent on each other and consequently a vast literature on inequalities has been

* Corresponding author.
E-mail addresses: m_amer_latif@hotmail.com (M.A. Latif), sever.dragomir@vu.edu.au (S.S. Dragomir), ebrahim.momoniat@gmail.com (E. Momoniat).
Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2016.07.001
1018-3647

produced by a number of researchers using convex functions, see Dragomir and Pearce (2000), Dragomir and Agarwal (1998) and Ion (2007). The Hermite–Hadamard inequalities are extensively studied during past three decades and the following inequalities, known as Hermite–Hadamard inequalities, provide a necessary and sufficient condition for a continuous function $f: I \subset \mathbb{R} \to \mathbb{R}$ to be convex on $[a, b]$, where $a, b \in I$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For further reading on integral inequalities using classical convexity and other important inequalities we refer our reader to Sudsutad et al. (2015) and Tariboon and Ntouyas (2014). Most recently, Noor et al. (2015a,b,c) and Zhuang et al. (2016) have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analogue of Hermite–Hadamard inequality through q-differentiable convex and q-differentiable quasi-convex functions. Ghany (2009, 2012, 2013a,b) gave integral representations of basic completely monotone functions, integral representations of basic completely alternating functions, q-derivative of basic hypergeometric series with respect to parameters and discussed some properties of the derivatives of basic hypergeometric series with respect to parameters. Motivated by the recent progress in the field of quantum calculus, our aim is to further develop this theory for functions of two variables and to provide some quantum analogues of Hermite–Hadamard inequality of functions of two variables over finite rectangles. At the next step, we will also provide some quantum estimates for the right part of the q-analogue of Hermite–Hadamard inequality of functions of two variables using convexity and quasi-convexity on co-ordinates of the absolute value of the q_1q_2-partial derivatives.

2. Preliminaries

The readers are referred to Tariboon and Ntouyas (2013), Sudsutad et al. (2015), Kac and Cheung (2002) and Ernst (2012) for some q-calculus essentials and inequalities over finite intervals.

We will also use the following definite q-integrals to prove our results.

Lemma 1 Sudsutad et al., 2015. *Let* $0 < q < 1$, *the following hold*

$$\Delta_q := \int_0^1 t|1-(1+q)t|\ {}_0d_qt = \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3},$$

$$\Psi_q := \int_0^1 (1-t)|1-(1+q)t|\ {}_0d_qt = \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}$$

and

$$\Phi_q := \int_0^1 |1-(1+q)t|\ {}_0d_qt = \frac{2q}{(1+q)^2}.$$

In what follows we introduce q-partial derivatives and definite q-integrals for functions of two variables.

Definition 1. Let $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables and $0 < q_1 < 1, 0 < q_2 < 1$, the partial q_1-derivatives, q_2-derivatives and q_1q_2-derivatives at $(x, y) \in [a, b] \times [c, d]$ can be defined as follows

$$\frac{{}_a\partial_{q_1}f(x,y)}{{}_a\partial_{q_1}x} = \frac{f(q_1x+(1-q_1)a, y) - f(x,y)}{(1-q_1)(x-a)}, \quad x \neq a$$

$$\frac{{}_c\partial_{q_2}f(x,y)}{{}_c\partial_{q_2}y} = \frac{f(x, q_2y+(1-q_2)c) - f(x,y)}{(1-q_2)(y-c)}, \quad y \neq c$$

and

$$\begin{aligned}\frac{{}_{a,c}\partial^2_{q_1,q_2}f(x,y)}{{}_a\partial_{q_1}x\,{}_c\partial_{q_2}y} = &\frac{1}{(1-q_1)(1-q_2)(y-c)(x-a)}\\ &\times [f(q_1x+(1-q_1)a, q_2y+(1-q_2)c)\\ &- f(q_1x+(1-q_1)a, y) - f(x, q_2y+(1-q_2)c)\\ &+ f(x,y)], \quad x \neq a,\ y \neq c.\end{aligned}$$

The function $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be partially q_1-, q_2- and q_1q_2-differentiable on $[a, b] \times [c, d]$ if $\frac{{}_a\partial_{q_1}f(x,y)}{{}_a\partial_{q_1}x}$, $\frac{{}_c\partial_{q_2}f(x,y)}{{}_c\partial_{q_2}y}$ and $\frac{{}_{a,c}\partial^2_{q_1,q_2}f(x,y)}{{}_a\partial_{q_1}x_c\ \partial_{q_2}y}$ exist for all $(x, y) \in [a, b] \times [c, d]$. We can similarly define higher order partial derivatives.

Definition 2. Suppose that $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Then the definite q_1q_2-integral on $[a, b] \times [c, d]$ is defined by

$$\begin{aligned}\int_c^y\int_a^x f(x,y)_ad_{q_1}x_cd_{q_2}y = &(x-a)(y-c)(1-q_1)(1-q_2)\\ &\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\left(q_1^nx+\left(1-q_1^n\right)a, q_2^my\right.\\ &\left.+\left(1-q_2^m\right)c\right)\end{aligned} \tag{2.1}$$

for $(x, y) \in [a, b] \times [c, d]$. It is clear from (2.1) that

$$\int_c^y\int_a^x f(x,y)_ad_{q_1}x_ad_{q_2}y = \int_a^x\int_c^y f(x,y)_cd_{q_2}y_ad_{q_1}x.$$

If $(x_1, y_1) \in (a, x) \times (c, y)$, then

$$\begin{aligned}&\int_{y_1}^y\int_{x_1}^x f(x,y)_ad_{q_1}x_ad_{q_2}y\\ &= \int_{y_1}^y\int_a^x f(x,y)_ad_{q_1}x_ad_{q_2}y - \int_{y_1}^y\int_a^{x_1} f(x,y)_ad_{q_1}x_ad_{q_2}y\\ &= \int_c^y\int_a^x f(x,y)_ad_{q_1}x_ad_{q_2}y - \int_c^{y_1}\int_a^x f(x,y)_ad_{q_1}x_ad_{q_2}y\\ &\quad - \int_c^y\int_a^{x_1} f(x,y)_ad_{q_1}x_ad_{q_2}y + \int_c^{y_1}\int_a^{x_1} f(x,y)_ad_{q_1}x_ad_{q_2}y.\end{aligned} \tag{2.2}$$

From (2.2), we also note that

$$\begin{aligned}\int_c^y\int_a^x f(x,y)_ad_{q_1}x_cd_{q_2}y &= \int_c^y\left(\int_a^x f(x,y)_ad_{q_1}x\right)\ {}_cd_{q_2}y\\ &= \int_a^x\left(\int_c^y f(x,y)_cd_{q_2}y\right)\ {}_ad_{q_1}x.\end{aligned}$$

Remark 1. It is easy to observe that the Definition 2 contains the Definition 2.3 in Tariboon and Ntouyas (2014) as special case when f is a function of single variable.

The following theorems hold for definite q_1q_2-double integrals.

Theorem 1. *Let* $f:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a continuous function. Then*

(1) $\frac{{}_{a,c}\partial^2_{q_1,q_2}}{{}_a\partial_{q_1}x_c\ \partial_{q_2}y}\int_c^y\int_a^x f(t,s)_a d_{q_1}t_c d_{q_2}s = f(x,y)$
(2) $\int_c^y\int_a^x \frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}t_c\ \partial_{q_2}s}\ {}_a d_{q_1}t_c d_{q_2}s = f(x,y)$
(3) $\int_{y_1}^y\int_{x_1}^x \frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}t_c\ \partial_{q_2}s}\ {}_a d_{q_1}t_c d_{q_2}s = f(x,y)-f(x,y_1)-f(x_1,y)+f(x_1,y_1),\quad (x_1,y_1)\in(a,x)\times(c,y).$

Proof

(1) By Definition 2 and the definition of partial q_1q_2-derivatives, we have

$$\begin{aligned}
&\frac{{}_{a,c}\partial^2_{q_1,q_2}}{{}_a\partial_{q_1}x\ {}_c\partial_{q_2}y}\Bigg[(x-a)(y-c)(1-q_1)(1-q_2)\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)\Bigg]\\
&=\frac{1}{(1-q_1)(1-q_2)(y-c)(x-a)}\\
&\quad\times\Bigg[q_1q_2(1-q_1)(1-q_2)(y-c)(x-a)\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\big(q_1^{n+1}x+(1-q_1^{n+1})a,q_2^{m+1}y+(1-q_2^{m+1})c\big)\\
&\quad-q_1(x-a)(y-c)(1-q_1)(1-q_2)\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\big(q_1^{n+1}x+(1-q_1^{n+1})a,q_2^{m}y+(1-q_2^{m})c\big)\\
&\quad-q_2(x-a)(y-c)(1-q_1)(1-q_2)\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\big(q_1^{n}x+(1-q_1^{n})a,q_2^{m+1}y+(1-q_2^{m+1})c\big)\\
&\quad+(x-a)(y-c)(1-q_1)(1-q_2)\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\big(q_1^{n}x+(1-q_1^{n})a,q_2^{m}y+(1-q_2^{m})c\big)\Bigg]\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_1^nq_2^mf\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)\\
&\quad-\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}q_1^nq_2^mf\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)\\
&\quad-\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)\\
&\quad+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)=f(x,y)
\end{aligned}$$

(2) By the definition of partial q_1q_2-derivatives and Definition 2, we have

$$\begin{aligned}
&\int_c^y\int_a^x\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}t_c\ \partial_{q_2}s}\ {}_cd_{q_2}s_ad_{q_1}t\\
&=\int_c^y\int_a^x\frac{1}{(1-q_1)(1-q_2)(s-c)(t-a)}\\
&\quad\times[f(q_1t+(1-q_1)a,q_2s+(1-q_2)c)-f(q_1t+(1-q_1)a,s)\\
&\quad-f(t,q_2s+(1-q_2)c)+f(t,s)]_cd_{q_2}s_ad_{q_1}t\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[f\big(q_1^{n+1}x+(1-q_1^{n+1})a,q_2^{m+1}y+(1-q_2^{m+1})c\big)\\
&\quad-f\big(q_1^{n+1}x+(1-q_1^{n+1})a,q_2^{m}y+(1-q_2^{m})c\big)\\
&\quad-f\big(q_1^{n}x+(1-q_1^{n})a,q_2^{m+1}y+(1-q_2^{m+1})c\big)\\
&\quad+f\big((q_1^nx+(1-q_1^n)a),(q_2^my+(1-q_2^m)c)\big)]\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)\\
&\quad-\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}f\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)\\
&\quad-\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}f\big(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\big)\\
&\quad+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f\big((q_1^nx+(1-q_1^n)a),(q_2^my+(1-q_2^m)c)\big)=f(x,y).
\end{aligned}$$

(3) Using (2.2) and applying the result (2), we obtain

$$\begin{aligned}
&\int_{y_1}^y\int_{x_1}^x\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_ad_{q_1}t_cd_{q_2}s\\
&=\int_c^y\int_a^x\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_ad_{q_1}t_cd_{q_2}s\\
&\quad-\int_c^{y_1}\int_a^x\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_ad_{q_1}t_cd_{q_2}s\\
&\quad-\int_c^{y}\int_a^{x_1}\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_ad_{q_1}t_cd_{q_2}s\\
&\quad+\int_c^{y_1}\int_a^{x_1}\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_ad_{q_1}t_cd_{q_2}s\\
&=f(x,y)-f(x,y_1)-f(x_1,y)+f(x_1,y_1).\qquad\square
\end{aligned}$$

Theorem 2. *Suppose that* $f,g:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ *are continuous functions,* $\alpha\in\mathbb{R}$*. Then, for* $(x,y)\in[a,b]\times[c,d]$*,*

(1) $\int_c^y\int_a^x[f(t,s)+g(t,s)]_ad_{q_1}t_cd_{q_2}s=\int_c^y\int_a^x f(t,s)_ad_qt+\int_c^y\int_a^x g(t,s)_ad_{q_1}t_cd_{q_2}s.$
(2) $\int_c^y\int_a^x\alpha f(t,s)_ad_{q_1}t_cd_{q_2}s=\alpha\int_c^y\int_a^x f(t,s)_ad_{q_1}t_cd_{q_2}s.$
(3) *The following integration by parts formula for iterated* q_1q_2*-double integrals holds:*

$$\int_{y_1}^{y}\int_{x_1}^{x} f(t,s)\frac{{}_{a,c}\partial^2_{q_1,q_2}g(t,s)}{{}_a\partial_{q_1}x\,{}_c\partial_{q_2}y}\,{}_ad_{q_1}t\,{}_cd_{q_2}s$$
$$=f(x,y)g(x,y)-f(x,y_1)g(x,y_1)-f(x_1,y)g(x_1,y)$$
$$+f(x_1,y_1)g(x_1,y_1)-\int_{y_1}^{y} g(x,q_2s+(1-q_2)c)\frac{{}_c\partial_{q_2}f(x,s)}{{}_c\partial_{q_2}s}\,{}_cd_{q_2}s$$
$$+\int_{y_1}^{y} g(x_1,q_2s+(1-q_2)c)\frac{{}_c\partial_{q_2}f(x_1,s)}{{}_c\partial_{q_2}s}\,{}_cd_{q_2}s$$
$$-\int_{x_1}^{x} g(q_1t+(1-q_1)a,y)\frac{{}_a\partial_{q_1}f(t,y)}{{}_a\partial_{q_1}t}\,{}_ad_{q_1}t$$
$$+\int_{a}^{x} g(q_1t+(1-q_1)a,y_1)\frac{{}_a\partial_{q_1}f(t,y_1)}{{}_a\partial_{q_1}t}\,{}_ad_{q_1}t$$
$$+\int_{y_1}^{y}\int_{x_1}^{x} g(q_1t+(1-q_1)a,q_2s+(1-q_2)c)$$
$$\times\frac{{}_{c,a}\partial^2_{q_2,q_1}f(t,s)}{{}_c\partial_{q_2}s\,{}_a\partial_{q_1}t}\,{}_ad_{q_1}t\,{}_cd_{q_2}s,\ (x_1,y_1)\in(a,x)\times(c,y).$$

Proof. The proof of (1) and (2) follows by definition of definite q_1q_2-double integrals. (3) By applying (3) of Theorem 3.3 (Tariboon and Ntouyas, 2013), we have

$$\int_{y_1}^{y}\left(\int_{x_1}^{x} f(t,s)\frac{{}_{a,c}\partial^2_{q_1,q_2}g(t,s)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_ad_{q_1}t\right){}_cd_{q_2}s$$
$$=\int_{y_1}^{y}\left[f(x,s)\frac{{}_c\partial_{q_2}g(x,s)}{{}_c\partial_{q_2}s}-f(x_1,s)\frac{{}_c\partial_{q_2}g(a,s)}{{}_c\partial_{q_2}s}\right.$$
$$\left.-\int_{x_1}^{x}\frac{{}_c\partial_{q_2}g(q_1t+(1-q_1)a,s)}{{}_c\partial_{q_2}s}\frac{{}_a\partial_{q_1}f(t,s)}{{}_a\partial_{q_1}t}\,{}_ad_{q_1}t\right]{}_cd_{q_2}s$$
$$=\int_{y_1}^{y} f(x,s)\frac{{}_c\partial_{q_2}g(x,s)}{{}_c\partial_{q_2}s}\,{}_cd_{q_2}s-\int_{y_1}^{y} f(x_1,s)\frac{{}_c\partial_{q_2}g(x_1,s)}{{}_c\partial_{q_2}s}\,{}_cd_{q_2}s$$
$$-\int_{x_1}^{x}\left(\int_{y_1}^{y}\frac{{}_c\partial_{q_2}g(q_1t+(1-q_1)a,s)}{{}_c\partial_{q_2}s}\frac{{}_a\partial_{q_1}f(t,s)}{{}_a\partial_{q_1}t}\,{}_cd_{q_2}s\right){}_ad_{q_1}t$$
$$=f(x,y)g(x,y)-f(x,y_1)g(x,y_1)$$
$$-\int_{y_1}^{y} g(x,q_2s+(1-q_2)c)\frac{{}_c\partial_{q_2}f(x,s)}{{}_c\partial_{q_2}s}\,{}_cd_{q_2}s$$
$$-f(x_1,y)g(x_1,y)+f(x_1,y_1)g(x_1,y_1)$$
$$+\int_{y_1}^{y} g(x_1,q_2s+(1-q_2)c)\frac{{}_c\partial_{q_2}f(x_1,s)}{{}_c\partial_{q_2}s}\,{}_cd_{q_2}s$$
$$-\int_{x_1}^{x} g(q_1t+(1-q_1)a,y)\frac{{}_a\partial_{q_1}f(t,y)}{{}_a\partial_{q_1}t}\,{}_ad_{q_1}t$$
$$+\int_{x_1}^{x} g(q_1t+(1-q_1)a,y_1)\frac{{}_a\partial_{q_1}f(t,y_1)}{{}_a\partial_{q_1}t}\,{}_ad_{q_1}t$$
$$+\int_{y_1}^{y}\int_{x_1}^{x} g(q_1t+(1-q_1)a,q_2$$
$$+(1-q_2)c)\frac{{}_{c,a}\partial^2_{q_2,q_1}f(t,s)}{{}_c\partial_{q_2}s\,{}_a\partial_{q_1}t}\,{}_ad_{q_1}t\,{}_cd_{q_2}s$$

which is the expected result. □

3. Main results

Before we proceed to prove the main results of this section, we refer the readers to Dragomir (2001), Latif and Alomari (2009) and Özdemir et al. (2012) to study the basic properties of convex and quasi-convex functions on the co-ordinates on $[a,b]\times[c,d]$. In this section, we first prove Hermite–Hadamard type inequality for functions of two variable which are convex on the co-ordinates on $[a,b]\times[c,d]$.

Theorem 3. *Let $f:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ be convex on co-ordinates on $[a,b]\times[c,d]$, the following inequalities holds*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leqslant\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right){}_ad_{q_1}x$$
$$+\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right){}_cd_{q_2}y$$
$$\leqslant\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,{}_cd_{q_2}y\,{}_ad_{q_1}x$$
$$\leqslant\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\,{}_cd_{q_2}y$$
$$+\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\,{}_cd_{q_2}y$$
$$+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\,{}_ad_{q_1}x$$
$$+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\,{}_ad_{q_1}x$$
$$\leqslant\frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}.$$

Proof. Since $f:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ is convex on co-ordinates on $[a,b]\times[c,d]$, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2},\frac{c+d}{2}\right)$$
$$\leqslant\frac{1}{2}f\left(ta+(1-t)b,\frac{c+d}{2}\right)$$
$$+\frac{1}{2}f\left(ta+(1-t)b,\frac{c+d}{2}\right)$$

The q_1-integration with respect to t over $[0,1]$, q_2-integration with respect to y over $[c,d]$ on both sides of the above inequality and by the change of variables, give

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right){}_ad_{q_1}x. \tag{3.1}$$

Now

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=f\left(\frac{a+b}{2},\frac{cs+(1-s)d+sd+(1-s)c}{2}\right)$$
$$\leqslant\frac{1}{2}f\left(\frac{a+b}{2},cs+(1-s)d\right)$$
$$+\frac{1}{2}f\left(\frac{a+b}{2},cd+(1-s)c\right)$$

The q_1-integration with respect to x over $[a,b]$, q_2-integration with respect to s over $[0,1]$ on both sides of the above inequality and by the change of variables, give

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leqslant\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right){}_cd_{q_2}y. \tag{3.2}$$

Adding (3.1) and (3.2) and dividing both sides by 2, we get

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leqslant \frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)\,{}_ad_{q_1}x + \frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\,{}_cd_{q_2}y. \quad (3.3)$$

Consider now

$$\begin{aligned}\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right) &= \frac{1}{2(b-a)}\int_a^b f\left(x,\frac{cs+(1-s)d+sd+(1-s)c}{2}\right)\,{}_ad_{q_1}x\\ &\leqslant \frac{1}{4(b-a)}\int_a^b f(x,cs+(1-s)d)\,{}_ad_{q_1}x\\ &\quad+\frac{1}{4(b-a)}\int_a^b f(x,sd+(1-s)c)\,{}_ad_{q_1}x\end{aligned}$$

The q_2-integration with respect to s over $[0,1]$, yields

$$\begin{aligned}\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right) &\leqslant \frac{1}{4(b-a)}\int_a^b\int_c^d f(x,y)_cd_{q_2}y_ad_{q_1}x\\ &\quad+\frac{1}{4(b-a)}\int_a^b\int_c^d f(x,y)\,{}_cd_{q_2}y_ad_{q_1}x\\ &= \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_cd_{q_2}y_ad_{q_1}x.\end{aligned} \quad (3.4)$$

Similarly

$$\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\,{}_cd_{q_2}y \leqslant \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_cd_{q_2}y_ad_{q_1}x. \quad (3.5)$$

Addition of (3.4) and (3.5), gives

$$\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\,{}_cd_{q_2}y \leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_cd_{q_2}y_ad_{q_1}x. \quad (3.6)$$

We also observe that

$$\begin{aligned}\int_a^b\int_c^d f(x,y)_cd_{q_2}y_ad_{q_1}x &= (b-a)\int_0^1\int_c^d f(tb+(1-t)a,y)_cd_{q_2}y_0d_{q_1}t\\ &\leqslant (b-a)\int_0^1\int_c^d (1-t)f(a,y)_cd_{q_2}y_0d_{q_1}t\\ &\quad+(b-a)\int_0^1\int_c^d tf(b,y)_cd_{q_2}y_0d_{q_1}t\\ &= \frac{q_1(b-a)}{1+q_1}\int_c^d f(a,y)_cd_{q_2}y+\frac{(b-a)}{1+q_1}\int_c^d f(b,y)_cd_{q_2}y. \end{aligned}\quad (3.7)$$

and

$$\begin{aligned}\int_a^b\int_c^d f(x,y)_cd_{q_2}y_ad_{q_1}x &= (d-c)\int_a^b\int_0^1 f(x,sd+(1-s)c)_cd_{q_2}s_0d_{q_1}x\\ &\leqslant (d-c)\int_a^b\int_0^1 sf(x,d)_cd_{q_2}s_0d_{q_1}x\\ &\quad+(d-c)\int_a^b\int_c^d (1-s)f(x,c)_cd_{q_2}s_0d_{q_1}x\\ &= \frac{(d-c)}{1+q_2}\int_a^b f(x,d)_ad_{q_1}x+\frac{q_2(d-c)}{1+q_2}\int_a^b f(x,c)_ad_{q_1}x.\end{aligned} \quad (3.8)$$

Adding (3.7) and (3.8) and multiplying the resulting inequality by $\frac{1}{2(b-a)(d-c)}$, we get

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_cd_{q_2}y_ad_{q_1}x &\leqslant \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)_cd_{q_2}y\\ &\quad+\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)_cd_{q_2}y\\ &\quad+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)_ad_{q_1}x\\ &\quad+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)_ad_{q_1}x.\end{aligned} \quad (3.9)$$

Lastly, we have

$$\begin{aligned}&\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)_cd_{q_2}y+\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)_cd_{q_2}y\\ &\quad+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)_ad_{q_1}x\\ &\quad+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)_ad_{q_1}x\\ &\leqslant \frac{q_1}{2(1+q_1)}\int_0^1 f(a,sd+(1-s)c)_0d_{q_2}s\\ &\quad+\frac{1}{2(1+q_1)}\int_0^1 f(b,sd+(1-s)c)_0d_{q_2}s\\ &\quad+\frac{1}{2(1+q_2)}\int_0^1 f(tb+(1-t)a,d)_0d_{q_1}t\\ &\quad+\frac{q_2}{2(1+q_2)}\int_0^1 f(tb+(1-t)a,c)_0d_{q_1}t\\ &\leqslant \frac{q_1f(a,d)}{2(1+q_1)}\int_0^1 s_0d_{q_2}s+\frac{q_1f(a,c)}{2(1+q_1)}\int_0^1 (1-s)\,{}_0d_{q_2}s\\ &\quad+\frac{f(b,d)}{2(1+q_1)}\int_0^1 s_0d_{q_2}s+\frac{f(b,c)}{2(1+q_1)}\int_0^1 (1-s)\,{}_0d_{q_2}s\\ &\quad+\frac{f(b,d)}{2(1+q_2)}\int_0^1 t_0d_{q_1}t+\frac{f(a,d)}{2(1+q_2)}\int_0^1 (1-t)_0d_{q_1}t\\ &\quad+\frac{q_2f(b,c)}{2(1+q_2)}\int_0^1 t_0d_{q_1}t+\frac{q_2f(a,c)}{2(1+q_2)}\int_0^1 (1-t)_0d_{q_1}t\\ &= \frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}. \quad \square\end{aligned} \quad (3.10)$$

Remark 2. When $q_1 \to 1^-$ and $q_2 \to 1^-$, Theorem 3 becomes Theorem 1 from Dragomir (2001, page 778).

We need the following results to prove our main results.

Theorem 4 (Hölder inequality for double sums). *Suppose $(a_{nm})_{n,m\in\mathbb{N}}, (b_{nm})_{n,m\in\mathbb{N}}$ with $a_{nm}, b_{nm} \in \mathbb{R}$ or $\mathbb{C}$ and $\frac{1}{p}+\frac{1}{p'}=1, p, p' > 1$, the following Hölder inequality for double sums holds*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}b_{nm}| \leqslant \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}|^{p}\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|b_{nm}|^{p'}\right)^{\frac{1}{p'}},$$

where all the sums are assumed to be finite.

Theorem 5 (q_1q_2-Hölder inequality for functions of two variables). *Let f and g be functions defined on $[a,b]\times[c,d]$ and $0 < q_1, q_2 < 1$. If $\frac{1}{r_1}+\frac{1}{r_2}=1$ with $r_1, r_2 > 1$, the following q_1q_2-Hölder inequality holds*

$$\int_a^x\int_c^y |f(x,y)g(x,y)|\ {}_cd_{q_2}y\ {}_ad_{q_1}x \leqslant \left(\int_a^b\int_c^d |f(x,y)|^{r_1}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_1}}\left(\int_a^b\int_c^d |g(x,y)|^{r_2}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_2}}. \quad (3.11)$$

Proof. By the definition of q_1q_2-integral and applying Theorem 4, we have

$$\begin{aligned}
&\int_a^b\int_c^d |f(x,y)g(x,y)|\ {}_cd_{q_2}y\ {}_ad_{q_1}x \\
&= (1-q_1)(1-q_2)(x-a)(y-c) \\
&\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\left|f\left(q_1^nx+\left(1-q_1^n\right)a, q_2^my+\left(1-q_2^m\right)c\right)\right| \\
&\quad\times\left|g\left(q_1^nx+\left(1-q_1^n\right)a, q_2^my+\left(1-q_2^m\right)c\right)\right| \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[(1-q_1)(1-q_2)(x-a)(y-c)]^{\frac{1}{r_1}}\left(q_1^nq_2^m\right)^{\frac{1}{r_1}} \\
&\quad\times\left|f\left(q_1^nx+\left(1-q_1^n\right)a, q_2^my+\left(1-q_2^m\right)c\right)\right| \\
&\quad\times[(1-q_1)(1-q_2)(x-a)(y-c)]^{\frac{1}{r_2}}\left(q_1^nq_2^m\right)^{\frac{1}{r_2}} \\
&\quad\times\left|g\left(q_1^nx+\left(1-q_1^n\right)a, q_2^my+\left(1-q_2^m\right)c\right)\right| \\
&\leqslant\Bigg((1-q_1)(1-q_2)(x-a)(y-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m \\
&\quad\times\left|f\left(q_1^nx+\left(1-q_1^n\right)a, q_2^my+\left(1-q_2^m\right)c\right)\right|^{r_1}\Bigg)^{\frac{1}{r_1}} \\
&\quad\times\Bigg((1-q_1)(1-q_2)(x-a)(y-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m \\
&\quad\times\left|g\left(q_1^nx+\left(1-q_1^n\right)a, q_2^my+\left(1-q_2^m\right)c\right)\right|^{r_2}\Bigg)^{\frac{1}{r_2}} \\
&= \left(\int_a^b\int_c^d |f(x,y)|^{r_1}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_1}} \\
&\quad\times\left(\int_a^b\int_c^d |g(x,y)|^{r_2}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_2}}. \qquad \square
\end{aligned}$$

Lemma 2. *Let $f : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° for $0 < q_1, q_2 < 1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$, then the following equality holds*

$$\begin{aligned}
\Upsilon_{q_1,q_2}(a,b,c,d)(f) &:= \frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)} \\
&\quad-\frac{q_2}{(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_ad_{q_1}x \\
&\quad-\frac{1}{(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_ad_{q_1}x \\
&\quad-\frac{q_1}{(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_cd_{q_2}y \\
&\quad-\frac{1}{(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_cd_{q_2}y \\
&\quad+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x \\
&=\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\int_0^1\int_0^1(1-(1+q_1)t)(1-(1+q_2)s) \\
&\quad\times\frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_1}t\ {}_0d_{q_2}s. \qquad (3.12)
\end{aligned}$$

Proof. By the definition of partial q_1q_2-derivatives and definite q_1q_2-integrals, we have

$$\begin{aligned}
&\int_0^1\int_0^1(1-(1+q_1)t)(1-(1+q_2)s) \\
&\quad\times\frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t_c\partial_{q_1}s}\ {}_0d_{q_1}t\ {}_0d_{q_2}s \\
&=\frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)}\int_0^1\int_0^1\frac{(1-(1+q_1)t)(1-(1+q_2)s)}{st} \\
&\quad\times[f(tq_1b+(1-tq_1)a,sq_2d+(1-sq_2)c)-f(tq_1b+(1-tq_1)a,sd+(1-s)c) \\
&\quad-f(tb+(1-t)a,q_2sd+(1-q_2s)c)+f(tb+(1-t)a,sd+(1-s)c)]\ {}_0d_{q_1}t\ {}_0d_{q_2}s \\
&=\frac{1}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(1-(1+q_1)q_1^n\right)\left(1-(1+q_2)q_2^m\right) \\
&\quad\times\left[f\left(q_1^{n+1}b+\left(1-q_1^{n+1}\right)a,q_2^{m+1}d+\left(1-q_2^{m+1}\right)c\right)\right. \\
&\quad-f\left(q_1^{n+1}b+\left(1-q_1^{n+1}\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad\left.-f\left(q_1^nb+\left(1-q_1^n\right)a,q_2^{m+1}d+\left(1-q_2^{m+1}\right)c\right)+f\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right)\right] \\
&=\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad-\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}f\left(q_1^nb+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^md\right) \\
&\quad-\frac{1}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad+\frac{1}{(d-c)(b-a)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad-\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^nf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad+\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^nf\left(q_1^nb+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^md\right) \\
&\quad+\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^nf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad-\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md\right. \\
&\quad\left.+\left(1-q_2^m\right)c\right)-\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad+\frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^md\right) \\
&\quad+\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad-\frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad+\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad-\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^md\right) \\
&\quad-\frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right) \\
&\quad+\frac{(1+q_1)(1+q_2)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mf\left(q_1^nb+\left(1-q_1^n\right)a,q_2^md+\left(1-q_2^m\right)c\right). \qquad (3.13)
\end{aligned}$$

We observe that

$$\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=\frac{f(a,c)}{(b-a)(d-c)}+\frac{1}{(b-a)(d-c)}\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right),\qquad(3.14)$$

$$-\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right)$$
$$=-\frac{f(a,d)}{(b-a)(d-c)}-\frac{1}{(b-a)(d-c)}\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right),\qquad(3.15)$$

$$-\frac{1}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=-\frac{f(b,c)}{(b-a)(d-c)}-\frac{1}{(b-a)(d-c)}\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right),\qquad(3.16)$$

$$\frac{1}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=\frac{f(b,d)}{(b-a)(d-c)}+\frac{1}{(b-a)(d-c)}\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right),\qquad(3.17)$$

$$-\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=\frac{(1+q_1)f(b,c)}{q_1(b-a)(d-c)}-\frac{(1+q_1)}{q_1(b-a)(d-c)}\times\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,c\right)-\frac{(1+q_1)}{q_1(b-a)(d-c)}\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right),\qquad(3.18)$$

$$\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right)$$
$$=-\frac{(1+q_1)f(b,d)}{q_1(b-a)(d-c)}+\frac{(1+q_1)}{q_1(b-a)(d-c)}\times\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,d\right)+\frac{(1+q_1)}{q_1(b-a)(d-c)}\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right),\qquad(3.19)$$

$$\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=\frac{(1+q_1)}{(b-a)(d-c)}\times\left[-\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,d\right)+\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,c\right)\right]+\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right),\qquad(3.20)$$

$$-\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=\frac{(1+q_2)f(a,d)}{q_2(b-a)(d-c)}-\frac{(1+q_2)}{q_2(b-a)(d-c)}\times\sum_{m=0}^{\infty}q_2^m f\left(a,q_2^m d+\left(1-q_2^m\right)c\right)-\frac{(1+q_2)}{q_2(b-a)(d-c)}\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right),\qquad(3.21)$$

$$\frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right)$$
$$=-\frac{(1+q_2)}{(b-a)(d-c)}\times\left[-\sum_{m=0}^{\infty}q_2^m f\left(b,\left(1-q_2^m\right)c+q_2^m d\right)+\sum_{m=0}^{\infty}q_2^m f\left(a,\left(1-q_2^m\right)c+q_2^m d\right)\right]+\frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right),\qquad(3.22)$$

$$\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=-\frac{(1+q_2)f(b,d)}{q_2(b-a)(d-c)}+\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(b,q_2^m d+\left(1-q_2^m\right)c\right)+\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right),\qquad(3.23)$$

$$\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=-\frac{(1+q_1)(1+q_2)f(b,d)}{q_1q_2(b-a)(d-c)}-\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,d\right)-\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(b,q_2^m d+\left(1-q_2^m\right)c\right)+\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)\qquad(3.24)$$

$$-\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right)$$
$$=\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(b,\left(1-q_2^m\right)c+q_2^m d\right)-\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,\left(1-q_2^m\right)c+q_2^m d\right)\qquad(3.25)$$

and

$$-\frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right)$$
$$=\frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+\left(1-q_1^n\right)a,d\right)-\frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+\left(1-q_1^n\right)a,q_2^m d+\left(1-q_2^m\right)c\right).\qquad(3.26)$$

Using (3.14)–(3.26) in (3.13) and simplifying, we get

$$\int_0^1\int_0^1 (1-(1+q_1)t)(1-(1+q_2)s) \times \frac{{}_{a,c}\partial^2_{q_1,q_2} f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}x\,{}_c\partial_{q_2}y}\ {}_0d_{q_1}t\ {}_0d_{q_2}s$$
$$= \frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{q_1q_2(b-a)(d-c)} - \frac{(1+q_1)(1-q_1)}{q_1(b-a)(d-c)}\sum_{n=0}^{\infty} q_1^n f(q_1^n b+(1-q_1^n)a,c) - \frac{(1+q_1)(1-q_1)}{q_1q_2(b-a)(d-c)}\sum_{n=0}^{\infty} q_1^n f(q_1^n b+(1-q_1^n)a,d) - \frac{(1+q_2)(1-q_2)}{q_2(b-a)(d-c)}\sum_{m=0}^{\infty} q_2^m f(a,q_2^m d+(1-q_2^m)c) - \frac{(1+q_2)(1-q_2)}{q_1q_2(b-a)(d-c)}\sum_{m=0}^{\infty} q_2^m f(b,(1-q_2^m)c+q_2^m d) + \frac{(1+q_1)(1+q_2)(1-q_1)(1-q_2)}{q_1q_2(b-a)(d-c)} \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c)$$
$$= \frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{q_1q_2(b-a)(d-c)} - \frac{(1+q_1)}{q_1(b-a)^2(d-c)}\int_a^b f(x,c)\ {}_ad_{q_1}x - \frac{(1+q_1)}{q_1q_2(b-a)^2(d-c)}\int_a^b f(x,d)\ {}_ad_{q_1}x - \frac{(1+q_2)}{q_2(b-a)(d-c)^2}\int_c^d f(a,y)\ {}_cd_{q_2}y - \frac{(1+q_2)}{q_1q_2(b-a)(d-c)^2}\int_c^d f(b,y)\ {}_cd_{q_2}y + \frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)^2(d-c)^2}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x. \quad (3.27)$$

Multiplying both sides of (3.27) by $\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}$, we get the desired equality. □

Remark 3. As $q_1 \to 1^-$ and $q_2 \to 1^-$, $\Upsilon_{q_1,q_2}(a,b,c,d)(f) \to \Upsilon(a,b,c,d)(f)$, where

$$\Upsilon(a,b,c,d)(f) := \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)}\int_a^b f(x,c)dx - \frac{1}{2(b-a)}\int_a^b f(x,d)dx - \frac{1}{2(d-c)}\int_c^d f(a,y)dy - \frac{1}{2(d-c)}\int_c^d f(b,y)dy + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

and hence the result of Lemma 2 becomes the result of Lemma 1 proved in Sarikaya et al. (2012, page 139).

Now, we can present some integral inequalities for functions whose partial q_1q_2-derivatives satisfy the assumptions of convexity on co-ordinates on $[a,b]\times[c,d]$.

Theorem 6. *Let $f : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0 < q_1 < 1$ and $0 < q_2 < 1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d] \subseteq \Lambda^\circ$ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ is convex on co-ordinates on $[a,b]\times[c,d]$ for $r \geqslant 1$, then the following inequality holds*

$$\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right| \leqslant \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\Phi_{q_1}\Phi_{q_2}\right)^{1-\frac{1}{r}} \times \left\{\Psi_{q_1}\Psi_{q_2}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r + \Delta_{q_1}\Psi_{q_2}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r + \Delta_{q_2}\Psi_{q_1}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r + \Delta_{q_1}\Delta_{q_2}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\right\}^{\frac{1}{r}}. \quad (3.28)$$

Proof. Taking the absolute value on both sides of the equality of Lemma 2, using the q_1q_2-Hölder inequality for functions of two variables and convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ on co-ordinates on $[a,b]\times[c,d]$, we have

$$\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right| \leqslant \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)} \times \left(\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{1-\frac{1}{r}} \times \left(\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|\right) \times \left[\left((1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r + (1-t)s\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r + (1-s)t\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r + st\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\right]\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r}}. \quad (3.29)$$

From Lemma 1, we observe that

$$\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s = \left(\int_0^1 |(1-(1+q_1)t)|\ {}_0d_{q_1}t\right)\left(\int_0^1 |(1-(1+q_2)s)|\ {}_0d_{q_2}s\right) = \Phi_{q_1}\Phi_{q_2},$$

$$\int_0^1\int_0^1 (1-t)(1-s)|(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s = \left(\int_0^1 (1-t)|1-(1+q_1)t|\ {}_0d_{q_1}t\right) \times \left(\int_0^1 (1-s)|1-(1+q_2)s|\ {}_0d_{q_2}s\right) = \Psi_{q_1}\Psi_{q_2},$$

$$\int_0^1\int_0^1 (1-t)s|(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s = \left(\int_0^1 (1-t)|1-(1+q_1)t|\ {}_0d_{q_1}t\right) \times \left(\int_0^1 s|1-(1+q_2)s|\ {}_0d_{q_2}s\right) = \Delta_{q_1}\Psi_{q_2},$$

$$\int_0^1\int_0^1 t(1-s)|(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s$$
$$=\left(\int_0^1 t|1-(1+q_1)t|\ {}_0d_{q_1}t\right)$$
$$\times\left(\int_0^1 (1-s)|1-(1+q_2)s|\ {}_0d_{q_2}s\right)=\Delta_{q_2}\Psi_{q_1}$$

and

$$\int_0^1\int_0^1 ts|(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s$$
$$=\left(\int_0^1 t|1-(1+q_1)t|\ {}_0d_{q_1}t\right)$$
$$\times\left(\int_0^1 (1-s)|1-(1+q_2)s|\ {}_0d_{q_2}s\right)=\Delta_{q_1}\Delta_{q_2}.$$

Using the values of the above q_1q_2-integrals, we get the required inequality. □

Remark 4. When $q_1\to 1^-$ and $q_2\to 1^-$ in Theorem 6, we get the result proved in Theorem 4 in Sarikaya et al. (2012, page 146).

Theorem 7. *Let $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}$ is convex on co-ordinates on $[a,b]\times[c,d]$ for $r_1>1$, then the following inequality holds*

$$\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right|\leqslant\frac{q_1q_2(b-a)(d-c)}{[(1+q_1)(1+q_2)]^{1+\frac{1}{r_1}}}\left(A_{q_1}(r_2)A_{q_2}(r_2)\right)^{\frac{1}{r_2}}$$
$$\times\left\{q_1q_2\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}+q_1\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right.$$
$$\left.+q_2\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right\}^{\frac{1}{r_1}},\tag{3.30}$$

where $\frac{1}{r_1}+\frac{1}{r_2}=1$.

Proof. Taking the absolute value on both sides of the equality of Lemma 2, using the q_1q_2-Hölder inequality for functions of two variables and convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r}$ on co-ordinates on $[a,b]\times[c,d]$, we have

$$\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right|\leqslant\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}$$
$$\times\left(\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|^{r_2}\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_0^1\int_0^1\left[(1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right.\right.$$
$$+(1-t)s\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}+(1-s)t\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}$$
$$\left.\left.+st\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right]\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r_1}}.\tag{3.31}$$

We observe that

$$\int_0^1 |1-(1+q_1)t|^{r_2}\ {}_0d_{q_1}t=\int_0^{\frac{1}{1+q_1}}(1-(1+q_1)t)^{r_2}\ {}_0d_{q_1}t+\int_{\frac{1}{1+q_1}}^1((1+q_1)t-1)^{r_2}\ {}_0d_{q_1}t.\tag{3.32}$$

Consider the first q_1-integral from (3.32) and making use of the substitution $1-(1+q_1)t=s$, we obtain

$$\int_0^{\frac{1}{1+q_1}}(1-(1+q_1)t)^{r_2}\ {}_0d_{q_1}t=-\frac{1}{1+q_1}\int_1^0 s^{r_2}\ {}_0d_{q_1}s=\frac{1}{1+q_1}\int_0^1 s^{r_2}\ {}_0d_{q_1}s=\frac{1-q_1}{(1+q_1)\left(1-q_1^{r_2+1}\right)}.\tag{3.33}$$

Consider the second q_1-integral from (3.32) and making use of the substitution $(1+q_1)t-1=s$, we get

$$\int_{\frac{1}{1+q_1}}^1((1+q_1)t-1)^{r_2}\ {}_0d_{q_1}t=\frac{1}{1+q_1}\int_0^{q_1}s^{r_2}\ {}_0d_{q_1}s=\frac{(1-q_1)q_1^{r_2+1}}{(1+q_1)\left(1-q_1^{r_2+1}\right)}.\tag{3.34}$$

Substitution of (3.33) and (3.34) in (3.32) gives

$$\int_0^1|1-(1+q_1)t|^{r_2}\ {}_0d_{q_1}t=\frac{(1-q_1)\left(1+q_1^{r_2+1}\right)}{(1+q_1)\left(1-q_1^{r_2+1}\right)}=A_{q_1}(r_2).\tag{3.35}$$

Similarly, one can have

$$\int_0^1|1-(1+q_2)s|^{r_2}\ {}_0d_{q_2}s=\frac{(1-q_2)\left(1+q_2^{r_2+1}\right)}{(1+q_2)\left(1-q_2^{r_2+1}\right)}=A_{q_2}(r_2).\tag{3.36}$$

Finally, we also have

$$\int_0^1(1-t)\ {}_0d_{q_1}t=\frac{q_1}{1+q_1},\ \int_0^1(1-s)\ {}_0d_{q_2}s=\frac{q_2}{1+q_2},$$
$$\int_0^1 t\ {}_0d_{q_1}t=\frac{1}{1+q_1}\ \text{and}\ \int_0^1 s\ {}_0d_{q_2}s=\frac{1}{1+q_2}.$$ □

Remark 5. When $q_1\to 1^-$ and $q_2\to 1^-$ in Theorem 7, we get the following inequality proved in Sarikaya et al. (2012, page 144).

$$|\Upsilon(a,b,c,d)(f)|\leqslant\frac{(b-a)(d-c)}{4}\left(\frac{1}{r_2+1}\right)^{\frac{2}{r_2}}\times\left\{\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^{r_1}+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^{r_1}+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^{r_1}+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^{r_1}}{4}\right\}^{\frac{1}{r_1}}.\tag{3.37}$$

Indeed, the inequality (3.37) follows by applying L'Hospital rule to the limits

$$\lim_{q_1\to 1^-}\frac{1-q_1}{1-q_1^{r_2+1}}\ \text{and}\ \lim_{q_2\to 1^-}\frac{1-q_2}{1-q_2^{r_2+1}}.$$

The next two results are for quasi-convex functions on co-ordinates on $[a,b]\times[c,d]$.

Theorem 8. *Let $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r$ is quasi-convex on co-ordinates on $[a,b]\times[c,d]$ for $r\geqslant 1$, then the following inequality holds*

$$\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right|\leqslant\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\frac{4q_1q_2}{(1+q_1)^2(1+q_2)^2}\right)\times\sup\left\{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|\right\}. \tag{3.38}$$

Proof. Lemma 2, an application of the q_1q_2-Hölder inequality and quasi-convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r$ on $[a,b]\times[c,d]$, yield

$$\begin{aligned}\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right|&\leqslant\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\\&\times\left(\int_0^1\int_0^1|(1-(1+q_1)t)(1-(1+q_2)s)|\,{}_0d_{q_1}t\,{}_0d_{q_2}s\right)^{1-\frac{1}{r}}\\&\times\left(\int_0^1\int_0^1|(1-(1+q_1)t)(1-(1+q_2)s)|\right.\\&\times\left.\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r{}_0d_{q_1}t\,{}_0d_{q_2}s\right)^{\frac{1}{r}}\\&=\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\\&\quad\times\left(\int_0^1\int_0^1|(1-(1+q_1)t)(1-(1+q_2)s)|\,{}_0d_{q_1}t\,{}_0d_{q_2}s\right)\\&\quad\times\left(\sup\left\{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r,\right.\right.\\&\quad\times\left.\left.\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r\right\}\right)^{\frac{1}{r}}.\end{aligned} \tag{3.39}$$

Now using the properties of supremum and Lemma 1, we get the required result from (3.39). □

Theorem 9. *Let $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^{r_1}$ is convex on co-ordinates on $[a,b]\times[c,d]$ for $r_1>1$, then the following inequality holds*

$$\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right|\leqslant\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(A_{q_1}(r_2)A_{q_2}(r_2)\right)^{\frac{1}{r_2}}\times\sup\left\{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|\right\}. \tag{3.40}$$

Proof. With the similar reasoning as in proving (3.38), we notice that

$$\begin{aligned}\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right|&\leqslant\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\\&\times\left(\int_0^1\int_0^1|(1-(1+q_1)t)(1-(1+q_2)s)|^{r_1}\,{}_0d_{q_1}t\,{}_0d_{q_2}s\right)^{\frac{1}{r_1}}\\&\times\left(\int_0^1\int_0^1\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^{r_2}{}_0d_{q_1}t\,{}_0d_{q_2}s\right)^{\frac{1}{r_2}}.\end{aligned} \tag{3.41}$$

Using the properties of the supremum, the quasi-convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r$ on $[a,b]\times[c,d]$, (3.35) and (3.36), we get (3.40). □

Remark 6. As $q_1\to 1^-$ and $q_2\to 1^-$ in Theorems 8 and 9, we get the corresponding results of classical calculus of functions of two variables.

4. Conclusion

In this manuscript partial q_1q_2-derivative and definite q_1q_2-integrals over the finite rectangles are discussed for the first time. Some q-analogues of integral inequalities for functions of two variables are presented using the notion of q-calculus of functions of two variables over the finite rectangles and the concept of two types of convexity on co-ordinates. The results of this paper have very clear physical understanding of minimizing the error bounds in the two variable trapezoidal rule.

Competing interests

The author declares that he has no competing interests.

Authors' Contributions

All the authors have contributed equality in preparing the manuscript. All the authors have approved the final version of the manuscript.

Acknowledgement

The authors are very thankful to the anonymous referees for their very useful comments/suggestions which have been incorporated in the final version of the manuscript.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.jksus.2016.07.001.

References

Dragomir, S.S., 2001. On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. Taiwan. J. Math. 4, 775–788.

Dragomir, S.S., Agarwal, R.P., 1998. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. 11 (9), 91–95.

Dragomir, S.S., Pearce, C.E.M., 2000. Selected topics on Hermite–Hadamard inequalities and applications. RGMIA Monographs. Victoria University.

Ernst, T., 2012. A Comprehensive Treatment of q-Calculus. Springer, Basel.

Gauchman, H., 2004. Integral inequalities in q-calculus. Comput. Math. Appl. 47, 281–300.

Ghany, H.A., 2009. q-derivative of basic hypergeometric series with respect to parameters. Int. J. Math. Anal. 33, 1617.

Ghany, H.A., 2012. Basic completely monotone functions as coefficients and solutions of linear q-difference equations with some application. Phys. Essays 25, 38.

Ghany, H.A., 2013a. Integral representation for basic completely alternating functions and some of its related functions. Int. J. Pure Appl. Math. 87 (5), 681–688.

Ghany, H.A., 2013b. Lévy–Khinchin type formula for basic completely monotone functions. Int. J. Pure Appl. Math. 87 (5), 689–697.

Ion, D.A., 2007. Some estimates on the Hermite–Hadamard inequality through quasi-convex functions. Ann. Univ. Craiova Ser. Mat. Inform. 34, 83–88.

Jackson, F.H., 1910. On a q-definite integrals. Q. J. Pure Appl. Math. 41, 193–203.

Kac, V., Cheung, P., 2002. Quantum Calculus, Universitext. Springer, New York.

Latif, M.A., Alomari, M., 2009. Hadamard-type inequalities for product of two convex functions on the co-ordinates. Int. Math. Forum 4 (47), 2327–2338.

Noor, M.A., Noor, K.I., Awan, M.U., 2015a. Some quantum estimates for Hermite–Hadamard inequalities. Appl. Math. Comput. 251, 675–679.

Noor, M.A., Noor, K.I., Awan, M.U., 2015b. Some quantum integral inequalities via preinvex functions. Appl. Math. Comput. 269, 242–251.

Noor, M.A., Noor, K.I., Awan, M.U., 2015c. Quantum analogues of Hermite–Hadamard type inequalities for generalized convexity. In: Daras, N., Rassias, M.T. (Eds.), Computation, Cryptography and Network Security, pp. 413–439.

Özdemir, M.E., Akdemir, A.O., Ağri, Yildiz, C., Erzurum, 2012. On co-ordinated quasi-convex functions. Czechoslovak Math. J. 62 (137), 889–900.

Sarikaya, M.Z., Set, E., Özdemir, M.E., Dragomir, S.S., 2012. New some Hadamard's type inequalities for co-ordinated convex functions. Tamsui Oxford J. Inf. Math. Sci. 28 (2), 137–152.

Sudsutad, W., Ntouyas, S.K., Tariboon, J., 2015. Quantum integral inequalities for convex functions. J. Math. Inequal. 9 (3), 781–793.

Tariboon, J., Ntouyas, S.K., 2013. Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. 282, 19 pp..

Tariboon, J., Ntouyas, S.K., 2014. Quantum integral inequalities on finite intervals. J. Inequal. Appl. 121, 13 pp..

Zhuang, H., Liu, W., Park, J., 2016. Some quantum estimates of Hermite–Hadmard inequalities for quasi-convex functions. Miskolc Math. Notes (to appear).