SYMMETRIZED CONVEXITY AND HERMITE–HADAMARD TYPE INEQUALITIES

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Abstract. In this paper we extend the Hermite-Hadamard inequality to the class of symmetrized convex functions. The corresponding version for \( h \)-convex functions is also investigated. Some examples of interest are provided as well.

1. Introduction

The following inequality holds for any convex function \( f \) defined on \( \mathbb{R} \)

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}, \quad a, b \in \mathbb{R}, \ a \neq b.
\]  

(1.1)

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [43]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite’s note in *Mathesis* [43]. Since (1.1) was known as Hadamard’s inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]–[19], [21]–[24], [31]–[34] and [46].

In this paper we show that the Hermite-Hadamard inequality can be extended to a larger class of functions containing the convex functions. The corresponding version for \( h \)-convex functions is also investigated. Some examples of interest are provided as well.


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2. Symmetrized convexity

For a function $f : [a, b] \to \mathbb{C}$ we consider the symmetrical transform of $f$ on the interval $[a, b]$, denoted by $\tilde{f}_{[a,b]}$ or simply $\tilde{f}$, when the interval $[a, b]$ is implicit, which is defined by

$$\tilde{f}(t) := \frac{1}{2} [f(t) + f(a + b - t)], \quad t \in [a, b].$$

The anti-symmetrical transform of $f$ on the interval $[a, b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t) := \frac{1}{2} [f(t) - f(a + b - t)], \quad t \in [a, b].$$

It is obvious that for any function $f$ we have $\tilde{f} + \tilde{f} = f$.

If $f$ is convex on $[a, b]$, then for any $t_1, t_2 \in [a, b]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have

$$\tilde{f}(\alpha t_1 + \beta t_2) = \frac{1}{2} [f(\alpha t_1 + \beta t_2) + f(a + b - \alpha t_1 - \beta t_2)]$$

$$= \frac{1}{2} [f(\alpha t_1 + \beta t_2) + f(\alpha (a + b - t_1) + \beta (a + b - t_2))]$$

$$\leq \frac{1}{2} [\alpha f(t_1) + \beta f(t_2) + \alpha f(a + b - t_1) + \beta f(a + b - t_2)]$$

$$= \frac{1}{2} \alpha [f(t_1) + f(a + b - t_1)] + \frac{1}{2} \beta [f(t_2) + f(a + b - t_2)]$$

$$= \alpha \tilde{f}(t_1) + \beta \tilde{f}(t_2),$$

which shows that $\tilde{f}$ is convex on $[a, b]$.

Consider the real numbers $a < b$ and define the function $f_0 : [a, b] \to \mathbb{R}$, $f_0(t) = t^3$. We have

$$f_0(t) := \frac{1}{2} \left[ t^3 + (a + b - t)^3 \right] = \frac{3}{2} (a + b) t^2 - \frac{3}{2} (a + b)^2 t + \frac{1}{2} (a + b)^3$$

for any $t \in \mathbb{R}$.

Since the second derivative $(f_0)''(t) = 3(a + b)$, $t \in \mathbb{R}$, then $f_0$ is strictly convex on $[a, b]$ if $\frac{a+b}{2} > 0$ and strictly concave on $[a, b]$ if $\frac{a+b}{2} < 0$. Therefore if $a < 0 < b$ with $\frac{a+b}{2} > 0$, then we can conclude that $f_0$ is not convex on $[a, b]$ while $\tilde{f}_0$ is convex on $[a, b]$.

We can introduce the following concept of convexity.

**DEFINITION 1.** We say that the function $f : [a, b] \to \mathbb{R}$ is symmetrized convex (concave) on the interval $[a, b]$ if the symmetrical transform $\tilde{f}$ is convex (concave) on $[a, b]$. 

Now, if we denote by $\text{Con}[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $\text{SCon}[a,b]$ the class of symmetrized convex functions, then from the above remarks we can conclude that

$$\text{Con}[a,b] \subsetneq \text{SCon}[a,b].$$ \hspace{1cm} (2.1)

Also, if $[c,d] \subset [a,b]$ and $f \in \text{SCon}[a,b]$, then this does not imply in general that $f \in \text{SCon}[c,d]$.

**Theorem 1.** Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}. $$ \hspace{1cm} (2.2)

**Proof.** Since $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$, then by writing the Hermite-Hadamard inequality for the function $\tilde{f}$ we have

$$\tilde{f} \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b \tilde{f}(t)dt \leq \frac{\tilde{f}(a) + \tilde{f}(b)}{2}. $$ \hspace{1cm} (2.3)

However

$$\tilde{f} \left( \frac{a+b}{2} \right) = f \left( \frac{a+b}{2} \right), \quad \frac{\tilde{f}(a) + \tilde{f}(b)}{2} = \frac{f(a) + f(b)}{2},$$

and

$$\int_a^b \tilde{f}(t)dt = \frac{1}{2} \int_a^b [f(t) + f(a+b-t)]dt = \int_a^b f(t)dt.$$ Then by (2.3) we get (2.2). \hfill \Box

For similar results see [36].

The following result holds:

**Theorem 2.** Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x \in [a,b]$ we have the bounds

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{2} [f(x) + f(a+b-x)] \leq \frac{f(a) + f(b)}{2}. $$ \hspace{1cm} (2.4)

**Proof.** Since $\tilde{f}$ is convex on $[a,b]$ then for any $x \in [a,b]$ we have

$$\frac{\tilde{f}(x) + \tilde{f}(a+b-x)}{2} \geq \tilde{f} \left( \frac{a+b}{2} \right)$$

and since

$$\frac{\tilde{f}(x) + \tilde{f}(a+b-x)}{2} = \frac{1}{2} [f(x) + f(a+b-x)]$$
while
\[ f\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right), \]
we get the first inequality in (2.4).

Also, by the convexity of \( \tilde{f} \) we have for any \( x \in [a, b] \) that
\[
\tilde{f}(x) \leq \frac{x-a}{b-a} \cdot \tilde{f}(b) + \frac{b-x}{b-a} \cdot \tilde{f}(a)
\]
\[
= \frac{x-a}{b-a} \cdot \frac{f(a) + f(b)}{2} + \frac{b-x}{b-a} \cdot \frac{f(a) + f(b)}{2}
\]
\[
= \frac{f(a) + f(b)}{2},
\]
which proves the second part of (2.4). \( \square \)

**Remark 1.** If \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \( [a, b] \), then we have the bounds
\[
\inf_{x \in [a, b]} \tilde{f}(x) = \tilde{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)
\]
and
\[
\sup_{x \in [a, b]} \tilde{f}(x) = \tilde{f}(a) = \tilde{f}(b) = \frac{f(a) + f(b)}{2}.
\]

**Corollary 1.** If \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \( [a, b] \) and \( w : [a, b] \to [0, \infty) \) is integrable on \( [a, b] \), then
\[
f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \leq \frac{1}{2} \int_a^b w(t) [f(t) + f(a+b-t)] dt \tag{2.5}
\]
\[
\leq \frac{f(a) + f(b)}{2} \int_a^b w(t) dt.
\]

Moreover, if \( w \) is symmetric almost everywhere on \( [a, b] \), i.e. \( w(t) = w(a+b-t) \) for almost every \( t \in [a, b] \), then
\[
f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \leq \int_a^b w(t)f(t) dt \leq \frac{f(a) + f(b)}{2} \int_a^b w(t) dt. \tag{2.6}
\]

**Proof.** The inequality (2.5) follows by (2.4) written for \( x = t \), multiplying by \( w(t) \geq 0 \) and integrating over \( t \) on \( [a, b] \).

By changing the variable, we have
\[
\int_a^b w(t)f(a+b-t) dt = \int_a^b w(a+b-t)f(t) dt.
\]
Since \( w \) is symmetric almost everywhere on \([a, b]\), then
\[
\int_a^b w(a + b - t)f(t) dt = \int_a^b w(t)f(t) dt.
\]

Therefore
\[
\frac{1}{2} \int_a^b w(t)\left[f(t) + f(a + b - t)\right] dt
= \frac{1}{2} \left[\int_a^b w(t)f(t) dt + \int_a^b w(t)f(a + b - t) dt\right]
= \frac{1}{2} \left[\int_a^b w(t)f(t) dt + \int_a^b w(t)f(t) dt\right] = \int_a^b w(t)f(t) dt
\]
and by (2.5) we get (2.6).  □

**Remark 2.** The inequality (2.6) was obtained by L. Fejér in 1906 for convex functions \( f \) and symmetric weights \( w \). It has been shown now that this inequality remains valid for the larger class of symmetrized convex functions \( f \) on the interval \([a, b]\).

The following result also holds.

**Theorem 3.** Assume that \( f : [a,b] \rightarrow \mathbb{R} \) is symmetrized convex on the interval \([a, b]\). Then for any \( x, y \in [a, b] \) with \( x \neq y \) we have the Hermite-Hadamard inequalities
\[
\frac{1}{2} \left[f \left(\frac{x+y}{2}\right) + f \left(a + b - \frac{x+y}{2}\right)\right] \leq \frac{1}{2(y-x)} \left[\int_x^y f(t) dt + \int_{a+b-y}^{a+b-x} f(t) dt\right] \leq \frac{1}{4} [f(x) + f(a + b - x) + f(y) + f(a + b - y)].
\]  (2.7)

**Proof.** Since \( \tilde{f}_{[a,b]} \) is convex on \([a, b]\), then \( \tilde{f}_{[a,b]} \) is also convex on any subinterval \([x, y]\) (or \([y, x]\)) where \( x, y \in [a, b] \).

By Hermite-Hadamard inequalities for convex functions we have
\[
\tilde{f}_{[a,b]} \left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y \tilde{f}_{[a,b]}(t) dt \leq \frac{\tilde{f}_{[a,b]}(x) + \tilde{f}_{[a,b]}(y)}{2}
\]  (2.8)

for any \( x, y \in [a, b] \) with \( x \neq y \).

We have
\[
\tilde{f}_{[a,b]} \left(\frac{x+y}{2}\right) = \frac{1}{2} \left[f \left(\frac{x+y}{2}\right) + f \left(a + b - \frac{x+y}{2}\right)\right],
\]
\[
\int_x^y f_{[a,b]}(t)\,dt = \frac{1}{2} \int_x^y [f(t) + f(a + b - t)]\,dt \\
= \frac{1}{2} \int_x^y f(t)\,dt + \frac{1}{2} \int_x^y f(a + b - t)\,dt \\
= \frac{1}{2} \int_x^y f(t)\,dt + \frac{1}{2} \int_{a+b-y}^{a+b-x} f(t)\,dt
\]
and
\[
\frac{f_{[a,b]}(x) + f_{[a,b]}(y)}{2} = \frac{1}{4} [f(x) + f(a + b - x) + f(y) + f(a + b - y)].
\]

Then by (2.8) we deduce the desired result (2.7). \(\square\)

**Remark 3.** If we take \(x = a\) and \(y = b\) in (2.7), then we get (2.2).

If, for a given \(x \in [a,b]\), we take \(y = a + b - x\), then from (2.7) we get
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left(\frac{a+b}{2} - x\right) \int_x^{a+b-x} f(t)\,dt \leq \frac{1}{2} [f(x) + f(a+b-x)],
\]
where \(x \neq \frac{a+b}{2}\), provided that \(f : [a,b] \to \mathbb{R}\) is symmetrized convex on the interval \([a,b]\).

Integrating this inequality over \(x\) we get the following refinement of the first part of (2.2)
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)} \int_a^b \left[ \frac{1}{\left(\frac{a+b}{2} - x\right)} \int_x^{a+b-x} f(t)\,dt \right] dx
\]
\[
\leq \frac{1}{b-a} \int_a^b f(t)\,dt,
\]
provided that \(f : [a,b] \to \mathbb{R}\) is symmetrized convex on the interval \([a,b]\).

When the function is convex, we have the following inequalities as well:

**Remark 4.** If \(f : [a,b] \to \mathbb{R}\) is convex, then from (2.7) we have the inequalities
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[ f\left(\frac{x+y}{2}\right) + f\left(a + b - \frac{x+y}{2}\right) \right]
\]
\[
\leq \frac{1}{2(y-x)} \left[ \int_x^y f(t)\,dt + \int_{a+b-y}^{a+b-x} f(t)\,dt \right]
\]
\[
\leq \frac{1}{4} [f(x) + f(a+b-x) + f(y) + f(a+b-y)]
\]
for any \(x,y \in [a,b], x \neq y\).

If we integrate (2.11) over \((x,y)\) on the square \([a,b]^2\) and divide by \((b-a)^2\), then we get the following refinement of the first Hermite-Hadamard inequality for convex
functions
\[
\begin{align*}
f \left( \frac{a + b}{2} \right) & \leq \frac{1}{2(b - a)^2} \left[ \int_a^b \int_a^b f \left( \frac{x + y}{2} \right) \, dx \, dy + \int_a^b \int_a^b f \left( a + b - \frac{x + y}{2} \right) \, dx \, dy \right] \\
& \leq \frac{1}{2(b - a)^2} \int_a^b \int_a^b \frac{1}{y - x} \left[ \int_x^y f(t) \, dt + \int_{a+b-y}^{a+b-x} f(t) \, dt \right] \, dx \, dy \\
& \leq \frac{1}{b - a} \int_a^b f(t) \, dt.
\end{align*}
\]

We notice that, the second and the third inequalities also hold for the more general case of symmetrized convex functions on the interval \([a, b]\).

A concept of weaker symmetrized convexity can be introduced as follows:

**Definition 2.** We say that the function \(f : [a, b] \rightarrow \mathbb{R}\) is weak symmetrized convex (concave) on the interval \([a, b]\) if the symmetrical transform \(\tilde{f}\) is convex (concave) on the interval \([a, \frac{a+b}{2}]\).

We denote this class by \(WSCon[a, b]\).

It is clear that any symmetrized convex function on \([a, b]\) is weak symmetrized convex on that interval. Also, there are weak symmetrized convex functions on \([a, b]\) that are not symmetrized convex on \([a, b]\).

If we consider the function \(f_0 : [a, b] \rightarrow \mathbb{R}\) defined by

\[
f_0(t) = \begin{cases} 
  t^2, & t \in \left[ a, \frac{a+b}{2} \right], \\
  (a + b - t)^2, & t \in \left( \frac{a+b}{2}, b \right], 
\end{cases}
\]

then we observe that \(f_0\) is convex on \([a, \frac{a+b}{2}]\) and \([\frac{a+b}{2}, b]\) but not convex on the whole interval \([a, b]\). We also observe that \(\tilde{f}_0\) is a symmetrical function on \([a, b]\) and then \(\tilde{f}_0 = f_0\). Therefore \(f_0\) is weak symmetrized convex function on \([a, b]\) but not symmetrized convex on that interval.

We have the following strict inclusion

\[
SCon[a, b] \subsetneq WSCon[a, b].
\]

We also notice that \(f\) is weak symmetrized convex function on \([a, b]\) if and only if \(\tilde{f}\) is convex on the second half of the interval \([a, b]\), namely \([\frac{a+b}{2}, b]\).

**Theorem 4.** Assume that \(f : [a, b] \rightarrow \mathbb{R}\) is weak symmetrized convex on the interval \([a, b]\). Then for any \(x, y \in \left[ a, \frac{a+b}{2} \right], x \neq y\) we have the Hermite-Hadamard inequalities (2.7).
In particular, we have

\[
\frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] \leq \frac{1}{b-a} \int_{a}^{b} f(t) dt \leq \frac{1}{2} \left[ f \left( \frac{a+b}{2} \right) + f \left( \frac{a}{2} \right) + f \left( \frac{b}{2} \right) \right]. \tag{2.14}
\]

Proof. The first part follows from the proof of Theorem 3 for \( x, y \in \left[a, \frac{a+b}{2}\right] \). The second part follows from the inequality (2.7) by taking \( x = a \) and \( y = \frac{a+b}{2} \). \( \square \)

REMARK 5. We observe that if \( f : [a, b] \rightarrow \mathbb{R} \) is weak symmetrized convex on the interval \([a, b]\), then the inequality (2.9) holds for any \( x \in [a, \frac{a+b}{2}] \) and integrating on \([a, \frac{a+b}{2}]\) we also have

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{2(b-a)} \int_{a}^{b} \left[ \frac{1}{a+b-x} \int_{x}^{a+b-x} f(t) dt \right] dx \tag{2.15}
\]

We can state in general the following result for symmetrized convex functions.

PROPOSITION 1. Any inequality that holds for convex functions \( f \) defined on the interval \([a, b]\) will hold for symmetrized convex functions by replacing \( f \) with \( \tilde{f}_{[a,b]} \) and performing the required calculations.

We can illustrate this fact with two simple examples.

It is known that, see [19], if \( f : [a, b] \rightarrow \mathbb{R} \) is differentiable convex on \((a, b)\), then for any \( x, y \in (a, b) \) with \( x \neq y \) we have

\[
0 \leq \frac{1}{y-x} \int_{x}^{y} f(t) - f \left( \frac{x+y}{2} \right) \leq \frac{1}{8} \left( f' \left( y \right) - f' \left( x \right) \right) (y-x). \tag{2.16}
\]

Now, if \( f : [a, b] \rightarrow \mathbb{R} \) is differentiable and symmetrized convex on \((a, b)\), then by writing (2.16) for \( \tilde{f}_{[a,b]} \) we have

\[
0 \leq \frac{1}{y-x} \int_{x}^{y} \tilde{f}_{[a,b]}(t) - \tilde{f}_{[a,b]} \left( \frac{x+y}{2} \right) \leq \frac{1}{8} \left( \left( \tilde{f}_{[a,b]} \right)' \left( y \right) - \left( \tilde{f}_{[a,b]} \right)' \left( x \right) \right) (y-x). \tag{2.17}
\]

However

\[
\frac{1}{y-x} \int_{x}^{y} \tilde{f}_{[a,b]}(t) = \frac{1}{2(y-x)} \left[ \int_{x}^{y} f(t) dt + \int_{a+b-y}^{a+b-x} f(t) dt \right],
\]

\[
\tilde{f}_{[a,b]} \left( \frac{x+y}{2} \right) = \frac{1}{2} \left[ f \left( \frac{x+y}{2} \right) + f \left( a+b - \frac{x+y}{2} \right) \right]
\]
and
\[
(f_{a,b}^f)'(y) - (f_{a,b}^f)'(x) = \frac{1}{2} \left( f'(y) - f'(a+b-y) - f'(x) + f'(a+b-x) \right).
\]

Then by (2.17) we get
\[
0 \leq \frac{1}{2(y-x)} \left[ \int_x^y f(t) dt + \int_{a+b-y}^{a+b-x} f(t) dt \right] - \frac{1}{2} \left[ f \left( \frac{x+y}{2} \right) + f \left( a + b - \frac{x+y}{2} \right) \right] \leq \frac{1}{16} \left[ f'(y) - f'(a+b-y) - f'(x) + f'(a+b-x) \right] (y-x)
\]
that holds for any \( x, y \in (a,b) \) with \( x \neq y \).

From this inequality, by taking \( y = a+b-x \), we get
\[
0 \leq \frac{1}{2} \left( \frac{a+b}{2} - x \right) \int_x^{a+b-x} f(t) dt - f \left( \frac{a+b}{2} \right)
\]
\[
\leq \frac{1}{4} \left[ f'(a+b-x) - f'(x) \right] \left( \frac{a+b}{2} - x \right)
\]
for any \( x \in (a,b) \) with \( x \neq \frac{a+b}{2} \).

If \( f : [a,b] \to \mathbb{R} \) is differentiable convex on \( (a,b) \), then for any \( x, y \in (a,b) \) with \( x \neq y \) we also have [20]
\[
0 \leq \frac{f(x) + f(y)}{2} - \frac{1}{y-x} \int_x^y f(t) dt \leq \frac{1}{8} \left( f'(y) - f'(x) \right) (y-x).
\]

Now, if \( f : [a,b] \to \mathbb{R} \) is differentiable and symmetrized convex on \( (a,b) \), then by a similar argument as above we have
\[
0 \leq \frac{1}{4} \left[ f(x) + f(a+b-x) + f(y) + f(a+b-y) \right]
\]
\[
- \frac{1}{2(y-x)} \left[ \int_x^y f(t) dt + \int_{a+b-y}^{a+b-x} f(t) dt \right]
\]
\[
\leq \frac{1}{16} \left[ f'(y) - f'(a+b-y) - f'(x) + f'(a+b-x) \right] (y-x)
\]
for any \( x, y \in (a,b) \) with \( x \neq y \).

In particular, we have
\[
0 \leq \frac{1}{2} \left[ f(x) + f(a+b-x) \right] - \frac{1}{2} \left( \frac{a+b}{2} - x \right) \int_x^{a+b-x} f(t) dt
\]
\[
\leq \frac{1}{4} \left[ f'(a+b-x) - f'(x) \right] \left( \frac{a+b}{2} - x \right)
\]
for any \( x \in (a,b) \) with \( x \neq \frac{a+b}{2} \).
3. Symmetrized $h$-convexity

We recall here some concepts of convexity that are well known in the literature. Let $I$ be an interval in $\mathbb{R}$.

**Definition 3.** ([38]) We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that $f$ belongs to the class $Q(I)$ if $f$ is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y). \quad (3.1)$$

Some further properties of this class of functions can be found in [27], [28], [30], [44], [47] and [48]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

**Definition 4.** ([30]) We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y). \quad (3.2)$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and quasi convex functions, i.e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \quad (3.3)$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on $P$-functions see [30] and [45] while for quasi convex functions, the reader can consult [29].

**Definition 5.** ([7]) Let $s$ be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s$-convex (in the second sense) or Breckner $s$-convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)s f(y) \quad (3.4)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [25], [26], [39], [41] and [50].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of $h$-convex functions as follows.

Assume that $I$ and $J$ are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions $h$ and $f$ are real non-negative functions defined in $J$ and $I$, respectively.

**Definition 6.** ([53]) Let $h : J \to [0,\infty)$ with $h$ not identical to 0. We say that $f : I \to [0,\infty)$ is an $h$-convex function if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \quad (3.4)$$

for all $t \in (0,1)$. 
For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

We can introduce now another class of functions.

**Definition 7.** We say that the function \( f : I \to [0, \infty) \) is of \( s \)-Godunova-Levin type, with \( s \in [0, 1] \), if

\[
f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y),
\]

for all \( t \in (0, 1) \) and \( x, y \in I \).

We observe that for \( s = 0 \) we obtain the class of \( P \)-functions while for \( s = 1 \) we obtain the class of Godunova-Levin. If we denote by \( Q_s(I) \) the class of \( s \)-Godunova-Levin functions defined on \( I \), then we obviously have

\[
P(I) = Q_0(I) \subseteq Q_1(I) \subseteq Q_s(I) \subseteq Q_1(I) = Q(I)
\]

for \( 0 \leq s_1 \leq s_2 \leq 1 \).

The following inequality of Hermite-Hadamard type holds [49].

**Theorem 5.** Assume that the function \( f : I \to [0, \infty) \) is an \( h \)-convex function with \( h \in L[0, 1] \). Let \( y, x \in I \) with \( y \neq x \) and assume that the mapping \([0, 1] \ni t \mapsto f((1-t)x + ty)\) is Lebesgue integrable on \([0, 1]\). Then

\[
\frac{1}{2h(x+y)} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \left[ f(x) + f(y) \right] \int_0^1 h(t) \, dt.
\]

If we write (3.6) for \( h(t) = t \), then we get the classical Hermite-Hadamard inequality for convex functions

\[
f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{2}.
\]

If we write (3.6) for the case of \( P \)-type functions \( f : I \to [0, \infty) \), i.e., \( h(t) = 1, t \in [0, 1] \), then we get the inequality

\[
\frac{1}{2} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq f(x) + f(y),
\]

that has been obtained for functions of real variable in [30].

If \( f \) is Breckner \( s \)-convex on \( I \), for \( s \in (0, 1) \), then by taking \( h(t) = t^s \) in (3.6) we get

\[
2^{s-1} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{s+1},
\]

that was obtained for functions of a real variable in [25].
If \( f : I \to [0, \infty) \) is of \( s \)-Godunova-Levin type, with \( s \in [0, 1) \), then
\[
\frac{1}{2^{s+1}} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{1-s}. \tag{3.10}
\]
We notice that for \( s = 1 \) the first inequality in (3.10) still holds [30], i.e.
\[
\frac{1}{4} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du. \tag{3.11}
\]
We can introduce the following concept generalizing the notion of \( h \)-convexity.

**Definition 8.** Assume that \( h \) is as in Definition 6. We say that the function \( f : [a, b] \to [0, \infty) \) is \( h \)-symmetrized convex (concave) on the interval \( [a, b] \) if the symmetrical transform \( \tilde{f} \) is \( h \)-convex (concave) on \( [a, b] \).

Now, if we denote by \( \text{Con}_h[a,b] \) the closed convex cone of \( h \)-convex functions defined on \( [a, b] \) and by \( S\text{Con}_h[a,b] \) the class of \( h \)-symmetrized convex, then, as in the previous section, we can conclude in general that
\[
\text{Con}_h[a,b] \subset S\text{Con}_h[a,b]. \tag{3.12}
\]

**Definition 9.** Assume that \( h \) is as in Definition 6. We say that the function \( f : [a, b] \to \mathbb{R} \) is \( h \)-weak symmetrized convex (concave) on the interval \( [a, b] \) if the symmetrical transform \( \tilde{f} \) is \( h \)-convex (concave) on the interval \( [a, \frac{a+b}{2}] \).

We denote this class by \( \text{WSCon}_h[a,b] \). As in the previous section, we can conclude in general that
\[
S\text{Con}_h[a,b] \subset \text{WSCon}_h[a,b]. \tag{3.13}
\]

Utilising Theorem 5 and a similar proof to that of Theorem 3, we can state the following result as well:

**Theorem 6.** Assume that the function \( f : [a, b] \to [0, \infty) \) is \( h \)-symmetrized convex on the interval \( [a, b] \) with \( h \) integrable on \( [0, 1] \) and \( f \) integrable on \( [a, b] \). Then for any \( x, y \in [a, b] \) we have the Hermite-Hadamard inequalities
\[
\frac{1}{4h \left( \frac{1}{2} \right)} \left[ f \left( \frac{x+y}{2} \right) + f \left( a+b - \frac{x+y}{2} \right) \right] \leq \frac{1}{2(y-x)} \left[ \int_x^y f(t) \, dt + \int_{a+b-x}^{a+b-y} f(t) \, dt \right] \\
\leq \frac{1}{2} [f(x) + f(a+b-x) + f(y) + f(a+b-y)] \int_0^1 h(t) \, dt. \tag{3.14}
\]

In particular, we have
\[
\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq [f(a) + f(b)] \int_0^1 h(t) \, dt. \tag{3.15}
\]
Remark 6. If, for a given \( x \in [a, b] \), we take \( y = a + b - x \), then from \((3.14)\) we get
\[
\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(\frac{a+b-x}{2}\right) \int_x^{a+b-x} f(t) dt \leq [f(x) + f(a + b - x)] \int_0^1 h(t) dt,
\]
where \( x \neq \frac{a+b}{2} \), provided that \( f : [a, b] \to \mathbb{R} \) is \( h \)-symmetrized convex and integrable on the interval \([a, b]\).

Integrating on \([a, b]\) over \( x \) we get
\[
\frac{1}{4h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{4(b-a)} \int_a^b \left[ \frac{1}{(a+b-x)} \int_x^{a+b-x} f(t) dt \right] dx \leq \frac{1}{b-a} \int_a^b f(x) dx \int_0^1 h(t) dt.
\]

We have the following result as well:

Theorem 7. Assume that \( h \) is as in Definition 6. If the function \( f : [a, b] \to [0, \infty) \) is \( h \)-symmetrized convex on the interval \([a, b]\), then we have the bounds
\[
\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{f(x) + f(a + b - x)}{2} \leq \left[ h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right) \right] \frac{f(a) + f(b)}{2}
\]
for any \( x \in [a, b] \).

Proof. Since \( \tilde{f} \) is \( h \)-convex on \([a, b]\) then for any \( x \in [a, b] \) we have
\[
h\left(\frac{1}{2}\right) [\tilde{f}(x) + \tilde{f}(a + b - x)] \geq \tilde{f}\left(\frac{a+b}{2}\right)
\]
and since
\[
\frac{\tilde{f}(x) + \tilde{f}(a + b - x)}{2} = \frac{1}{2} [f(x) + f(a + b - x)]
\]
while
\[
\tilde{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),
\]
we get the first inequality in \((2.4)\).
Also, by the convexity of $\tilde{f}$ we have for any $x \in [a, b]$ that

$$\tilde{f}(x) \leq h \left( \frac{x-a}{b-a} \right) \cdot \tilde{f}(b) + h \left( \frac{b-x}{b-a} \right) \cdot \tilde{f}(a)$$

$$= h \left( \frac{x-a}{b-a} \right) \cdot \frac{f(a) + f(b)}{2} + h \left( \frac{b-x}{b-a} \right) \cdot \frac{f(a) + f(b)}{2}$$

$$= \left[ h \left( \frac{b-x}{b-a} \right) + h \left( \frac{x-a}{b-a} \right) \right] \frac{f(a) + f(b)}{2},$$

which proves the second part of (3.18). $\square$

**Corollary 2.** Assume that the function $f : [a, b] \to [0, \infty)$ is $h$-symmetrized convex on the interval $[a, b]$ with $h$ integrable on $[0, 1]$ and $f$ integrable on $[a, b]$. If $w : [a, b] \to [0, \infty)$ is integrable on $[a, b]$, then

$$\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{a+b}{2} \right) \int_a^b w(t) dt$$

$$\leq \frac{1}{2} \int_a^b w(t) [f(t) + f(a+b-t)] dt$$

$$\leq \frac{f(a) + f(b)}{2} \int_a^b h \left( \frac{t-a}{b-a} \right) [w(t) + w(a+b-t)] dt.$$ 

Moreover, if $w$ is symmetric almost everywhere on $[a, b]$, then

$$\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{a+b}{2} \right) \int_a^b w(t) dt \leq \int_a^b w(t) f(t) dt$$

$$\leq [f(a) + f(b)] \int_a^b h \left( \frac{t-a}{b-a} \right) w(t) dt.$$

**Proof.** From (3.18) we have

$$\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{a+b}{2} \right) \leq \frac{f(t) + f(a+b-t)}{2}$$

$$\leq \left[ h \left( \frac{b-t}{b-a} \right) + h \left( \frac{t-a}{b-a} \right) \right] \frac{f(a) + f(b)}{2}$$

for any $t \in [a, b]$.

Multiplying with $w(t) \geq 0$ and integrating over $t \in [a, b]$ we get

$$\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{a+b}{2} \right) \int_a^b w(t) dt$$

$$\leq \frac{1}{2} \int_a^b w(t) [f(t) + f(a+b-t)] dt$$

$$\leq \frac{f(a) + f(b)}{2} \int_a^b \left[ h \left( \frac{b-t}{b-a} \right) + h \left( \frac{t-a}{b-a} \right) \right] w(t) dt.$$
Observe that, by changing the variable 
\[ t = a + b - s, \quad s \in [a, b], \]
we have
\[ \int_a^b h \left( \frac{b - t}{b - a} \right) w(t) dt = \int_a^b h \left( \frac{s - a}{b - a} \right) w(a + b - s) ds, \]
then we get
\[ \int_a^b \left[ h \left( \frac{b - t}{b - a} \right) + h \left( \frac{t - a}{b - a} \right) \right] w(t) dt \]
\[ = \int_a^b h \left( \frac{t - a}{b - a} \right) [w(t) + w(a + b - t)] dt \]
and by (3.21) we obtain the second part of (3.19).

Utilising the previous examples of \( h \)-convex functions the reader may state various
inequalities of Hermite-Hadamard type.

For instance, if we assume that the functions
\[ f : [a, b] \rightarrow [0, \infty) \]
is integrable and
of symmetrized Godunova-Levin type, then for the symmetric weight
\[ w : [a, b] \rightarrow [0, \infty), \quad w(t) = (t - a)(b - t) \]
we have from (3.20) that
\[ \frac{1}{4} f \left( \frac{a + b}{2} \right) \int_a^b (t - a)(b - t) dt \leq \int_a^b (t - a)(b - t) f(t) dt \]
\[ \leq [f(a) + f(b)] (b - a) \int_a^b (b - t) dt \]
and since
\[ \int_a^b (t - a)(b - t) dt = \frac{1}{6} (b - a)^3, \quad \int_a^b (b - t) dt = \frac{1}{2} (b - a)^2, \]
then we get the following inequality of interest:
\[ \frac{1}{24} f \left( \frac{a + b}{2} \right) (b - a)^3 \leq \int_a^b (t - a)(b - t) f(t) dt \leq \frac{f(a) + f(b)}{2} (b - a)^3. \quad (3.22) \]

Moreover, if we assume that the function \( f : [a, b] \rightarrow [0, \infty) \) is integrable and symmetrized Breckner \( s \)-convex with \( s \in (0, 1) \), then for the symmetric weight
\[ w : [a, b] \rightarrow [0, \infty), \quad w(t) = (t - a)(b - t) \]
we have from (3.20) that
\[ \frac{1}{2^{1-s} f \left( \frac{a + b}{2} \right)} \int_a^b (t - a)(b - t) dt \]
\[ \leq \int_a^b (t - a)(b - t) f(t) dt \]
\[ \leq \frac{f(a) + f(b)}{(b - a)^s} \int_a^b (t - a)^{s+1} (b - t) dt \]
and since
\[ \int_a^b (t-a)^{s+1} (b-t) \ dt = \frac{(b-a)^{s+3}}{(s+2)(s+3)} \]
then we get the following inequality of interest:
\[ \frac{1}{2^{s+3}} f\left(\frac{a+b}{2}\right)(b-a)^3 \leq \int_a^b (t-a)(b-t)f(t)\ dt \tag{3.23} \]
\[ \leq \frac{f(a) + f(b)}{(s+2)(s+3)} (b-a)^3. \]

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