Robust $l_2-l_\infty$ Filtering for Discrete-Time Delay Systems

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Abstract

The problem of robust $l_2-l_\infty$ filtering for discrete-time system with interval time-varying delay and uncertainty is investigated, where the time delay and uncertainty considered are varying in a given interval and norm-bounded, respectively. The filtering problem based on the $l_2-l_\infty$ performance is to design a filter such that the filtering error system is asymptotically stable with minimizing the peak value of the estimation error for all possible bounded energy disturbances. Firstly, sufficient $l_2-l_\infty$ performance analysis condition is established in terms of linear matrix inequalities (LMIs) for discrete-time delay systems by utilizing reciprocally convex approach. Then a less conservative result is obtained by introducing some variables to decouple the Lyapunov matrices and the filtering error system variables. Moreover, the robust $l_2-l_\infty$ filter is designed for systems with time-varying delay and uncertainty. Finally, a numerical example is given to demonstrate the effectiveness of the filter design method.

1. Introduction

The uncertainty is unavoidable in practical engineering due to the parameter drafting, modeling error, and component aging. The controllers or filtering obtained based on nominal systems cannot be employed to get the desired performance. Therefore, more and more researchers are devoted to robust control or robust filtering problems; see, for instance, [1–4]. On the other hand, time-delay often exists in the practical engineering systems and is the main reason of the instability and poor performance of the systems. Time-delay systems have been widely studied during the past two decades [5–7]. In order to get less conservative results, more and more approaches have been proposed to develop delay-dependent conditions for discrete-time system with time-varying delay. For examples, Jensen’s inequality is proposed in [8]; delay-partitioning method is utilized in [9]; improved results are obtained by using convex combination approach in [10].

In some practical applications, the peak value of the estimation error is required to be within a certain range and the aim of the $l_2-l_\infty$ (energy-to-peak) filtering is to minimize the peak values of the filtering error for any bounded energy disturbance, which has received many attention. By using a parameter-dependent approach, the robust energy-to-peak filtering problem is considered in [11]. An improved robust energy-to-peak filtering condition is proposed by increasing the flexible dimensions in the solution space in [12]. The robust $L_2-L_\infty$ filtering for stochastic systems and the exponential $L_2-L_\infty$ filtering for Markovian jump systems are investigated in [13, 14], respectively. Compared with the corresponding continuous-time systems, discrete-time systems with time-varying delay have more stronger application background [15]. For discrete-time Markovian jumping systems, the reduced-order filter is designed in [16] such that the filtering error system satisfies an energy-to-peak performance. When time-delay appears, the robust energy-to-peak filtering problem for networked systems is tackled in [17]. For discrete-time switched systems with time-varying delay, an improved robust energy-to-peak filtering design method is proposed in [18].
In this paper we consider the problem of robust $l_2$-$l_\infty$ filtering for uncertain discrete-time systems with time-varying delay. The filter is employed by utilizing the reciprocally convex approach proposed in [19] such that the filtering error system is asymptotically stable with an $l_2$-$l_\infty$ performance. Firstly, a sufficient condition of the $l_2$-$l_\infty$ performance analysis for nominal systems is obtained in terms of LMIs for systems with time-varying delay and uncertainty. Based on this criterion, by introducing some slack matrices, a less conservative result is obtained. Moreover, the desired filter for nominal systems with time-varying delay is obtained by solving a set of LMIs. Then the result is extended to the uncertain systems. A numerical example is given to illustrate the effectiveness of the presented results.

**Notation.** The notation used throughout the paper is given as follows. $\mathbb{R}^n$ is the $n$-dimensional Euclidean space and $P > 0$ ($\geq 0$) denotes that matrix $P$ is real symmetric and positive definite (semidefinite); $I$ and $0$ present the identity matrix and zero matrix with compatible dimensions, respectively; $\ast$ means the symmetric terms in a symmetric matrix and sym$(A)$ stands for $A + A^T$; $l_2$ means the space of square summable infinite vector sequences; for any real function $x \in l_2$, we define $\|x\|_\infty = \sup_k \sqrt{x^T(k)x(k)}$ and $\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} x^T(k)x(k)}$; $\|\cdot\|$ refer to the Euclidean vector norm. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

## 2. Problem Statement

Consider a class of uncertain discrete-time systems with time-varying delay described by

\[
\begin{align*}
x(k+1) &= A(\sigma)x(k) + A_d(\sigma)x(k-d(k)) + B(\sigma)u(k), \\
y(k) &= C(\sigma)x(k) + C_d(\sigma)x(k-d(k)) + D(\sigma)w(k), \\
z(k) &= L(\sigma)x(k) + L_d(\sigma)x(k-d(k)) + G(\sigma)w(k), \\
x(k) &= \phi(k), \quad k = -d_2, -d_2 + 1, \ldots, 0,
\end{align*}
\]

(1)

where $x(k) \in \mathbb{R}^n$ is the state vector; $y(k) \in \mathbb{R}^m$ is the measured output; $z(k) \in \mathbb{R}^p$ represents the signal to be estimated; $u(k) \in \mathbb{R}^l$ is assumed to be an arbitrary noise belonging to $l_2$ and $\phi(k)$ is a given initial condition sequence; $d(k)$ is a time-varying delay satisfying

\[
1 \leq d_1 \leq d(k) \leq d_2 < \infty, \quad k = 1, 2, \ldots
\]

(2)

$A(\sigma), A_d(\sigma), B(\sigma), C(\sigma), C_d(\sigma), D(\sigma), L(\sigma), L_d(\sigma),$ and $G(\sigma)$ are system matrices and satisfy

\[
A(\sigma) = A + \Delta A(\sigma), \quad A_d(\sigma) = A_d + \Delta A_d(\sigma), \quad B(\sigma) = B + \Delta B(\sigma), \quad C(\sigma) = C + \Delta C(\sigma), \quad C_d(\sigma) = C_d + \Delta C_d(\sigma), \quad D(\sigma) = D + \Delta D(\sigma),
\]

(3)

Matrices $\Delta A(\sigma), \Delta A_d(\sigma), \Delta B(\sigma), \Delta C(\sigma), \Delta C_d(\sigma), \Delta D(\sigma), \Delta L(\sigma), \Delta L_d(\sigma),$ and $\Delta G(\sigma)$ are unknown time-invariant matrices representing the uncertainty of the system satisfying the following conditions:

\[
\begin{align*}
[\Delta A(\sigma) \Delta A_d(\sigma) \Delta B(\sigma) \Delta C(\sigma) \Delta C_d(\sigma) \Delta D(\sigma)] &= M_1 \Delta_1(\sigma) [N_A \; N_{A_d} \; N_B], \\
\Delta_1(\sigma) \Delta_1(\sigma) &\leq I, \\
[\Delta C(\sigma) \Delta C_d(\sigma) \Delta D(\sigma)] &= M_2 \Delta_2(\sigma) [N_C \; N_{C_d} \; N_D], \\
\Delta_2(\sigma) \Delta_2(\sigma) &\leq I, \\
[\Delta L(\sigma) \Delta L_d(\sigma) \Delta G(\sigma)] &= M_3 \Delta_3(\sigma) [N_L \; N_{L_d} \; N_G], \\
\Delta_3(\sigma) \Delta_3(\sigma) &\leq I
\end{align*}
\]

(4)

where $\sigma \in \Theta$ and $\Theta$ is a compact set in $\mathbb{R}$. The system in (1) is assumed to be asymptotically stable. Our purpose is to design a full order linear filter for the estimate of $z(k)$:

\[
\begin{align*}
\hat{x}(k+1) &= A_f \hat{x}(k) + B_f y(k), \quad \hat{x}(0) = 0, \\
\hat{z}(k) &= C_f \hat{x}(k) + D_f y(k),
\end{align*}
\]

(5)

where $A_f, B_f, C_f,$ and $D_f$ are filter gains to be determined.

Let the augmented state vector $\tilde{x}(k) = [x^T(k) \; \hat{x}^T(k)]^T$ and $\tilde{z}(k) = z(k) - \hat{z}(k).$ Then the filtering error system is described as

\[
\begin{align*}
\tilde{x}(k+1) &= \tilde{A}(\sigma) \tilde{x}(k) + \tilde{A}_d(\sigma) \Phi \tilde{x}(k-d(k)) + \tilde{B}(\sigma)w(k), \\
\tilde{z}(k) &= \tilde{L}(\sigma) \tilde{x}(k) + \tilde{L}_d(\sigma) \Phi \tilde{x}(k-d(k)) + \tilde{G}(\sigma)w(k), \\
\tilde{x}(k) &= [\phi^T(k) \; 0]^T, \quad k = -d_2, -d_2 + 1, \ldots, 0,
\end{align*}
\]

(6)

where $\Phi = [I \; 0]$ and

\[
\begin{align*}
\tilde{A}(\sigma) &= \begin{bmatrix} A(\sigma) & 0 \\ B_f C(\sigma) & A_f \end{bmatrix}, \quad \tilde{A}_d(\sigma) = \begin{bmatrix} A_d(\sigma) & \ast \\ B_f C_d(\sigma) & \ast \end{bmatrix}, \\
\tilde{B} &= \begin{bmatrix} B(\sigma) \\ B_f D(\sigma) \end{bmatrix}, \\
\tilde{L}(k) &= L(\sigma) - D_f C(\sigma) - C_f, \\
\tilde{L}_d(\sigma) &= L_d(\sigma) - D_f C_d(\sigma), \quad \tilde{G}(\sigma) = G(\sigma) - D_f D(\sigma).
\end{align*}
\]

(7)
The nominal system of (6) is system (6) without uncertainty; that is, $\Delta A(\sigma) = 0$, $\Delta A_d(\sigma) = 0$, $\Delta B(\sigma) = 0$, $\Delta C(\sigma) = 0$, $\Delta C_d(\sigma) = 0$, $\Delta D(\sigma) = 0$, $\Delta L(\sigma) = 0$, $\Delta L_d(\sigma) = 0$, and $\Delta G(\sigma) = 0$.

The following lemmas and definition will be utilized in the derivation of the main results.

**Lemma 1** (see [20]). For any matrices $U$ and $V > 0$, the following inequality holds:

$$UV^{-1}U^T \succeq U + U^T - V.$$  \hspace{1cm} (8)

**Lemma 2** (see [19]). Let $f_1, f_2, \ldots, f_N : \mathbb{R}^m \to \mathbb{R}$ have positive values in a subset $D$ of $\mathbb{R}^m$. Then, the reciprocally convex combination of $f_i$ over $D$ satisfies

$$\min_{\{a_i, a > 0, \sum \alpha_i = 1\}} \sum_{i} \alpha_i f_i(k) \geq \sum_{i} \alpha_i \left[ \sum_{j} g_{ij}(k) + \max_{i \in [i]} \sum_{j \neq i} g_{ij}(k) \right].$$  \hspace{1cm} (9)

**Lemma 3.** For any constant matrix $M > 0$, integers $a \leq b$, vector function $w : [a, a + 1, \ldots, b] \to \mathbb{R}^n$, then

$$-(b - a + 1) \sum_{i=a}^{b} w^T(i)Mw(i) \leq \left( \sum_{i=a}^{b} w(i) \right)^T M \left( \sum_{i=a}^{b} w(i) \right).$$  \hspace{1cm} (11)

**Lemma 4.** Given a symmetric matrix $Q$ and matrices $H, E$ with appropriate dimensions, then

$$Q + \text{sym}(HFE) < 0,$$  \hspace{1cm} (12)

for all $F^T F \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that

$$Q + \epsilon H E + \epsilon^{-1} HH^T < 0.$$  \hspace{1cm} (13)

**Definition 5.** Given a scalar $\gamma > 0$, the filtering error $\tilde{z}(k)$ in (6) is said to satisfy the $l_2-l_\infty$ disturbance attenuation level $\gamma$ under zero initial state, and the following condition is satisfied:

$$\|\tilde{z}\|_{l_\infty} < \gamma \|w\|_2.$$  \hspace{1cm} (14)

Our aim is to design a filter in the form of (5) such that the filtering error system in (6) is asymptotically stable and satisfies the $l_2-l_\infty$ performance defined in Definition 5.

### 3. Main Results

In this section, the sufficient $l_2-l_\infty$ performance analysis condition is first derived for nominal filtering error system of (6). Then an equivalent result is obtained by introducing three slack matrices. Based on these results, a desired filter is designed to render the nominal system of (6) asymptotically stable with an $l_2-l_\infty$ performance. Then the result is extended to the uncertain system in (6).

#### 3.1. $l_2-l_\infty$ Performance Analysis

In this subsection, we first give the result of $l_2-l_\infty$ performance analysis for nominal system of (6).

**Theorem 6.** Given a scalar $\gamma > 0$, the nominal system of (6) is asymptotically stable with an $l_2-l_\infty$ performance if there exist matrices $Q_1 > 0$, $Q_2 > 0$, $i = 1, 2$, $S_j > 0$, $j = 1, 2$, and $M$ such that the following LMIs hold:

$$\begin{bmatrix} S_2 & M \\ * & S_2 \end{bmatrix} \succeq 0,$$  \hspace{1cm} (15)

$$\begin{bmatrix} P & 0 & 0 & \tilde{I}^T \\ * & Q_3 & \tilde{T}_d & 0 \\ * & * & I & \tilde{C}^T \\ * & * & * & \gamma^2 I \end{bmatrix} \succeq 0,$$  \hspace{1cm} (16)

where

$$\begin{bmatrix} \Pi_{11} & \Phi^T S_1 & 0 & 0 & 0 & 0 & \Phi^T S_1 & \Phi \\ \Pi_{12} & \Pi_{21} & M^T & 0 & 0 & 0 & 0 & S_2 - M^T \\ \Pi_{13} & \Pi_{31} & \Pi_{33} & \Pi_{44} & 0 & d_1 A_d^T S_1 & d_1 A_d^T S_2 & \Pi_{44}^T A_d^T P \\ \Pi_{14} & \Pi_{41} & \Pi_{43} & \Pi_{44} & 0 & 0 & 0 & S_2 - M^T \\ \Pi_{22} & \Pi_{22} & \Pi_{22} & \Pi_{22} & \Pi_{22} & \Pi_{22} & \Pi_{22} & \Pi_{22} \\ \Pi_{23} & \Pi_{23} & \Pi_{23} & \Pi_{23} & \Pi_{23} & \Pi_{23} & \Pi_{23} & \Pi_{23} \\ \Pi_{32} & \Pi_{32} & \Pi_{32} & \Pi_{32} & \Pi_{32} & \Pi_{32} & \Pi_{32} & \Pi_{32} \\ \Pi_{33} & \Pi_{33} & \Pi_{33} & \Pi_{33} & \Pi_{33} & \Pi_{33} & \Pi_{33} & \Pi_{33} \end{bmatrix} \succeq 0,$$  \hspace{1cm} (17)

$$\begin{bmatrix} \phi & \Phi^T Q_4 \Phi - \Phi^T S_1 \Phi, \\ \phi & -Q_3 - 2S_2 + \text{sym}(M), \\ \phi & -Q_3 - 2S_2 + \text{sym}(M), \\ \phi & -Q_3 - 2S_2 + \text{sym}(M), \\ \phi & -Q_3 - 2S_2 + \text{sym}(M), \\ \phi & -Q_3 - 2S_2 + \text{sym}(M), \\ \phi & -Q_3 - 2S_2 + \text{sym}(M), \\ \phi & -Q_3 - 2S_2 + \text{sym}(M), \end{bmatrix} \succeq 0,$$  \hspace{1cm} (18)

$$Q_i = \text{diag}\{Q_i, Q_i\}, \quad i = 1, 2, \quad \tilde{d} = d_2 - d_1.$$  \hspace{1cm} (19)

**Proof.** First, the asymptotic stability of the nominal system of (6) is proved. We denote $\tilde{y}(k) = \tilde{z}(k + 1) - \tilde{z}(k)$ and the following Lyapunov functional is chosen:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k),$$  \hspace{1cm} (19)

where

$$V_1(k) = \tilde{z}^T(k) P \tilde{z}(k),$$  \hspace{1cm} (20)

$$V_2(k) = \sum_{j=1}^{3} \sum_{i=k-d_j}^{k-1} \tilde{x}^T(i) Q_j \tilde{x}(i),$$  \hspace{1cm} (21)
\[ V_3(k) = \sum_{i=k-d(k)}^{k-d(k)} \tilde{x}^T(i) \Phi^T Q_3 \Phi \tilde{x}(i) \]
\[ + \sum_{j=-d_1}^{-d_1} \sum_{i=k+j}^{k-1} \tilde{x}^T(i) \Phi^T Q_3 \Phi \tilde{x}(i), \]
\[ V_4(k) = \sum_{j=-d_1}^{-d_1} \sum_{i=k+j}^{k-1} d_1 \eta^T(i) \Phi^T Q_3 \eta(i), \]
\[ V_5(k) = \sum_{j=-d_1}^{-d_1} \sum_{i=k+j}^{k-1} d_2 \eta^T(i) \Phi^T S_2 \eta(i). \]

Calculating the forward difference of \( V(k) \) along the trajectories of filtering error system (6) with \( w(k) = 0 \) yields
\[ \Delta V_1(k) = \tilde{x}^T(k+1) P \tilde{x}(k+1) - \tilde{x}^T(k) P \tilde{x}(k) \]
\[ = \left( \tilde{\Phi}(k) + \tilde{\Phi}(k-d(k)) \right)^T \times P \left( \tilde{\Phi}(k) + \tilde{\Phi}(k-d(k)) \right) \]
\[ - \tilde{x}^T(k) P \tilde{x}(k), \]
\[ \Delta V_2(k) = \sum_{j=1}^{2} \tilde{x}^T(k) Q_j \tilde{x}(k) \]
\[ = \sum_{j=1}^{2} \left( \tilde{x}(k-d_j) Q_j \tilde{x}(k-d_j) \right) \]
\[ \leq \sum_{j=1}^{2} \left( \tilde{x}^T(k) Q_j \tilde{x}(k) \right) \]
\[ - \sum_{j=1}^{2} \left( \tilde{x}(k-d_j) \Phi^T Q_j \Phi \tilde{x}(k-d_j) \right), \]
\[ \Delta V_3(k) = (\tilde{d} + 1) \tilde{x}^T(k) \Phi^T Q_3 \Phi \tilde{x}(k) \]
\[ + \sum_{i=k+1}^{k-1} \tilde{x}^T(i) \Phi^T Q_3 \Phi \tilde{x}(i) \]
\[ - \sum_{i=k+1}^{k-1} \tilde{x}^T(i) \Phi^T Q_3 \Phi \tilde{x}(i) \]
\[ - \tilde{x}^T(k-d(k)) \Phi^T Q_3 \Phi \tilde{x}(k-d(k)) \]
\[ - \sum_{i=k-d_1+1}^{k-d_1} \tilde{x}^T(i) \Phi^T Q_3 \Phi \tilde{x}(i), \]
\[ \Delta V_4(k) = d_1 \eta^T(k) \Phi^T S_1 \Phi \eta(k) \]
\[ = d_1 \sum_{i=k-d_1+1}^{k-1} \eta^T(i) \Phi^T S_1 \Phi \eta(i) \]
\[ \leq d_1 \left( (A-I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k-d(k)) \right)^T \]
\[ \times S_1 \left( (A-I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k-d(k)) \right) \]
\[ - \left( \Phi \tilde{x}(k-d_1) - \Phi \tilde{x}(k-d(k)) \right)^T \]
\[ \times S_1 \left( \Phi \tilde{x}(k-d_1) - \Phi \tilde{x}(k-d(k)) \right), \]
\[ \Delta V_5(k) = d_2 \eta^T(k) \Phi^T S_2 \eta(k) \]
\[ = d_2 \sum_{i=k-d_2+1}^{k-1} \eta^T(i) \Phi^T S_2 \eta(i) \]
\[ \leq d_2 \left( (A-I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k-d(k)) \right)^T \]
\[ \times S_2 \left( (A-I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k-d(k)) \right) \]
\[ - \left( \Phi \tilde{x}(k-d_1) - \Phi \tilde{x}(k-d(k)) \right)^T \]
\[ \times S_2 \left( \Phi \tilde{x}(k-d_1) - \Phi \tilde{x}(k-d(k)) \right), \]
\[ \text{Since } \left[ \begin{array}{cc} S_1 & M \\ S_2 & S_2 \end{array} \right] \geq 0, \text{ the following inequality holds:} \]
\[ \begin{bmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} \\ -\sqrt{\alpha_1} & \sqrt{\alpha_2} \end{bmatrix} \begin{bmatrix} x(k-d(k)) - x(k-d_1) \\ x(k-d(k)) - x(k-d_1) \end{bmatrix} \]
\[ \times \begin{bmatrix} S_1 & M \\ S_2 & S_2 \end{bmatrix} \begin{bmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} \\ -\sqrt{\alpha_1} & \sqrt{\alpha_2} \end{bmatrix} \begin{bmatrix} x(k-d(k)) - x(k-d_1) \\ x(k-d(k)) - x(k-d_1) \end{bmatrix} \geq 0, \]
where \( \alpha_1 = (d_2 - d(k))/\bar{d} \) and \( \alpha_2 = (d(k) - d_1)/\bar{d} \). Then employing Lemma 2, for \( d_1 < d(k) < d_2 \), we have

\[
\Delta V_2(k) = \tilde{d}^2 \overline{\eta}^T(k) \Phi^T S_2 \Phi \eta(k) \\
- \frac{\tilde{d}}{d_2 - d(k)} \left( \sum_{i=k-d_1}^{k-d_2-1} \Phi \eta(i) \right) S_2 \left( \sum_{i=k-d_1}^{k-d_2-1} \Phi \eta(i) \right)^T \\
- \frac{\tilde{d}}{d(k) - d_1} \left( \sum_{i=k-d(k)}^{k-d_1-1} \Phi \eta(i) \right) S_2 \left( \sum_{i=k-d(k)}^{k-d_1-1} \Phi \eta(i) \right)^T \\
\leq \tilde{d}^2 \overline{\eta}^T(k) \Phi^T S_2 \Phi \eta(k) \\
- \frac{\tilde{d}}{d_2 - d(k)} \left( \sum_{i=k-d_1}^{k-d_2-1} \Phi \eta(i) \right) S_2 \left( \sum_{i=k-d_1}^{k-d_2-1} \Phi \eta(i) \right)^T \\
- \frac{\tilde{d}}{d(k) - d_1} \left( \sum_{i=k-d(k)}^{k-d_1-1} \Phi \eta(i) \right) S_2 \left( \sum_{i=k-d(k)}^{k-d_1-1} \Phi \eta(i) \right)^T \\
\leq \tilde{d}^2 ((A - I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k - d(k)))^T \\
\times S_2 ((A - I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k - d(k))) \\
- \left[ x(k - d(k)) - x(k - d_1) \right]^T \left[ S_1 \ M \right] \\
\times \left[ x(k - d(k)) - x(k - d_1) \right] \\
\times \left[ x(k - d(k)) - x(k - d_2) \right] \\
\times \left[ x(k - d(k)) - x(k - d_2) \right]. \\
\tag{26}
\]

Note that when \( d(k) = d_1 \) or \( d(k) = d_2 \), it yields \( x(k - d_1) - x(k - d(k)) = 0 \) or \( x(k - d(k)) - x(k - d_2) = 0 \). Hence, the inequality in (24) still holds. Combining the conditions from (21) to (24), we have

\[
\Delta V(k) = \xi^T(k) \Pi_1 \xi(k), \\
\tag{27}
\]

where

\[
\xi(k) = \overline{x}^T(k) \overline{x}^T(k - d_1) \Phi^T \overline{x}^T(k - d(k)) \Phi \overline{x}^T(k - d_2) \Phi^T, \\
\Pi_1 = \begin{bmatrix} \overline{\Pi}_{11} & 0 & 0 & \overline{\Pi}_{16} \\ * & \overline{\Pi}_{22} & \overline{\Pi}_{23} & M^T \\ * & * & \overline{\Pi}_{33} & \overline{\Pi}_{34} \\ * & * & * & \overline{\Pi}_{44} \end{bmatrix} \begin{bmatrix} \overline{\Pi}_{16} \\ \overline{\Pi}_{17} \\ \overline{\Pi}_{18} \\ \overline{\Pi}_{19} \end{bmatrix} \begin{bmatrix} \overline{\Pi}_{17} \\ \overline{\Pi}_{18} \\ \overline{\Pi}_{19} \\ A_t \end{bmatrix} \begin{bmatrix} A_t^T \\ A_t^T \\ A_t^T \\ A_t^T \end{bmatrix}. \\
\tag{28}
\]

On the other hand, the following inequality can be obtained from (17):

\[
\Pi_{d1} = \begin{bmatrix} \overline{\Pi}_{11} & \Phi^T S_1 & 0 & 0 & \overline{\Pi}_{16} & \overline{\Pi}_{17} & 0 & 0 & \overline{\Pi}_{19} & A_t^T \end{bmatrix} \begin{bmatrix} \overline{\Pi}_{16} & \overline{\Pi}_{17} \\ \overline{\Pi}_{18} & \overline{\Pi}_{19} \end{bmatrix} \begin{bmatrix} A_t^T \\ A_t^T \\ A_t^T \\ A_t^T \end{bmatrix} < 0. \\
\tag{29}
\]

which is equivalent to \( \Pi_{d1} < 0 \). Hence, \( \Delta V(k) < 0 \) which implies that the filtering error system in (6) with \( w(k) = 0 \) is asymptotically stable.

Next, we show the \( l_2-l_\infty \) performance of system (6). To this end, we define

\[
J(k) = V(k) - \sum_{i=0}^{k-1} w^T(i) w(i). \\
\tag{30}
\]

Then under the zero initial condition, that is, \( x(k) = 0, k = -d_2, -d_2 + 1, \ldots, 0 \), it can be shown that for any nonzero \( w(k) \in l_2[0, \infty) \),

\[
J(k) = \sum_{i=0}^{k-1} \left[ \Delta V(i) - w^T(i) w(i) \right] \\
= \sum_{i=0}^{k-1} \xi^T(i) \left( \Pi + \Pi_{d1} p \Pi_{d1} + d_{12}^T \Pi_{23} S_1 \Pi_{24} + d_{12}^T \Pi_{23} S_2 \Pi_{24} \right) \xi(i), \\
\tag{31}
\]

where

\[
\xi(i) = \left[ \overline{x}^T(i) \overline{x}^T(i - d_1) \Phi^T \overline{x}^T(i - d_0) \Phi \overline{x}^T(i - d_2) \Phi^T \right] w(i), \\
\Pi = \begin{bmatrix} \overline{\Pi}_{11} & \Phi^T S_1 & 0 & 0 & \overline{\Pi}_{16} \\ * & \overline{\Pi}_{22} & \overline{\Pi}_{23} & M^T & 0 \\ * & * & \overline{\Pi}_{33} & \overline{\Pi}_{34} & -T_d S \\ * & * & * & \overline{\Pi}_{44} & 0 \\ * & * & * & * & \overline{\Pi}_{55} \end{bmatrix} \\
\Pi_{d1} = \begin{bmatrix} \overline{\Pi}_{11} \\ \overline{\Pi}_{12} \\ \overline{\Pi}_{13} \\ \overline{\Pi}_{14} \end{bmatrix} \begin{bmatrix} \overline{\Pi}_{16} \\ \overline{\Pi}_{17} \\ \overline{\Pi}_{18} \\ \overline{\Pi}_{19} \end{bmatrix} \begin{bmatrix} \overline{\Pi}_{17} \\ \overline{\Pi}_{18} \\ \overline{\Pi}_{19} \\ A_t \end{bmatrix} \begin{bmatrix} A_t^T \\ A_t^T \\ A_t^T \\ A_t^T \end{bmatrix}. \\
\tag{32}
\]

By using Schur complement equivalence, the inequality in (17) is equivalent to \( \Pi + \Pi_{d1} p \Pi_{d1} + d_{12}^T \Pi_{23} S_1 \Pi_{24} + d_{12}^T \Pi_{23} S_2 \Pi_{24} < 0 \). Then we have \( J(k) < 0 \); that is,

\[
V(k) < \sum_{i=0}^{k-1} w^T(i) w(i). \\
\tag{33}
\]
On the other hand, it yields from (16) and (33) that
\[
\tilde{z}^T(k) \tilde{z}(k) = \eta^T(k) \begin{bmatrix} L & L_d & G \end{bmatrix}^T \begin{bmatrix} L & L_d & G \end{bmatrix} \eta(k)
\]
\[
\leq \gamma^2 \eta^T(k) \begin{bmatrix} P & 0 & 0 \\ * & Q_3 & 0 \\ * & * & I \end{bmatrix} \eta(k)
\]
\[
\leq \gamma^2 \left( V(k) + w^T(k) w(k) \right)
\]
\[
\leq \gamma^2 \sum_{i=0}^{\infty} w^T(i) w(i)
\]
where
\[
\eta(k) = \begin{bmatrix} \tilde{x}(k) \\ \Phi \tilde{x}(k-d(k)) \\ w(k) \end{bmatrix}.
\]

Then, we have \( \|\tilde{z}\|_\infty < \gamma \|w\|_2 \) by taking the supremum over time \( k > 0 \). By Definition 5, the filtering error \( \tilde{z}(k) \) satisfies a given \( l^2 - l^\infty \) disturbance attenuation level. This completes the proof.

**Remark 7.** The advantage of the results benefits from utilizing the reciprocally convex combination approach proposed in [19]. For the extensively used Jensen inequality [8], the integral term
\[
- \sum_{i=k-d_1}^{k-d_1-1} (d_2 - d_1) \eta^T(i) S \eta(i)
\]
\[
= - \sum_{i=k-d_2}^{k-(d_2-1)} (d_2 - d_1) \eta^T(i) S \eta(i)
\]
with \( d_1 \leq d(k) \leq d_2 \), \( \eta(i) = x(i+1) - x(i) \) by the term
\[
- \left[ x(k-d_1) - x(k-d(d(k))) \right]^T S \left[ x(k-d_1) - x(k-d(d(k))) \right]
\]
\[
- \left[ x(k-d(d(k))) - x(k-d_2) \right]^T S \left[ x(k-d(d(k))) - x(k-d_2) \right]
\]
which is a special case of the following term with \( M = 0 \)
\[
\begin{bmatrix} \gamma \left( V(k) + w^T(k) w(k) \right) \leq \gamma \sum_{i=0}^{\infty} w^T(i) w(i) \end{bmatrix}
\]
with \( \left[ S M \right] \geq 0 \), which is one of the advantages of reciprocally convex combination approach. On the other hand, the delay partitioning method is widely applied to reduce the conservatism of the results [9, 21, 22]. Also, the method can be extended to the problem considered in this paper. However, it will rise significant computation cost with the partitioning number increasing. Therefore, the reciprocally convex method needs less decision variables and can be seen as a tradeoff between the conservatism and the computation cost.

Then, an equivalent condition of LMI (17) is obtained by introducing three slack matrices \( H_1, H_2, \) and \( T \), which is presented in the following theorem.

**Theorem 8.** Given a scalar \( \gamma > 0 \), the nominal system of (6) is asymptotically stable with an \( l^2 - l^\infty \) performance if there exist matrices \( P > 0 \), \( Q_i > 0 \), \( i = 1,2,3 \), \( S_j > 0 \), \( H_j \), \( j = 1,2 \), and \( M \), such that the following LMIs hold:

\[
\begin{bmatrix} S_2 & M \\ * & S_2 \end{bmatrix} \geq 0,
\]
\[
\begin{bmatrix} P & 0 & 0 \\ 0 & Q_3 & 0 \\ * & * & I \end{bmatrix} \geq 0,
\]
\[
\begin{bmatrix} \Pi_{11} & \Phi S_1 & 0 & 0 & 0 & \Pi_{16} H_1^T & \Pi_{17} H_2^T & \bar{A}^T T^T \\ \ast & \Pi_{22} & \Pi_{23} M^T & 0 & 0 & 0 & 0 & 0 \\ \ast & \ast & \Pi_{33} & \Pi_{34} & 0 & d_1 \bar{A}_d H_1^T & \bar{d} A_3 H_2^T & \bar{A}_d T^T \\ \ast & \ast & \ast & \Pi_{44} & 0 & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & -I & d_1 \bar{B}_d H_1^T & \bar{d} B_3 H_2^T & \bar{B}_d T^T \\ \ast & \ast & \ast & \ast & \ast & S_1 - H_1^T - H_1 & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast & S_2 - H_2^T - H_2 & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & P - T^T - T \end{bmatrix} \leq 0,
\]
where $\Pi_i$, $i = 1, \ldots, 4$, $\Pi_{16}$, $\Pi_{17}$, $\Pi_{23}$, and $\Pi_{34}$ are defined in (17).

Proof. On one hand, if (17) holds, then there exist $H_j = H_j^T = S_j$, $j = 1, 2$, and $T = T^T = P$ such that (41) holds. On the other hand, if (41) holds, we have the following inequality based on Lemma 1:

$$\begin{bmatrix} 11 \Phi S_1 & 0 & 0 & 0 & \Pi_{16} H_1^T & \Pi_{17} H_1^T & \Pi_{23} H_1^T & \Pi_{34} H_1^T \\ * & \Pi_{16} H_1 & M_1^T & 0 & 0 & 0 & 0 & 0 \\ * & * & \Pi_{17} H_1 & M_2^T & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{23} H_1 & M_3^T & 0 & 0 & 0 \\ * & * & * & * & \Pi_{34} H_1 & M_4^T & 0 & 0 \\ * & * & * & * & * & \Pi_{16} H_1 & M_5^T & 0 \\ * & * & * & * & * & * & \Pi_{17} H_1 & M_6^T \\ * & * & * & * & * & * & * & \Pi_{23} H_1 \\ * & * & * & * & * & * & * & \Pi_{34} H_1 \end{bmatrix} < 0. \quad (42)$$

In addition, matrices $H_j$, $j = 1, 2$, and $T$ are nonsingular due to $S_j - H_j^T - H_j < 0$, $j = 1, 2$, and $P - T^T - T < 0$. Then, pre- and postmultiplying (42) by diag(1, 1, 1, 1, 1, 1, 1, 1) and its transpose yields (17). Therefore, the equivalence between (41) and (17) is proved.

### 3.2. Robust Filter Design.

In this subsection, the filter in the form of (5) is firstly designed such that the nominal filtering error system of (6) is asymptotically stable with an $l_2$-$l_{\infty}$ performance. Then the robust filtering problem is solved. Based on the result of Theorem 8, the filter design method for nominal system of (1) is presented in the following theorem.

**Theorem 9.** Given a scalar $\gamma > 0$, the nominal system of (6) is asymptotically stable with an $l_2$-$l_{\infty}$ performance if there exist matrices $[P_1, P_2, P_3, P_4] > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Q_4 > 0$, $L_1 = 1, 2$, $S_j > 0$, $H_j$, $F_j$, $j = 1, 2$, diagonal matrix $N > 0$, $T_1$, and $M$ such that the following set of LMIs hold:

$$\begin{bmatrix} S_2 & M \\ * & S_2 \end{bmatrix} > 0 \quad (43)$$

$$\Omega = \begin{bmatrix} \Xi & \Gamma \\ * & \Lambda \end{bmatrix} < 0 \quad (44)$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & -P_2 & S_1 & C^T N C_d & 0 & 0 \\ * & \Xi_{22} & 0 & 0 & 0 & 0 \\ * & * & \Xi_{33} & S_2 - M^T & M^T & 0 \\ * & * & * & \Xi_{44} & S_4 - M^T & 0 \\ * & * & * & * & \Xi_{55} & -\gamma^2 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & A^T T^T - C_d^T \mathbb{A}^T & A^T F_1^T & C_d^T \mathbb{A}^T F_1^T \\ 0 & 0 & 0 & 0 \\ d_1 A_1^{T-T} & d_1 A_2^{T-T} & d_1 A_3^{T-T} & d_1 A_4^{T-T} \\ 0 & 0 & 0 & 0 \\ d_1 A_1^{T-T} & d_1 A_2^{T-T} & d_1 A_3^{T-T} & d_1 A_4^{T-T} \\ 0 & 0 & 0 & 0 \\ S_1 - H_1 - H_1^T & S_2 - H_2 - H_2^T & 0 & 0 \\ * & * & P_1 - T_1 - T_1^T & P_2 - F_2 - F_2^T \\ * & * & * & P_1 - T_1 - T_1^T \\ * & * & * & P_2 - F_2 - F_2^T \end{bmatrix}.$$ 

Moreover, a suitable $l_2$-$l_{\infty}$ filter is given by

$$A_f = \overline{A}_f F_2^{-1}, \quad B_f = \overline{B}_f, \quad C_f = \overline{C}_f F_2^{-1}, \quad D_f = \overline{D}_f. \quad (45)$$

Proof. Firstly, we introduce four matrices $T_1$, $T_2$, $T_3$, and $T_4$ with $T_4$ invertible and define

$$I = \begin{bmatrix} I & 0 \\ 0 & T_1 T_4^{-1} \end{bmatrix}, \quad F_1 = T_2 T_4^{-1} T_3, \quad F_2 = T_2 T_4^{-1} T_3, \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \quad J = \text{diag} \{J_1, I, I, I, I, I, J_1, \}.$$ 

From (44), we have $F_2 + F_2^T = T_2 T_4^{-1} T_3^T + T_2 T_4^{-1} T_3^T > 0$ which implies that $T_2$ is nonsingular. Hence, $J$ and $J_2$ are
nonsingular. The inequality in (45) can be obtained by pre-
annomultiplying (33) with \( J_2 \) and \( J_2^T \), respectively. Noting
that
\[
\Omega = J_2 \Omega J_2^T
\]
we have \( \Omega < 0 \). On the other hand, because \( T_2 \) and \( T_4 \) cannot
be obtained from (44), we cannot determine the filters from
(48). However, we can construct an equivalent filter transfer function
by \( \gamma(k) \) to (52):
\[
T_{2y} = C_f (zI - A_f)^{-1} B_f + D_f
\]
\[
= \overline{C}_f T_2^{-1} T_4^T (zI - T_2^{-1} \overline{A}_f T_2^{-1} T_4^T)^{-1} T_2^{-1} \overline{B}_f + \overline{D}_f
\]
\[
= \overline{C}_f F_2^{-1} (zI - \overline{A}_f F_2^{-1})^{-1} \overline{B}_f + \overline{D}_f.
\]
Therefore, the desired filter can be obtained from (47). This
completes the proof.

Then the filter design result for uncertain system (6) is
presented in the following theorem:

**Theorem 10.** Given a scalar \( \gamma > 0 \), the system in (6) with
uncertainty is asymptotically stable with an \( l_2 - l_\infty \) performance
if there exist matrices \( \left[ \begin{array}{c} P_i \end{array} \right] > 0 \), \( Q_i > 0 \), \( \overline{Q}_i > 0 \), \( \overline{Q}_i > 0 \), \( l = 1, 2, S_j > 0 \), \( H_j, F_j, j = 1, 2 \), diagonal matrix \( N > 0 \), \( T_1, M \), and scalars \( \varepsilon_i > 0 \), \( i = 1, \ldots, 4 \) such that the
following set of LMIs hold:
\[
\begin{bmatrix}
S_2 & M \\
* & S_2
\end{bmatrix} \geq 0,
\]
\[
\begin{bmatrix}
\Omega + \varepsilon_1 \Omega_1 \Omega_4 + \varepsilon_2 \Omega_2 \Omega_4 \\
\ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\Omega_3 & \Omega_4 \\
\ast & \ast & \ast & \ast & \ast
\end{bmatrix}
< 0,
\]
\[
\begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & Y_{14} & \left( L - D_f C_f \right)^T \\
* & P_2 & 0 & 0 & -C_f^T \\
* & * & Y_{33} & Y_{34} & \left( L - D_f C_d \right)^T \\
* & * & * & Y_{44} & \left( G - D_f C_d \right)^T \\
* & * & * & * & \gamma^T I
\end{bmatrix}
\begin{bmatrix}
M_3 & 0 \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\geq 0,
\]
where \( \Omega \) is defined in (44) and
\[
\Omega = \begin{bmatrix} N_A & 0 & 0 & N_{Ad} & 0 \end{bmatrix},
\]
\[
\Omega_3 = \begin{bmatrix} N_C & 0 & 0 & N_{Cd} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]
\[
\Omega_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

Therefore, the desired filter can be obtained from (47). This
completes the proof.

Moreover, a suitable \( l_2 - l_\infty \) filter is given by
\[
A_f = \overline{A}_f F_2^{-1}, \quad B_f = \overline{B}_f, \quad C_f = \overline{C}_f F_2^{-1},
\]
\[
D_f = \overline{D}_f.
\]

Proof. Firstly, replace matrices \( A, A_d, B, C, C_d, \) and \( D \) in (44)
with \( A + \Delta A, A_d + \Delta A_d, B + \Delta B, C + \Delta C, C_d + \Delta C_d, \) and \( D + \Delta D \),
respectively, and the following inequality is obtained:
\[
\Omega + \text{sym} \left( \Omega_1^T \Delta_1 \Omega_1 \right) + \text{sym} \left( \Omega_2^T \Delta_2 \Omega_2 \right) < 0,
\]
where \( \Omega_i, i = 1, \ldots, 4 \) are defined in (52). Then by using
Lemma 4, the above inequality holds if and only if
\[
\Omega + \varepsilon_3 \Omega_1^T \Omega_1 + \varepsilon_4 \Omega_2^T \Omega_2 + \varepsilon_5 \Omega_1^T \Omega_2 + \varepsilon_6 \Omega_2^T \Omega_1 < 0.
\]
By following the similar line, the equivalence between (58) and (53) can be proved.

4. Illustrative Example

In this section, the following example is given to demonstrate the effectiveness of the proposed approach.

Example 1. Firstly, consider a nominal discrete-time delay system in (1) with the following parameters:

\[
\begin{align*}
A &= \begin{bmatrix} 0.1 & -0.5 \\ 0.2 & 0.5 \end{bmatrix}, & A_d &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \\
B &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & C &= \begin{bmatrix} -0.1 & -1 \end{bmatrix}, & C_d &= \begin{bmatrix} -0.1 & 0.6 \end{bmatrix}, \\
L &= \begin{bmatrix} 1 & 0.2 \end{bmatrix}, & L_d &= \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, & D &= 0.1, \\
G &= -0.5.
\end{align*}
\]

For different delay cases, the different minima of \( \gamma \) can be calculated by solving the LMIs in Theorem 9. When the upper bound of the time-varying delay is 5, that is, \( d_2 = 5 \), the minima of \( \gamma \) for a given \( d_1 \) are listed in Table 1.

Moreover, when \( d_1 = 2, d_2 = 5 \), the corresponding \( l_2-l_\infty \) filter is given as follows:

\[
\begin{align*}
A_f &= \begin{bmatrix} 2.6553 & -1.9535 \\ 2.2783 & -1.5631 \end{bmatrix}, & B_f &= \begin{bmatrix} -1.5405 \\ -1.9024 \end{bmatrix}, \\
C_f &= \begin{bmatrix} -1.3230 \\ 1.0479 \end{bmatrix}, & D_f &= 0.2378.
\end{align*}
\]

When uncertainty appears in the system, Theorem 10 will be used for the desired filter design. The following uncertainty parameters are considered:

\[
\begin{align*}
M_1 &= \begin{bmatrix} 0.35 & 0 \\ -0.2 & 0.1 \end{bmatrix}, & N_A &= \begin{bmatrix} 0.2 & 0.4 \\ 0 & 0.5 \end{bmatrix}, \\
N_{Ad} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, & N_B &= \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \\
M_2 &= 0.2, & N_C &= \begin{bmatrix} 0.15 & -0.22 \end{bmatrix}, \\
N_{Cd} &= \begin{bmatrix} -0.3 & 0.2 \end{bmatrix}, & N_D &= -0.5, \\
M_3 &= -0.4, & N_L &= \begin{bmatrix} -0.25 & -0.2 \end{bmatrix}, \\
N_{Ld} &= \begin{bmatrix} 0.13 & 0.32 \end{bmatrix}, & N_D &= 0.2.
\end{align*}
\]

Similarly, the allowed minimal values of \( \gamma \) can be obtained by solving the LMIs in Theorem 10. For \( d_2 = 5 \), the different minimum allowed \( \gamma \) are listed in Table 2 for the uncertain system with different \( d_1 \).

Moreover, when \( d_1 = 2, d_2 = 5 \), the desired filter is given as follows:

\[
\begin{align*}
A_f &= \begin{bmatrix} -0.1261 & 0.2147 \\ -0.3333 & 0.5671 \end{bmatrix}, & B_f &= \begin{bmatrix} -0.1725 \\ -0.2622 \end{bmatrix}, \\
C_f &= \begin{bmatrix} -0.0076 \\ 0.0125 \end{bmatrix}, & D_f &= 0.1106.
\end{align*}
\]

5. Conclusions

The robust \( l_2-l_\infty \) filtering has been studied for uncertain discrete-time systems with time-varying delay in this paper. Based on reciprocally convex approach, the sufficient \( l_2-l_\infty \) performance analysis conditions in terms of LMIs have been proposed to render the filtering error systems asymptotically stable with an \( l_2-l_\infty \) performance. Then the desired filter has been designed for the filtering error system with time-varying delay. The results presented in this paper are in terms of strict LMIs which make the conditions more tractable. Finally, a numerical example is given to demonstrate the effectiveness of our methods. For future research topic, the results can be extended to the system with actuator/sensor failures which may lead to unsatisfactory performance and has attracted many researchers’ attention such as faulty diagnosis [23, 24] and fault tolerant control [25].

Conflict of Interests

None of the authors of the paper has declared any conflict of interests.

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References


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