



VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

Applications of Kato's inequality for n -tuples of operators in Hilbert spaces, (I)

This is the Published version of the following publication

Dragomir, Sever S, Cho, Yeol Je and Kim, Young-Ho (2013) Applications of Kato's inequality for n -tuples of operators in Hilbert spaces, (I). *Journal of Inequalities and Applications*, 2013. ISSN 1025-5834

The publisher's official version can be found at
<https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/1029-242X-2013-21>

Note that access to this version may require subscription.

Downloaded from VU Research Repository <https://vuir.vu.edu.au/35568/>

RESEARCH

Open Access

Applications of Kato's inequality for n -tuples of operators in Hilbert spaces, (I)

Sever S Dragomir^{1,2}, Yeol Je Cho^{3*} and Young-Ho Kim^{4*}

*Correspondence: yjcho@gnu.ac.kr;
yhkim@changwon.ac.kr

³Department of Mathematics
Education and the RINS,
Gyeongsang National University,
Chinju, 660-701, Republic of Korea

⁴Department of Mathematics,
Changwon National University,
Changwon, 641-773, Republic of
Korea

Full list of author information is
available at the end of the article

Abstract

In this paper, by the use of the famous Kato's inequality for bounded linear operators, we establish some inequalities for n -tuples of operators and apply them for functions of normal operators defined by power series as well as for some norms and numerical radii that arise in multivariate operator theory.

MSC: 47A63; 47A99

Keywords: bounded linear operators; functions of normal operators; inequalities for operators; norm and numerical radius inequalities; Kato's inequality

1 Introduction

The 'square root' of a positive bounded self-adjoint operator on H can be defined as follows (see, for instance, [1, p.240]).

If the operator $A \in \mathcal{B}(H)$ is self-adjoint and positive, then there exists a unique positive self-adjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$. If A is invertible, then so is B .

If $A \in \mathcal{B}(H)$, then the operator A^*A is self-adjoint and positive. Define the 'absolute value' operator by $|A| := \sqrt{A^*A}$.

In 1952, Kato [2] proved the following generalization of Schwarz inequality:

$$|\langle Tx, y \rangle|^2 \leq \langle (T^*T)^\alpha x, x \rangle \langle (TT^*)^{1-\alpha} y, y \rangle, \quad (1.1)$$

for any $x, y \in H$, $\alpha \in [0, 1]$ and T is a bounded linear operator on H .

Utilizing the modulus notation introduced before, we can write (1.1) as follows:

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle. \quad (1.2)$$

For results related to the Kato's inequality, see [2–18] and [19].

In the recent paper [20], by employing Kato's inequality (1.2), Dragomir established the following results for sequences of bonded linear operators on complex Hilbert spaces.

Theorem 1.1 Let $(T_1, \dots, T_n) \in \mathcal{B}(H) \times \dots \times \mathcal{B}(H) := \mathcal{B}^{(n)}(H)$ be an n -tuple of bounded linear operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $(p_1, \dots, p_n) \in \mathbb{R}_+^{*n}$ be an n -tuple of non-

negative weights not all of them equal to zero. Then we have

$$\sum_{j=1}^n p_j |\langle T_j x, y \rangle|^2 \leq \left\langle \sum_{j=1}^n p_j |T_j|^2 x, x \right\rangle^\alpha \left\langle \sum_{j=1}^n p_j |T_j^*|^2 y, y \right\rangle^{1-\alpha} \tag{1.3}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$.

He also obtained the following result.

Theorem 1.2 *With the assumptions in Theorem 1.1, we have*

$$\sum_{j=1}^n p_j |\langle T_j x, y \rangle| \leq \left\langle \sum_{j=1}^n p_j |T_j|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^n p_j |T_j^*|^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{1.4}$$

for any $x, y \in H$.

For various related results, see the papers [21–31].

Motivated by the above results, we establish in this paper other similar inequalities for n -tuples of bounded linear operators that can be obtained from Kato’s result (1.2) and apply them to functions of normal operators defined by power series as well as to some norms and numerical radii that can be associated with these n -tuples of bonded linear operators on Hilbert spaces.

2 Some inequalities for an n -tuple of linear operators

Employing Kato’s inequality (1.2), we can state the following new result.

Theorem 2.1 *Let $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$ be an n -tuple of bounded linear operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $(p_1, \dots, p_n) \in \mathbb{R}_+^{*n}$ be an n -tuple of nonnegative weights, not all of them equal to zero. Then we have*

$$\begin{aligned} \sum_{j=1}^n p_j |\langle T_j x, y \rangle| &\leq \left\langle \sum_{j=1}^n p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \\ &\quad \times \left\langle \sum_{j=1}^n p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2} \end{aligned} \tag{2.1}$$

for any $x, y \in H$, $\alpha \in [0, 1]$ and, in particular, for $\alpha = \frac{1}{2}$

$$\sum_{j=1}^n p_j |\langle T_j x, y \rangle| \leq \left\langle \sum_{j=1}^n p_j |T_j| x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^n p_j |T_j^*| y, y \right\rangle^{1/2} \tag{2.2}$$

for any $x, y \in H$.

Proof Utilizing Kato’s inequality, we have

$$|\langle T_j x, y \rangle| \leq (|T_j|^{2\alpha} x, x)^{1/2} (|T_j^*|^{2(1-\alpha)} y, y)^{1/2}$$

and by replacing α with $1 - \alpha$,

$$|\langle T_j x, y \rangle| \leq (|T_j|^{2(1-\alpha)} x, x)^{1/2} (|T_j^*|^{2\alpha} y, y)^{1/2},$$

which by summation gives

$$\begin{aligned} |\langle T_j x, y \rangle| &\leq \frac{1}{2} [(|T_j|^{2\alpha} x, x)^{1/2} (|T_j^*|^{2(1-\alpha)} y, y)^{1/2} \\ &\quad + (|T_j|^{2(1-\alpha)} x, x)^{1/2} (|T_j^*|^{2\alpha} y, y)^{1/2}] \end{aligned} \tag{2.3}$$

for any $j \in \{1, \dots, n\}$ and $x, y \in H$. By the elementary inequality

$$ab + cd \leq (a^2 + c^2)^{1/2} (b^2 + d^2)^{1/2}, \quad a, b, c, d \geq 0, \tag{2.4}$$

we have

$$\begin{aligned} &[(|T_j|^{2\alpha} x, x)^{1/2} (|T_j^*|^{2(1-\alpha)} y, y)^{1/2} + (|T_j|^{2(1-\alpha)} x, x)^{1/2} (|T_j^*|^{2\alpha} y, y)^{1/2}] \\ &\leq [((|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}) x, x)]^{1/2} [((|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}) y, y)]^{1/2}, \end{aligned}$$

which by (2.3) produces

$$|\langle T_j x, y \rangle| \leq \left\langle \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \left\langle \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2} \tag{2.5}$$

for any $j \in \{1, \dots, n\}$ and $x, y \in H$. Multiplying the inequalities (2.5) with the positive weights p_j , summing over j from 1 to n and utilizing the weighted Cauchy-Buniakowski-Schwarz inequality

$$\sum_{j=1}^n p_j a_j b_j \leq \left(\sum_{j=1}^n p_j a_j^2 \right)^{1/2} \left(\sum_{j=1}^n p_j b_j^2 \right)^{1/2},$$

where $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{R}_+^n$, we have

$$\begin{aligned} &\sum_{j=1}^n p_j |\langle T_j x, y \rangle| \\ &\leq \sum_{j=1}^n p_j \left\langle \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \left\langle \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2} \\ &\leq \left\langle \sum_{j=1}^n p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^n p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2} \end{aligned} \tag{2.6}$$

for any $x, y \in H$, and the inequality in (2.1) is proved. □

Remark 2.1 In order to provide some applications for functions of normal operators defined by power series, we need to state the inequality (2.1) for normal operators N_j , $j \in \{1, \dots, n\}$, namely,

$$\sum_{j=1}^n p_j |\langle N_j x, y \rangle| \leq \left\langle \sum_{j=1}^n p_j \left(\frac{|N_j|^{2\alpha} + |N_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \times \left\langle \sum_{j=1}^n p_j \left(\frac{|N_j|^{2\alpha} + |N_j|^{2(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2} \tag{2.7}$$

for any $\alpha \in [0, 1]$ and for any $x, y \in H$.

From a different perspective that involves quadratics, we can state the following result as well.

Theorem 2.2 Let $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$ be an n -tuple of bounded linear operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $(p_1, \dots, p_n) \in \mathbb{R}_+^{*n}$ be an n -tuple of nonnegative weights, not all of them equal to zero. Then we have

$$\begin{aligned} & \sum_{j=1}^n p_j |\langle T_j x, y \rangle|^2 \\ & \leq \frac{1}{2} \sum_{j=1}^n p_j (\|T_j x\|^{2\alpha} \|T_j^* y\|^{2(1-\alpha)} + \|T_j^* y\|^{2\alpha} \|T_j x\|^{2(1-\alpha)}) \\ & \leq \frac{1}{2} \left[\left(\sum_{j=1}^n p_j \|T_j x\|^2 \right)^\alpha \left(\sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^{1-\alpha} \right. \\ & \quad \left. + \left(\sum_{j=1}^n p_j \|T_j x\|^2 \right)^{1-\alpha} \left(\sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^\alpha \right] \\ & \leq \frac{1}{2} \sum_{j=1}^n p_j (\|T_j x\|^2 + \|T_j^* y\|^2) \end{aligned} \tag{2.8}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$.

Proof We must prove the inequalities only in the case $\alpha \in (0, 1)$, since the case $\alpha = 0$ or $\alpha = 1$ follows directly from the corresponding case of Kato's inequality.

Utilizing Kato's inequality for the operator T_j , $j \in \{1, \dots, n\}$, we have

$$|\langle T_j x, y \rangle|^2 \leq \langle |T_j|^{2\alpha} x, x \rangle \langle |T_j|^{2(1-\alpha)} y, y \rangle \tag{2.9}$$

and, by replacing α with $1 - \alpha$,

$$|\langle T_j x, y \rangle|^2 \leq \langle |T_j|^{2(1-\alpha)} x, x \rangle \langle |T_j|^{2\alpha} y, y \rangle \tag{2.10}$$

for any $x, y \in H$.

By the Hölder-McCarthy inequality $\langle P^r x, x \rangle \leq \langle P x, x \rangle^r$ that holds for the positive operator P , for $r \in (0, 1)$ and $x \in H$ with $\|x\| = 1$, we also have

$$\langle |T_j|^{2\alpha} x, x \rangle \langle |T_j^*|^{2(1-\alpha)} y, y \rangle \leq \langle |T_j|^2 x, x \rangle^\alpha \langle |T_j^*|^2 y, y \rangle^{1-\alpha} \tag{2.11}$$

and

$$\langle |T_j|^{2(1-\alpha)} x, x \rangle \langle |T_j^*|^{2\alpha} y, y \rangle \leq \langle |T_j|^2 x, x \rangle^{1-\alpha} \langle |T_j^*|^2 y, y \rangle^\alpha \tag{2.12}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, $j \in \{1, \dots, n\}$ and $\alpha \in (0, 1)$.

If we add (2.9) with (2.10) and make use of (2.11) and (2.12), we deduce

$$2|\langle T_j x, y \rangle|^2 \leq \langle |T_j|^2 x, x \rangle^\alpha \langle |T_j^*|^2 y, y \rangle^{1-\alpha} + \langle |T_j^*|^2 y, y \rangle^\alpha \langle |T_j|^2 x, x \rangle^{1-\alpha} \tag{2.13}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, $j \in \{1, \dots, n\}$ and $\alpha \in (0, 1)$.

Now, if we multiply (2.13) with $p_j \geq 0$, sum over j from 1 to n , we get

$$\begin{aligned} 2 \sum_{j=1}^n p_j |\langle T_j x, y \rangle|^2 &\leq \sum_{j=1}^n p_j \langle |T_j|^2 x, x \rangle^\alpha \langle |T_j^*|^2 y, y \rangle^{1-\alpha} \\ &\quad + \sum_{j=1}^n p_j \langle |T_j^*|^2 y, y \rangle^\alpha \langle |T_j|^2 x, x \rangle^{1-\alpha} \end{aligned} \tag{2.14}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in (0, 1)$.

Since $\langle |T_j|^2 x, x \rangle = \|T_j x\|^2$ and $\langle |T_j^*|^2 y, y \rangle = \|T_j^* y\|^2$, $j \in \{1, \dots, n\}$, then we get from (2.14) the first inequality in (2.8).

Now, on making use of the weighted Hölder discrete inequality

$$\sum_{j=1}^n p_j a_j b_j \leq \left(\sum_{j=1}^n p_j a_j^p \right)^{1/p} \left(\sum_{j=1}^n p_j b_j^q \right)^{1/q}, \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

where $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{R}_+^n$, we also have

$$\sum_{j=1}^n p_j \|T_j x\|^{2\alpha} \|T_j^* y\|^{2(1-\alpha)} \leq \left(\sum_{j=1}^n p_j \|T_j x\|^2 \right)^\alpha \left(\sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^{1-\alpha}$$

and

$$\sum_{j=1}^n p_j \|T_j^* y\|^{2\alpha} \|T_j x\|^{2(1-\alpha)} \leq \left(\sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^\alpha \left(\sum_{j=1}^n p_j \|T_j x\|^2 \right)^{1-\alpha}.$$

Summing these two inequalities, we deduce the second inequality in (2.8).

Finally, on utilizing the Hölder inequality

$$ab + cd \leq (a^p + c^p)^{1/p} (b^q + d^q)^{1/q}, \quad a, b, c, d \geq 0,$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} & \left(\sum_{j=1}^n p_j \|T_j x\|^2 \right)^\alpha \left(\sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^{1-\alpha} + \left(\sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^\alpha \left(\sum_{j=1}^n p_j \|T_j x\|^2 \right)^{1-\alpha} \\ & \leq \left(\sum_{j=1}^n p_j \|T_j x\|^2 + \sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^\alpha \left(\sum_{j=1}^n p_j \|T_j x\|^2 + \sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^{1-\alpha} \\ & = \sum_{j=1}^n p_j \|T_j x\|^2 + \sum_{j=1}^n p_j \|T_j^* y\|^2, \end{aligned}$$

and the proof is concluded. □

Remark 2.2 For $\alpha = \frac{1}{2}$, we get from (2.8) that

$$\begin{aligned} & \sum_{j=1}^n p_j |\langle T_j x, y \rangle|^2 \\ & \leq \sum_{j=1}^n p_j \|T_j x\| \|T_j^* y\| \leq \left(\sum_{j=1}^n p_j \|T_j x\|^2 \right)^{1/2} \left(\sum_{j=1}^n p_j \|T_j^* y\|^2 \right)^{1/2} \\ & \leq \frac{1}{2} \sum_{j=1}^n p_j (\|T_j x\|^2 + \|T_j^* y\|^2) \end{aligned} \tag{2.15}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

3 Inequalities for functions of normal operators

Now, by the help of power series $f(z) = \sum_{n=0}^\infty a_n z^n$, we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z) := \sum_{n=0}^\infty |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$, then $f_A = f$.

As some natural examples that are useful for applications, we can point out that if

$$\begin{aligned} f(z) &= \sum_{n=1}^\infty \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z}, \quad z \in D(0,1); \\ g(z) &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \quad z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \quad z \in \mathbb{C}; \\ l(z) &= \sum_{n=0}^\infty (-1)^n z^n = \frac{1}{1+z}, \quad z \in D(0,1), \end{aligned} \tag{3.1}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are as follows:

$$\begin{aligned}
 f_A(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \quad z \in D(0,1); \\
 g_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbb{C}; \\
 h_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \quad z \in \mathbb{C}; \\
 l_A(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in D(0,1).
 \end{aligned}
 \tag{3.2}$$

The following result is a functional inequality for normal operators that can be obtained from (2.1).

Theorem 3.1 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If N is a normal operator on the Hilbert space H , for $\alpha \in (0,1)$, we have that $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then we have the inequality*

$$\begin{aligned}
 |\langle f(N)x, y \rangle| &\leq \frac{1}{2} \langle [f_A(|N|^{2\alpha}) + f_A(|N|^{2(1-\alpha)})]x, x \rangle^{1/2} \\
 &\quad \times \langle [f_A(|N|^{2\alpha}) + f_A(|N|^{2(1-\alpha)})]y, y \rangle^{1/2}
 \end{aligned}
 \tag{3.3}$$

for any $x, y \in H$. In particular, if $\|N\| < R$, then

$$|\langle f(N)x, y \rangle| \leq \langle f_A(|N|)x, x \rangle^{1/2} \langle f_A(|N|)y, y \rangle^{1/2}
 \tag{3.4}$$

for any $x, y \in H$.

Proof If N is a normal operator, then for any $j \in \mathbb{N}$, we have that

$$|N^j|^2 = (N^*N)^j = |N|^{2j}.$$

Now, utilizing the inequality (2.9), we can write

$$\begin{aligned}
 \left| \left\langle \sum_{j=0}^n a_j N^j x, y \right\rangle \right| &\leq \sum_{j=0}^n |a_j| |\langle N^j x, y \rangle| \\
 &\leq \left\langle \sum_{j=0}^n |a_j| \left(\frac{|N|^{2j\alpha} + |N|^{2j(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \\
 &\quad \times \left\langle \sum_{j=0}^n |a_j| \left(\frac{|N|^{2j\alpha} + |N|^{2j(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2}
 \end{aligned}
 \tag{3.5}$$

for any $x, y \in H$ and $n \in \mathbb{N}$. Since $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then it follows that the series $\sum_{j=0}^{\infty} |a_j|(|N|^{2\alpha})^j$ and $\sum_{j=0}^{\infty} |a_j|(|N|^{2(1-\alpha)})^j$ are absolute convergent in $\mathcal{B}(H)$, and by taking the limit over $n \rightarrow \infty$ in (3.5), we deduce the desired result (3.3). \square

Remark 3.1 With the assumptions in Theorem 3.1, if we take the supremum over $y \in H, \|y\| = 1$, then we get the vector inequality

$$\begin{aligned} \|f(N)x\| &\leq \frac{1}{2} \langle [f_A(|N|^{2\alpha}) + f_A(|N|^{2(1-\alpha)})]x, x \rangle^{1/2} \\ &\quad \times \|f_A(|N|^{2\alpha}) + f_A(|N|^{2(1-\alpha)})\|^{1/2} \end{aligned} \tag{3.6}$$

for any $x \in H$, which in its turn produces the norm inequality

$$\|f(N)\| \leq \frac{1}{2} \|f_A(|N|^{2\alpha}) + f_A(|N|^{2(1-\alpha)})\| \tag{3.7}$$

for any $\alpha \in [0, 1]$. Making use of the examples in (3.1) and (3.2), we can state the vector inequalities

$$\begin{aligned} &|\langle \ln(1_H + N)^{-1}x, y \rangle| \\ &\leq \frac{1}{2} \langle [\ln(1_H - |N|^{2\alpha})^{-1} + \ln(1_H - |N|^{2(1-\alpha)})^{-1}]x, x \rangle^{1/2} \\ &\quad \times \langle [\ln(1_H - |N|^{2\alpha})^{-1} + \ln(1_H - |N|^{2(1-\alpha)})^{-1}]y, y \rangle^{1/2}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &|\langle (1_H + N)^{-1}x, y \rangle| \\ &\leq \frac{1}{2} \langle [(1_H - |N|^{2\alpha})^{-1} + (1_H - |N|^{2(1-\alpha)})^{-1}]x, x \rangle^{1/2} \\ &\quad \times \langle [(1_H - |N|^{2\alpha})^{-1} + (1_H - |N|^{2(1-\alpha)})^{-1}]y, y \rangle^{1/2} \end{aligned} \tag{3.9}$$

for any $x, y \in H$ and $\|N\| < 1$. We also have the inequalities

$$\begin{aligned} |\langle \sin(N)x, y \rangle| &\leq \frac{1}{2} \langle [\sinh(|N|^{2\alpha}) + \sinh(|N|^{2(1-\alpha)})]x, x \rangle^{1/2} \\ &\quad \times \langle [\sinh(|N|^{2\alpha}) + \sinh(|N|^{2(1-\alpha)})]y, y \rangle^{1/2} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |\langle \cos(N)x, y \rangle| &\leq \frac{1}{2} \langle [\cosh(|N|^{2\alpha}) + \cosh(|N|^{2(1-\alpha)})]x, x \rangle^{1/2} \\ &\quad \times \langle [\cosh(|N|^{2\alpha}) + \cosh(|N|^{2(1-\alpha)})]y, y \rangle^{1/2} \end{aligned} \tag{3.11}$$

for any $x, y \in H$ and N a normal operator.

If we utilize the following function as power series representations with nonnegative coefficients:

$$\begin{aligned} \exp(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad z \in \mathbb{C}; \\ \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0,1); \\ \sin^{-1}(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \quad z \in D(0,1); \\ \tanh^{-1}(z) &= \sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}, \quad z \in D(0,1); \\ {}_2F_1(\alpha, \beta, \gamma, z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \quad \alpha, \beta, \gamma > 0, z \in D(0,1), \end{aligned} \tag{3.12}$$

where Γ is the *gamma function*, then we can state the following vector inequalities:

$$\begin{aligned} |\langle \exp(N)x, y \rangle| &\leq \frac{1}{2} \left([\exp(|N|^{2\alpha}) + \exp(|N|^{2(1-\alpha)})] x, x \right)^{1/2} \\ &\quad \times \left([\exp(|N|^{2\alpha}) + \exp(|N|^{2(1-\alpha)})] y, y \right)^{1/2} \end{aligned} \tag{3.13}$$

for any $x, y \in H$ and N a normal operator. If $\|N\| < 1$, then we also have the inequalities

$$\begin{aligned} &\left| \left\langle \ln\left(\frac{1_H + N}{1_H - N}\right) x, y \right\rangle \right| \\ &\leq \frac{1}{2} \left\langle \left[\ln\left(\frac{1_H + |N|^{2\alpha}}{1_H - |N|^{2\alpha}}\right) + \ln\left(\frac{1_H + |N|^{2(1-\alpha)}}{1_H - |N|^{2(1-\alpha)}}\right) \right] x, x \right\rangle^{1/2} \\ &\quad \times \left\langle \left[\ln\left(\frac{1_H + |N|^{2\alpha}}{1_H - |N|^{2\alpha}}\right) + \ln\left(\frac{1_H + |N|^{2(1-\alpha)}}{1_H - |N|^{2(1-\alpha)}}\right) \right] y, y \right\rangle^{1/2}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &|\langle \tanh^{-1}(N)x, y \rangle| \\ &\leq \frac{1}{2} \left([\tanh^{-1}(|N|^{2\alpha}) + \tanh^{-1}(|N|^{2(1-\alpha)})] x, x \right)^{1/2} \\ &\quad \times \left([\tanh^{-1}(|N|^{2\alpha}) + \tanh^{-1}(|N|^{2(1-\alpha)})] y, y \right)^{1/2} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &|\langle {}_2F_1(\alpha, \beta, \gamma, N)x, y \rangle| \\ &\leq \frac{1}{2} \left([{}_2F_1(\alpha, \beta, \gamma, |N|^{2\alpha}) + {}_2F_1(\alpha, \beta, \gamma, |N|^{2(1-\alpha)})] x, x \right)^{1/2} \\ &\quad \times \left([{}_2F_1(\alpha, \beta, \gamma, |N|^{2\alpha}) + {}_2F_1(\alpha, \beta, \gamma, |N|^{2(1-\alpha)})] y, y \right)^{1/2} \end{aligned} \tag{3.16}$$

for any $x, y \in H$. From a different perspective, we also have

Theorem 3.2 *With the assumption of Theorem 3.1 and if N is a normal operator on the Hilbert space H and $z \in \mathbb{C}$ such that $\|N\|^2, |z|^2 < R$, then we have the inequalities*

$$\begin{aligned} |\langle f(zN)x, y \rangle|^2 &\leq \frac{1}{2} f_A(|z|^2) [\langle f_A(|N|^2)x, x \rangle^\alpha \langle f_A(|N|^2)y, y \rangle^{1-\alpha} \\ &\quad + \langle f_A(|N|^2)x, x \rangle^{1-\alpha} \langle f_A(|N|^2)y, y \rangle^\alpha] \\ &\leq \frac{1}{2} f_A(|z|^2) (\langle f_A(|N|^2)x, x \rangle + \langle f_A(|N|^2)y, y \rangle) \end{aligned} \quad (3.17)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$. In particular, for $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} |\langle f(zN)x, y \rangle|^2 &\leq f_A(|z|^2) \langle f_A(|N|^2)x, x \rangle^{1/2} \langle f_A(|N|^2)y, y \rangle^{1/2} \\ &\leq \frac{1}{2} f_A(|z|^2) (\langle f_A(|N|^2)x, x \rangle + \langle f_A(|N|^2)y, y \rangle) \end{aligned} \quad (3.18)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof If we use the second and third inequality from (2.8) for powers of operators, we have

$$\begin{aligned} &\sum_{j=0}^n |a_j| |\langle N^j x, y \rangle|^2 \\ &\leq \frac{1}{2} \left[\left(\sum_{j=0}^n |a_j| \|N^j x\|^2 \right)^\alpha \left(\sum_{j=0}^n |a_j| \|(N^*)^j y\|^2 \right)^{1-\alpha} \right. \\ &\quad \left. + \left(\sum_{j=0}^n |a_j| \|N^j x\|^2 \right)^{1-\alpha} \left(\sum_{j=0}^n |a_j| \|(N^*)^j y\|^2 \right)^\alpha \right] \\ &\leq \frac{1}{2} \sum_{j=0}^n |a_j| (\|N^j x\|^2 + \|(N^*)^j y\|^2) \end{aligned} \quad (3.19)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$. Since N is a normal operator on the Hilbert space H , then

$$\|N^j x\|^2 = \langle |N^j|^2 x, x \rangle = \langle |N|^{2j} x, x \rangle$$

and

$$\|(N^*)^j y\|^2 = \langle |(N^*)^j|^2 y, y \rangle = \langle |N^*|^{2j} y, y \rangle = \langle |N|^{2j} y, y \rangle$$

for any $j \in \{0, \dots, n\}$ and for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Then from (3.19), we have

$$\begin{aligned} &\sum_{j=0}^n |a_j| |\langle N^j x, y \rangle|^2 \\ &\leq \frac{1}{2} \left[\left(\left\langle \sum_{j=0}^n |a_j| |N|^{2j} x, x \right\rangle \right)^\alpha \left(\left\langle \sum_{j=0}^n |a_j| |N|^{2j} y, y \right\rangle \right)^{1-\alpha} \right] \end{aligned}$$

$$\begin{aligned}
 & + \left(\left\langle \sum_{j=0}^n |a_j| |N|^{2j} x, x \right\rangle \right)^{1-\alpha} \left(\left\langle \sum_{j=0}^n |a_j| |N|^{2j} y, y \right\rangle \right)^\alpha \Big] \\
 & \leq \frac{1}{2} \left(\left\langle \sum_{j=0}^n |a_j| |N|^{2j} x, x \right\rangle + \left\langle \sum_{j=0}^n |a_j| |N|^{2j} y, y \right\rangle \right) \tag{3.20}
 \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$. By the weighted Cauchy-Buniakowski-Schwarz inequality, we also have

$$\left| \left\langle \sum_{j=0}^n a_j z^j N^j x, y \right\rangle \right|^2 \leq \sum_{j=0}^n |a_j| |z|^{2j} \sum_{j=0}^n |a_j| |\langle N^j x, y \rangle|^2 \tag{3.21}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, since the series $\sum_{j=0}^\infty a_j z^j N^j$, $\sum_{j=0}^\infty |a_j| |z|^{2j}$, $\sum_{j=0}^\infty |a_j| |N|^{2j}$ are convergent, then by (3.20) and (3.21), on letting $n \rightarrow \infty$, we deduce the desired result (3.17). \square

Similar inequalities for some particular functions of interest can be stated. However, the details are left to the interested reader.

4 Applications for the Euclidean norm

In [29], the author has introduced the following norm on the Cartesian product $\mathcal{B}^{(n)}(H) := \mathcal{B}(H) \times \dots \times \mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H :

$$\|(T_1, \dots, T_n)\|_e := \sup_{(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n} \|\lambda_1 T_1 + \dots + \lambda_n T_n\|, \tag{4.1}$$

where $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$ and $\mathbb{B}_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \mid \sum_{j=1}^n |\lambda_j|^2 \leq 1\}$ is the Euclidean closed ball in \mathbb{C}^n .

It is clear that $\|\cdot\|_e$ is a norm on $\mathcal{B}^{(n)}(H)$ and, for any $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$, we have

$$\|(T_1, \dots, T_n)\|_e = \|(T_1^*, \dots, T_n^*)\|_e,$$

where T_j^* is the adjoint operator of T_j , $j \in \{1, \dots, n\}$. We call this the *Euclidean norm* of an n -tuple of operators $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$.

It has been shown in [29] that the following basic inequality for the Euclidean norm holds true:

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^n |T_j^*|^2 \right\|^{\frac{1}{2}} \leq \|(T_1, \dots, T_n)\|_e \leq \left\| \sum_{j=1}^n |T_j|^2 \right\|^{\frac{1}{2}} \tag{4.2}$$

for any n -tuple $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [29], the author has introduced the *Euclidean operator radius* of an n -tuple of operators (T_1, \dots, T_n) by

$$w_e(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \tag{4.3}$$

and proved that $w_e(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality

$$\frac{1}{2} \|(T_1, \dots, T_n)\|_e \leq w_e(T_1, \dots, T_n) \leq \|(T_1, \dots, T_n)\|_e \tag{4.4}$$

for each n -tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$.

As pointed out in [29], the Euclidean numerical radius also satisfies the double inequality

$$\frac{1}{2\sqrt{n}} \left\| \sum_{j=1}^n |T_j^*|^2 \right\|^{\frac{1}{2}} \leq w_e(T_1, \dots, T_n) \leq \left\| \sum_{j=1}^n |T_j^*|^2 \right\|^{\frac{1}{2}} \tag{4.5}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

In [30], by utilizing the concept of *hypo-Euclidean norm* on H^n , we obtained the following representation for the Euclidean norm.

Proposition 4.1 For any $(T_1, \dots, T_n) \in B^{(n)}(H)$, we have

$$\|(T_1, \dots, T_n)\|_e = \sup_{\|y\|=1, \|x\|=1} \left(\sum_{j=1}^n |\langle T_j y, x \rangle|^2 \right)^{\frac{1}{2}}. \tag{4.6}$$

We can state now the following result.

Theorem 4.1 For any $(T_1, \dots, T_n) \in B^{(n)}(H)$, we have

$$\begin{aligned} \|(T_1, \dots, T_n)\|_e^2 &\leq \frac{1}{2} \left[\left(\left\| \sum_{j=1}^n |T_j|^2 \right\| \right)^\alpha \left(\left\| \sum_{j=1}^n |T_j^*|^2 \right\| \right)^{1-\alpha} \right. \\ &\quad \left. + \left(\left\| \sum_{j=1}^n |T_j|^2 \right\| \right)^{1-\alpha} \left(\left\| \sum_{j=1}^n |T_j^*|^2 \right\| \right)^\alpha \right] \\ &\leq \frac{1}{2} \left[\left\| \sum_{j=1}^n |T_j|^2 \right\| + \left\| \sum_{j=1}^n |T_j^*|^2 \right\| \right] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} w_e^2(T_1, \dots, T_n) &\leq \frac{1}{2} \left[\sup_{\|x\|=1} \left\{ \left(\left\langle \sum_{j=1}^n |T_j|^2 x, x \right\rangle \right)^\alpha \left(\left\langle \sum_{j=1}^n |T_j^*|^2 x, x \right\rangle \right)^{1-\alpha} \right\} \right. \\ &\quad \left. + \sup_{\|x\|=1} \left\{ \left(\left\langle \sum_{j=1}^n |T_j|^2 x, x \right\rangle \right)^{1-\alpha} \left(\left\langle \sum_{j=1}^n |T_j^*|^2 x, x \right\rangle \right)^\alpha \right\} \right] \\ &\leq \frac{1}{2} \left[\left(\left\| \sum_{j=1}^n |T_j|^2 \right\| \right)^\alpha \left(\left\| \sum_{j=1}^n |T_j^*|^2 \right\| \right)^{1-\alpha} \right. \\ &\quad \left. + \left(\left\| \sum_{j=1}^n |T_j|^2 \right\| \right)^{1-\alpha} \left(\left\| \sum_{j=1}^n |T_j^*|^2 \right\| \right)^\alpha \right] \end{aligned} \tag{4.8}$$

for any $\alpha \in [0, 1]$.

Proof We have from the second inequality in (2.8)

$$\begin{aligned} \sum_{j=1}^n |\langle T_j x, y \rangle|^2 &\leq \frac{1}{2} \left[\left(\left\langle \sum_{j=1}^n |T_j|^2 x, x \right\rangle \right)^\alpha \left(\left\langle \sum_{j=1}^n |T_j^*|^2 y, y \right\rangle \right)^{1-\alpha} \right. \\ &\quad \left. + \left(\left\langle \sum_{j=1}^n |T_j|^2 x, x \right\rangle \right)^{1-\alpha} \left(\left\langle \sum_{j=1}^n |T_j^*|^2 y, y \right\rangle \right)^\alpha \right] \end{aligned} \tag{4.9}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$. Taking the supremum over $\|x\| = \|y\| = 1$, we have

$$\begin{aligned} &\|(T_1, \dots, T_n)\|_e^2 \\ &\leq \frac{1}{2} \left[\left(\sup_{\|x\|=1} \left\langle \sum_{j=1}^n |T_j|^2 x, x \right\rangle \right)^\alpha \left(\sup_{\|y\|=1} \left\langle \sum_{j=1}^n |T_j^*|^2 y, y \right\rangle \right)^{1-\alpha} \right. \\ &\quad \left. + \left(\sup_{\|x\|=1} \left\langle \sum_{j=1}^n |T_j|^2 x, x \right\rangle \right)^{1-\alpha} \left(\sup_{\|y\|=1} \left\langle \sum_{j=1}^n |T_j^*|^2 y, y \right\rangle \right)^\alpha \right] \\ &= \frac{1}{2} \left[\left(\left\| \sum_{j=1}^n |T_j|^2 \right\| \right)^\alpha \left(\left\| \sum_{j=1}^n |T_j^*|^2 \right\| \right)^{1-\alpha} + \left(\left\| \sum_{j=1}^n |T_j|^2 \right\| \right)^{1-\alpha} \left(\left\| \sum_{j=1}^n |T_j^*|^2 \right\| \right)^\alpha \right], \end{aligned}$$

which proves the first part of (4.7). The second part follows by the elementary inequality

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1 - \alpha)b$$

for $a, b \geq 0$ and $\alpha \in [0, 1]$. The inequality (4.8) follows from (4.9) by taking $y = x$ and then the supremum over $\|x\| = 1$. \square

5 Applications for s -1-norm and s -1-numerical radius

Following [20], we consider the s - p -norm of the n -tuple of operators $(T_1, \dots, T_n) \in B^{(n)}(H)$ by

$$\|(T_1, \dots, T_n)\|_{s,p} := \sup_{\|y\|=1, \|x\|=1} \left[\left(\sum_{j=1}^n |\langle T_j y, x \rangle|^p \right)^{\frac{1}{p}} \right]. \tag{5.1}$$

For $p = 2$, we get

$$\|(T_1, \dots, T_n)\|_{s,2} = \|(T_1, \dots, T_n)\|_e.$$

We are interested in this section in the case $p = 1$, namely, on the s -1-norm defined by

$$\|(T_1, \dots, T_n)\|_{s,1} := \sup_{\|y\|=1, \|x\|=1} \sum_{j=1}^n |\langle T_j y, x \rangle|.$$

Since for any $x, y \in H$ we have $\sum_{j=1}^n |\langle T_j y, x \rangle| \geq |\langle \sum_{j=1}^n T_j y, x \rangle|$, then by the properties of the supremum, we get the basic inequality

$$\left\| \sum_{j=1}^n T_j \right\| \leq \|(T_1, \dots, T_n)\|_{s,1} \leq \sum_{j=1}^n \|T_j\|. \tag{5.2}$$

Similarly, we can also consider the s - p -numerical radius of the n -tuple of operators $(T_1, \dots, T_n) \in B^{(n)}(H)$ by [20]

$$w_{s,p}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left[\left(\sum_{j=1}^n |\langle T_j x, x \rangle|^p \right)^{\frac{1}{p}} \right], \tag{5.3}$$

which for $p = 2$ reduces to the Euclidean operator radius introduced previously.

We observe that the s - p -numerical radius is also a norm on $B^{(n)}(H)$ for $p \geq 1$, and for $p = 1$ it satisfies the basic inequality

$$w \left(\sum_{j=1}^n T_j \right) \leq w_{s,1}(T_1, \dots, T_n) \leq \sum_{j=1}^n w(T_j). \tag{5.4}$$

We can state the following result.

Theorem 5.1 For any $(T_1, \dots, T_n) \in B^{(n)}(H)$, we have

$$\begin{aligned} & \|(T_1, \dots, T_n)\|_{s,1} \\ & \leq \left\| \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) \right\|^{1/2} \left\| \sum_{j=1}^n \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) \right\|^{1/2} \\ & \leq \frac{1}{2} \left[\left\| \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) \right\| + \left\| \sum_{j=1}^n \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) \right\| \right] \end{aligned} \tag{5.5}$$

and

$$w_{s,1}(T_1, \dots, T_n) \leq \left\| \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)} + |T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{4} \right) \right\|. \tag{5.6}$$

Proof From (2.1) we have

$$\begin{aligned} \sum_{j=1}^n |\langle T_j x, y \rangle| & \leq \left\langle \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \\ & \quad \times \left\langle \sum_{j=1}^n \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2} \end{aligned} \tag{5.7}$$

for any $x, y \in H$.

Taking the supremum over $\|y\| = 1, \|x\| = 1$ in (5.7), we have

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{s,1} &\leq \left[\sup_{\|x\|=1} \left\langle \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle \right]^{1/2} \\ &\quad \times \left[\sup_{\|y\|=1} \left\langle \sum_{j=1}^n \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) y, y \right\rangle \right]^{1/2} \\ &= \left\| \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) \right\|^{1/2} \\ &\quad \times \left\| \sum_{j=1}^n \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) \right\|^{1/2} \end{aligned}$$

and the first inequality in (5.5) is proved. The second part follows by the arithmetic mean-geometric mean inequality.

Now, if we take $y = x$ in (5.7), then we get

$$\begin{aligned} \sum_{j=1}^n |\langle T_j x, x \rangle| &\leq \left\langle \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \\ &\quad \times \left\langle \sum_{j=1}^n \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \\ &\leq \frac{1}{2} \left\langle \sum_{j=1}^n \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)} + |T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) x, x \right\rangle. \end{aligned}$$

Taking the supremum over $\|x\| = 1$, we deduce the desired result (5.6). \square

Remark 5.1 If we take $\alpha = \frac{1}{2}$ in the first inequality in (5.5), then we deduce

$$\|(T_1, \dots, T_n)\|_{s,1} \leq \left\| \sum_{j=1}^n |T_j| \right\|^{1/2} \left\| \sum_{j=1}^n |T_j^*| \right\|^{1/2}, \tag{5.8}$$

and then we get the following refinement of the generalized triangle inequality:

$$\begin{aligned} \left\| \sum_{j=1}^n T_j \right\| &\leq \|(T_1, \dots, T_n)\|_{s,1} \leq \left\| \sum_{j=1}^n |T_j| \right\|^{1/2} \left\| \sum_{j=1}^n |T_j^*| \right\|^{1/2} \\ &\leq \frac{1}{2} \left[\left\| \sum_{j=1}^n |T_j| \right\| + \left\| \sum_{j=1}^n |T_j^*| \right\| \right] \leq \sum_{j=1}^n \|T_j\|. \end{aligned}$$

From (5.6) we also have, for $\alpha = \frac{1}{2}$,

$$w_{s,1}(T_1, \dots, T_n) \leq \left\| \sum_{j=1}^n \left(\frac{|T_j| + |T_j^*|}{2} \right) \right\|. \tag{5.9}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Author details

¹School of Computer Science and Mathematics, Victoria University of Technology, P.O. Box 14428, MCMC, Melbourne, VIC 8001, Australia. ²School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg, 2050, South Africa. ³Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju, 660-701, Republic of Korea. ⁴Department of Mathematics, Changwon National University, Changwon, 641-773, Republic of Korea.

Acknowledgements

The authors wish to thank the anonymous referees for their valuable comments. Also, this research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-0008474).

Received: 21 August 2012 Accepted: 28 December 2012 Published: 16 January 2013

References

1. Helmberg, G: Introduction to Spectral Theory in Hilbert Space. Wiley, New York (1969)
2. Kato, T: Notes on some inequalities for linear operators. *Math. Ann.* **125**, 208-212 (1952)
3. Fujii, M, Lin, C-S, Nakamoto, R: Alternative extensions of Heinz-Kato-Furuta inequality. *Sci. Math.* **2**(2), 215-221 (1999)
4. Fujii, M, Furuta, T, Löwner-Heinz, C: Heinz-Kato inequalities. *Math. Jpn.* **38**(1), 73-78 (1993)
5. Fujii, M, Kamei, E, Kotari, C, Yamada, H: Furuta's determinant type generalizations of Heinz-Kato inequality. *Math. Jpn.* **40**(2), 259-267 (1994)
6. Fujii, M, Kim, YO, Seo, Y: Further extensions of Wielandt type Heinz-Kato-Furuta inequalities via Furuta inequality. *Arch. Inequal. Appl.* **1**(2), 275-283 (2003)
7. Fujii, M, Kim, YO, Tominaga, M: Extensions of the Heinz-Kato-Furuta inequality by using operator monotone functions. *Far East J. Math. Sci.: FJMS* **6**(3), 225-238 (2002)
8. Fujii, M, Nakamoto, R: Extensions of Heinz-Kato-Furuta inequality. *Proc. Am. Math. Soc.* **128**(1), 223-228 (2000)
9. Fujii, M, Nakamoto, R: Extensions of Heinz-Kato-Furuta inequality. II. *J. Inequal. Appl.* **3**(3), 293-302 (1999)
10. Furuta, T: Equivalence relations among Reid, Löwner-Heinz and Heinz-Kato inequalities, and extensions of these inequalities. *Integral Equ. Oper. Theory* **29**(1), 1-9 (1997)
11. Furuta, T: Determinant type generalizations of Heinz-Kato theorem via Furuta inequality. *Proc. Am. Math. Soc.* **120**(1), 223-231 (1994)
12. Furuta, T: An extension of the Heinz-Kato theorem. *Proc. Am. Math. Soc.* **120**(3), 785-787 (1994)
13. Kittaneh, F: Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24**(2), 283-293 (1988)
14. Kittaneh, F: Norm inequalities for fractional powers of positive operators. *Lett. Math. Phys.* **27**(4), 279-285 (1993)
15. Lin, C-S: On Heinz-Kato-Furuta inequality with best bounds. *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **15**(1), 93-101 (2008)
16. Lin, C-S: On chaotic order and generalized Heinz-Kato-Furuta-type inequality. *Int. Math. Forum* **2**(37-40), 1849-1858 (2007)
17. Lin, C-S: On inequalities of Heinz and Kato, and Furuta for linear operators. *Math. Jpn.* **50**(3), 463-468 (1999)
18. Lin, C-S: On Heinz-Kato type characterizations of the Furuta inequality. II. *Math. Inequal. Appl.* **2**(2), 283-287 (1999)
19. Uchiyama, M: Further extension of Heinz-Kato-Furuta inequality. *Proc. Am. Math. Soc.* **127**(10), 2899-2904 (1999)
20. Dragomir, SS: Some inequalities of Kato's type for sequences of operators in Hilbert spaces. *Publ. Res. Inst. Math. Sci.* **48**, 937-955 (2012).
21. Cho, YJ, Dragomir, SS, Pearce, CEM, Kim, SS: Cauchy-Schwarz functionals. *Bull. Aust. Math. Soc.* **62**, 479-491 (2000)
22. Dragomir, SS, Cho, YJ, Kim, SS: Some inequalities in inner product spaces related to the generalized triangle inequality. *Appl. Math. Comput.* **217**, 7462-7468 (2011)
23. Dragomir, SS, Cho, YJ, Kim, JK: Subadditivity of some functionals associated to Jensen's inequality with applications. *Taiwan. J. Math.* **15**, 1815-1828 (2011)
24. Lin, C-S, Cho, YJ: On Hölder-McCarthy-type inequalities with power. *J. Korean Math. Soc.* **39**, 351-361 (2002)
25. Lin, C-S, Cho, YJ: On norm inequalities of operators on Hilbert spaces. In: Cho, YJ, Kim, JK, Dragomir, SS (eds.) *Inequality Theory and Applications*, vol. 2, pp. 165-173. Nova Science Publishers, New York (2003)
26. Lin, C-S, Cho, YJ: On Kantorovich inequality and Hölder-McCarthy inequalities. *Dyn. Contin. Discrete Impuls. Syst.* **11**, 481-490 (2004)
27. Lin, C-S, Cho, YJ: Characteristic property for inequalities of bounded linear operators. In: Cho, YJ, Kim, JK, Dragomir, SS (eds.) *Inequality Theory and Applications*, vol. 4, pp. 85-92 (2007)
28. Lin, C-S, Cho, YJ: Characterizations of operator inequality $A \geq B \geq C$. *Math. Inequal. Appl.* **14**, 575-580 (2011)
29. Popescu, G: Unitary invariants in multivariable operator theory. *Mem. Am. Math. Soc.* **200**, Article ID 941 (2009)
30. Dragomir, SS: The hypo-Euclidean norm of an n -tuple of vectors in inner product spaces and applications. *J. Inequal. Pure Appl. Math.* **8**(2), Article ID 52 (2007)
31. McCarthy, CA: c_p . *Isr. J. Math.* **5**, 249-271 (1967)

doi:10.1186/1029-242X-2013-21

Cite this article as: Dragomir et al.: Applications of Kato's inequality for n -tuples of operators in Hilbert spaces, (I). *Journal of Inequalities and Applications* 2013 **2013**:21.