Trace inequalities of Shisha-Mond type for operators in Hilbert spaces

Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be two positive $n$-tuples with
\[ 0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \ldots, n\}, \tag{1} \]
then
\[ 0 \leq \left( \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \right)^{1/2} - \sum_{k=1}^{n} a_k b_k \leq \frac{(M - m)^2}{4(M + m)} \sum_{k=1}^{n} b_k^2. \tag{2} \]

The equality holds in (2) if and only if there exists a subsequence $(k_1, \ldots, k_p)$ of $\{1, \ldots, n\}$ such that
\[ \sum_{m=1}^{p} b_{k_m}^2 = \frac{M + 3m}{4(M + m)} \sum_{k=1}^{n} b_k^2, \]
\[ \frac{a_{k_m}}{b_{k_m}} = M \text{ for every } m = 1, \ldots, p \text{ and } \frac{a_k}{b_k} = m \text{ for every } k \text{ distinct from all } k_m. \]

Recall some other classical reverses of Cauchy-Bunyakovsky-Schwarz inequality when bounds for each $n$-tuple are available.

Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be two positive $n$-tuples with
\[ 0 < m_1 \leq a_i \leq M_1 < \infty \text{ and } 0 < m_2 \leq b_i \leq M_2 < \infty; \tag{3} \]
for each $i \in \{1, \ldots, n\}$, and some constants $m_1, m_2, M_1, M_2$.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

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a) Pólya-Szego’s inequality [51]:
\[ \frac{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}{(\sum_{k=1}^{n} a_kb_k)^2} \leq \frac{1}{4} \left( \sqrt{ \frac{M_1 M_2}{m_1 m_2} } + \sqrt{ \frac{m_1 m_2}{M_1 M_2} } \right)^2. \]

b) Shisha-Mond’s inequality [55]:
\[ \frac{\sum_{k=1}^{n} a_k^2}{\sum_{k=1}^{n} a_kb_k} - \frac{\sum_{k=1}^{n} a_kb_k}{\sum_{k=1}^{n} a_k^2} \leq \left[ \left( \frac{M_1}{m_1} \right)^{\frac{1}{2}} - \left( \frac{m_1}{M_1} \right)^{\frac{1}{2}} \right]^2. \]

c) Ozeki’s inequality [48]:
\[ \sum_{k=1}^{n} \frac{a_k^2}{w_k} \sum_{k=1}^{n} \frac{b_k^2}{w_k} - \left( \sum_{k=1}^{n} \frac{a_kb_k}{w_k} \right)^2 \leq \frac{n^2}{3} (M_1 M_2 - m_1 m_2)^2. \]

d) Diaz-Metcalf’s inequality [17]:
\[ \sum_{k=1}^{n} \frac{b_k^2}{w_k} + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^{n} \frac{a_k^2}{w_k} \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{k=1}^{n} \frac{a_kb_k}{w_k}. \]

If \( \mathbf{w} = (w_1, \ldots, w_n) \) is a positive sequence, then the following weighted inequalities also hold:

e) Cassels’ inequality [58]. If the positive real sequences \( \mathbf{a} = (a_1, \ldots, a_n) \) and \( \mathbf{b} = (b_1, \ldots, b_n) \) satisfy the condition (1), then
\[ \frac{\left( \sum_{k=1}^{n} w_k a_k^2 \right) \left( \sum_{k=1}^{n} w_k b_k^2 \right)}{\left( \sum_{k=1}^{n} w_k a_k b_k \right)^2} \leq \frac{(M + m)^2}{4mM}. \]

f) Greub-Reinboldt’s inequality [38]. We have
\[ \left( \sum_{k=1}^{n} w_k a_k^2 \right) \left( \sum_{k=1}^{n} w_k b_k^2 \right) \leq \left( \frac{M_1 M_2 + m_1 m_2}{4m_1 m_2 M_1 M_2} \right)^2 \left( \sum_{k=1}^{n} w_k a_k b_k \right)^2, \]
provided \( \mathbf{a} = (a_1, \ldots, a_n) \) and \( \mathbf{b} = (b_1, \ldots, b_n) \) satisfy the condition (3).

g) Generalized Diaz-Metcalf’s inequality [17], see also [46, p. 123]. If \( u, v \in [0, 1] \) and \( v \leq u, u + v = 1 \) and (1) holds, then one has the inequality
\[ u \sum_{k=1}^{n} w_k b_k^2 + v M m \sum_{k=1}^{n} w_k a_k^2 \leq (vm + uM) \sum_{k=1}^{n} w_k a_k b_k. \]

h) Klamkin-McLenaghan’s inequality [40]. If \( \mathbf{a}, \mathbf{b} \) satisfy (1), then
\[ \left( \sum_{i=1}^{n} w_i a_i^2 \right) \left( \sum_{i=1}^{n} w_i b_i^2 \right) - \left( \sum_{i=1}^{n} w_i a_i b_i \right)^2 \leq \left( M_2^2 - m_2^2 \right)^2 \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2. \] (4)

For other recent results providing discrete reverse inequalities, see the monograph online [19].

The following reverse of Schwarz’s inequality in inner product spaces holds [20].

**Theorem 1.2** (Dragomir, 2003, [20]). Let \( A, a \in \mathbb{C} \) and \( x, y \in H \), where \( H \) is a complex inner product space with the inner product \( \langle \cdot, \cdot \rangle \). If
\[ \text{Re} \langle Ay - x, x - ay \rangle \geq 0, \] (5)
or equivalently,
\[ \left\| x - \frac{a + A}{2} \cdot y \right\| \leq \frac{1}{2} \| A - a \| \| y \|, \] (6)
holds, then we have the inequality
\[
0 \leq \|x\|^2 \|y\|^2 - |(x, y)|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4. \tag{7}
\]
The constant \(\frac{1}{4}\) is sharp in (7).

In 1935, G. Grüss [39] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:
\[
\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma), \tag{8}
\]
where \(f, g : [a, b] \to \mathbb{R}\) are integrable on \([a, b]\) and satisfy the condition
\[
\phi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma \tag{9}
\]
for each \(x \in [a, b]\), where \(\phi, \Phi, \gamma, \Gamma\) are given real constants.
Moreover, the constant \(\frac{1}{4}\) is sharp in the sense that it cannot be replaced by a smaller one.

In [22], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

**Theorem 1.3** (Dragomir, 1999, [22]). Let \((H, \langle \cdot, \cdot \rangle)\) be an inner product space over \(\mathbb{K} (\mathbb{K} = \mathbb{R}, \mathbb{C})\) and \(e \in H, \|e\| = 1\). If \(\varphi, \gamma, \Phi, \Gamma\) are real or complex numbers and \(x, y\) are vectors in \(H\) such that the conditions
\[
\text{Re} \{\Phi e - x, x - \varphi e\} \geq 0 \quad \text{and} \quad \text{Re} \{\Gamma e - y, y - \gamma e\} \geq 0 \tag{10}
\]
hold, then we have the inequality
\[
|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma|. \tag{11}
\]
The constant \(\frac{1}{4}\) is best possible in the sense that it cannot be replaced by a smaller constant.

For other results of this type, see the recent monograph [25] and the references therein.

For other Grüss type results for integral and sums see the papers [1]-[3], [8]-[10], [11]-[13], [21]-[28], [35], [49], [62] and the references therein.

In order to state some reverses of Schwarz and Grüss type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

### 2 Some facts on trace of operators

Let \((H, \langle \cdot, \cdot \rangle)\) be a complex Hilbert space and \(\{e_i\}_{i \in I}\) an orthonormal basis of \(H\). We say that \(A \in \mathcal{B}(H)\) is a Hilbert-Schmidt operator if
\[
\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{12}
\]
It is well know that, if \(\{e_i\}_{i \in I}\) and \(\{f_j\}_{j \in J}\) are orthonormal bases for \(H\) and \(A \in \mathcal{B}(H)\) then
\[
\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in J} \|Af_j\|^2 = \sum_{j \in J} \|A^* f_j\|^2 \tag{13}
\]
showing that the definition (12) is independent of the orthonormal basis and \(A\) is a Hilbert-Schmidt operator iff \(A^*\) is a Hilbert-Schmidt operator.
Let $B_2(H)$ the set of Hilbert-Schmidt operators in $B(H)$. For $A \in B_2(H)$ we define

$$\|A\|_2 := \left( \sum_{i \in I} \|Ae_i\|^2 \right)^{1/2}$$

(14)

for $\{e_i\}_{i \in I}$ an orthonormal basis of $H$. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $B_2(H)$ is a vector space and that $\|\cdot\|_2$ is a norm on $B_2(H)$, which is usually called in the literature as the Hilbert-Schmidt norm.

Denote the modulus of an operator $A \in B(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\|A\| \leq \|A\|_2$ for all $x \in H$, $A$ is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \|A\|_2$.

From (13) we have that if $A \in B_2(H)$, then $A^* \in B_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

**Theorem 2.1.** We have

(i) $(B_2(H), \|\cdot\|_2)$ is a Hilbert space with inner product

$$\langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^* Ae_i, e_i \rangle$$

(15)

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$.

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2$$

(16)

for any $A \in B_2(H)$ and

$$\|AT\|_2 \cdot \|TA\|_2 \leq \|T\| \cdot \|A\|_2$$

(17)

for any $A \in B_2(H)$ and $T \in B(H)$;

(iii) $B_2(H)$ is an operator ideal in $B(H)$, i.e.

$$B(H)B_2(H)B(H) \subseteq B_2(H);$$

(iv) $B_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $B_2(H)$;

(v) $B_2(H) \subseteq K(H)$, where $K(H)$ denotes the algebra of compact operators on $H$.

If $\{e_i\}_{i \in I}$ an orthonormal basis of $H$, we say that $A \in B(H)$ is trace class if

$$\|A\|_1 := \sum_{i \in I} \|A|e_i, e_i\| < \infty.$$  

(18)

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $B_1(H)$ the set of trace class operators in $B(H)$.

The following proposition holds:

**Proposition 2.2.** If $A \in B(H)$, then the following are equivalent:

(i) $A \in B_1(H)$;

(ii) $|A|^{1/2} \in B_2(H)$;

(iii) $A$ (or $|A|$) is the product of two elements of $B_2(H)$.

The following properties are also well known:

**Theorem 2.3.** With the above notations:

(i) We have

$$\|A\|_1 = \|A^*\|_1 \quad \text{and} \quad \|A\|_2 \leq \|A\|_1$$

(19)

for any $A \in B_1(H)$.
(ii) \( B_1 (H) \) is an operator ideal in \( B (H) \), i.e.
\[
B (H) B_1 (H) B (H) \subseteq B_1 (H);
\]

(iii) We have
\[
B_2 (H) B_2 (H) = B_1 (H);
\]

(iv) We have
\[
\| A \|_1 = \sup \{ \langle A, B \rangle_2 \mid B \in B_2 (H), \| B \| \leq 1 \};
\]

(v) \((B_1 (H), \| \cdot \|_1)\) is a Banach space.

(iv) We have the following isometric isomorphisms
\[
B_1 (H) \cong K (H)^* \text{ and } B_1 (H)^* \cong B (H),
\]
where \( K (H)^* \) is the dual space of \( K (H) \) and \( B_1 (H)^* \) is the dual space of \( B_1 (H) \).

We define the trace of a trace class operator \( A \in B_1 (H) \) to be
\[
\text{tr} (A) := \sum_{i \in I} \langle A e_i, e_i \rangle,
\]
(20)
where \( \{ e_i \}_{i \in I} \) is an orthonormal basis of \( H \). Note that this coincides with the usual definition of the trace if \( H \) is finite-dimensional. We observe that the series (20) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 2.4.** We have
(i) If \( A \in B_1 (H) \) then \( A^* \in B_1 (H) \) and
\[
\text{tr} (A^*) = \overline{\text{tr} (A)};
\]
(21)
(ii) If \( A \in B_1 (H) \) and \( T \in B (H) \), then \( AT, TA \in B_1 (H) \) and
\[
\text{tr} (AT) = \text{tr} (TA) \text{ and } | \text{tr} (AT) | \leq \| A \|_1 \| T \| ;
\]
(22)
(iii) \( \text{tr} (\cdot) \) is a bounded linear functional on \( B_1 (H) \) with \( \| \text{tr} \| = 1; \)
(iv) If \( A, B \in B_2 (H) \) then \( AB, BA \in B_1 (H) \) and \( \text{tr} (AB) = \text{tr} (BA); \)
(v) \( B_{fin} (H) \) is a dense subspace of \( B_1 (H) \).

Utilising the trace notation we obviously have that
\[
\langle A, B \rangle_2 = \text{tr} (B^* A) = \text{tr} (AB^*) \text{ and } \| A \|_2^2 = \text{tr} (A^* A) = \text{tr} (|A|^2)
\]
for any \( A, B \in B_2 (H) \).

The following Hölder’s type inequality has been obtained by Ruskai in [52]
\[
| \text{tr} (AB) | \leq \text{tr} (|AB|) \leq \left[ \text{tr} \left( |A|^{1/\alpha} \right) \right]^{\alpha} \left[ \text{tr} \left( |B|^{1/(1-\alpha)} \right) \right]^{1-\alpha}
\]
(23)
where \( \alpha \in (0, 1) \) and \( A, B \in B (H) \) with \( |A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in B_1 (H) \).

In particular, for \( \alpha = \frac{1}{2} \) we get the Schwarz inequality
\[
| \text{tr} (AB) | \leq \text{tr} (|AB|) \leq \left[ \text{tr} \left( |A|^2 \right) \right]^{1/2} \left[ \text{tr} \left( |B|^2 \right) \right]^{1/2}
\]
(24)
with \( A, B \in B_2 (H) \).

For the theory of trace functionals and their applications the reader is referred to [56].

For some classical trace inequalities see [14], [16], [47] and [61], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [14], [36], [41], [42], [44], [53] and [57].

We denote by
\[
B_1^+ (H) := \{ P : P \in B_1 (H), P \text{ is selfadjoint and } P \geq 0 \}.
\]
We obtained recently the following result [33]:
Theorem 2.5. For any $A, C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_{1+}^+(H) \setminus \{0\}$ we have the inequality
\[
\frac{\left| \text{tr}(PAC) \right|}{\text{tr}(P)} - \frac{\text{tr}(PA) \text{tr}(PC)}{\text{tr}(P)} \leq \inf_{\lambda \in \mathbb{C}} \| A - \lambda \cdot 1_H \| \frac{1}{\text{tr}(P)} \text{tr}\left( \left( C - \frac{\text{tr}(PC)}{\text{tr}(P)} \cdot 1_H \right) P \right) \]
\[
\leq \inf_{\lambda \in \mathbb{C}} \| A - \lambda \cdot 1_H \| \frac{\text{tr}(P |C|^2)}{\text{tr}(P)} - \left( \frac{\text{tr}(PC)}{\text{tr}(P)} \right)^2 \right)^{1/2},
\]
where $\| \cdot \|$ is the operator norm.

We also have [33]:

Corollary 2.6. Let $\alpha, \beta \in \mathbb{C}$ and $A \in \mathcal{B}(H)$ such that
\[
\left\| A - \frac{\alpha + \beta}{2} \cdot 1_H \right\| \leq \frac{1}{2} \left| \beta - \alpha \right|.
\]

For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_{1+}^+(H) \setminus \{0\}$ we have the inequality
\[
\frac{\left| \text{tr}(PAC) \right|}{\text{tr}(P)} - \frac{\text{tr}(PA) \text{tr}(PC)}{\text{tr}(P)} \leq \frac{1}{2} \left| \beta - \alpha \right| \frac{1}{\text{tr}(P)} \text{tr}\left( \left( C - \frac{\text{tr}(PC)}{\text{tr}(P)} \cdot 1_H \right) P \right) \]
\[
\leq \frac{1}{2} \left| \beta - \alpha \right| \left[ \frac{\text{tr}(P |C|^2)}{\text{tr}(P)} - \left( \frac{\text{tr}(PC)}{\text{tr}(P)} \right)^2 \right]^{1/2}.
\]

In particular, if $C \in \mathcal{B}(H)$ is such that
\[
\left\| C - \frac{\alpha + \beta}{2} \cdot 1_H \right\| \leq \frac{1}{2} \left| \beta - \alpha \right|,
\]
then
\[
0 \leq \frac{\text{tr}(P |C|^2)}{\text{tr}(P)} - \left( \frac{\text{tr}(PC)}{\text{tr}(P)} \right)^2 \leq \frac{1}{2} \left| \beta - \alpha \right| \frac{1}{\text{tr}(P)} \text{tr}\left( \left( C - \frac{\text{tr}(PC)}{\text{tr}(P)} \cdot 1_H \right) P \right) \]
\[
\leq \frac{1}{2} \left| \beta - \alpha \right| \left[ \frac{\text{tr}(P |C|^2)}{\text{tr}(P)} - \left( \frac{\text{tr}(PC)}{\text{tr}(P)} \right)^2 \right]^{1/2} \leq \frac{1}{4} \left| \beta - \alpha \right|^2.
\]

Also
\[
\frac{\left| \text{tr}(PC^2) \right|}{\text{tr}(P)} - \left( \frac{\text{tr}(PC)}{\text{tr}(P)} \right)^2 \leq \frac{1}{2} \left| \beta - \alpha \right| \frac{1}{\text{tr}(P)} \text{tr}\left( \left( C - \frac{\text{tr}(PC)}{\text{tr}(P)} \cdot 1_H \right) P \right) \]
\[
\leq \frac{1}{2} \left| \beta - \alpha \right| \left[ \frac{\text{tr}(P |C|^2)}{\text{tr}(P)} - \left( \frac{\text{tr}(PC)}{\text{tr}(P)} \right)^2 \right]^{1/2} \leq \frac{1}{4} \left| \beta - \alpha \right|^2.
\]

For other related results see [33].
3 Shisha-Mond type trace inequalities

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$C_{\alpha, \beta} (T, U) = (T^* - \tilde{\alpha} U^*) (\beta U - T).$$

This transform generalizes the transform

$$C_{\alpha, \beta} (T) := (T^* - \tilde{\alpha} 1_H) (\beta 1_H - T) = C_{\alpha, \beta} (T, 1_H),$$

where $1_H$ is the identity operator, which has been introduced in [31] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator $T$ on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called **accretive** if

$$\Re \langle T y, y \rangle \geq 0$$

for any $y \in H$.

Utilizing the following identity

$$\Re \langle C_{\alpha, \beta} (T, U) x, x \rangle = \Re \langle C_{\beta, \alpha} (T, U) x, x \rangle$$

that holds for any scalars $\alpha, \beta$ and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

**Lemma 3.1.** For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:

- (i) The transform $C_{\alpha, \beta} (T, U)$ (or, equivalently, $C_{\beta, \alpha} (T, U)$) is accretive;
- (ii) We have the norm inequality

$$\left\| TX - \frac{\alpha + \beta}{2} U x \right\| \leq \frac{1}{2} |\beta - \alpha| \|Ux\|$$

for any $x \in H$;
- (iii) We have the following inequality in the operator order

$$\left\| T - \frac{\alpha + \beta}{2} U \right\|^2 \leq \frac{1}{4} |\beta - \alpha|^2 |U|^2.$$

As a consequence of the above lemma we can state:

**Corollary 3.2.** Let $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$. If $C_{\alpha, \beta} (T, U)$ is accretive, then

$$\left\| T - \frac{\alpha + \beta}{2} U \right\| \leq \frac{1}{2} |\beta - \alpha| \|U\|.$$

**Remark 3.3.** In order to give examples of linear operators $T, U \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha, \beta} (T, U)$ is accretive, it suffices to select two bounded linear operator $S$ and $V$ and the complex numbers $z, w$ ($w \neq 0$) with the property that $\|S x - z V x\| \leq |w| \|V x\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2} (z + w)$ and $\beta = \frac{1}{2} (z - w)$ we observe that $T$ and $U$ satisfy (30), i.e., $C_{\alpha, \beta} (T, U)$ is accretive.

The following result is useful in the sequel:

**Lemma 3.4.** Let, either $P \in B_+ (H)$, $A, B \in B_2 (H)$ or $P \in B_1^+ (H)$, $A, B \in B (H)$ and $\gamma, \Gamma \in \mathbb{C}$. Then

$$\Re \langle \text{tr} \left[ P \left( A^* - \overline{\gamma} B^* \right) (\Gamma B - A) \right] \rangle \geq 0$$

(32)
if and only if

\[ \text{tr} \left( P \left| A - \frac{\gamma + \Gamma}{2} B \right|^2 \right) \leq \frac{1}{4} |\Gamma - \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right). \]  

(33)

To simplify the writing, we say that \((A, B)\) satisfies the \(P-(\gamma, \Gamma)\)-trace property.

Proof. Doing the calculation, we have the equality

\[ \frac{1}{4} |\Gamma - \gamma|^2 P \left| B \right|^2 - P \left| A - \frac{\gamma + \Gamma}{2} B \right|^2 = P \left[ -|A|^2 + \frac{\gamma + \Gamma}{2} B^*A + \frac{\gamma + \Gamma}{2} A^*B - \text{Re}(\Gamma \gamma) \left| B \right|^2 \right] \]  

(34)

for any bounded operators \(A, B, P\) and the complex numbers \(\gamma, \Gamma \in \mathbb{C}\).

Taking the trace in (34) we get after some simple manipulation

\[ \frac{1}{4} |\Gamma - \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right) - \text{tr} \left( P \left| A - \frac{\gamma + \Gamma}{2} B \right|^2 \right) \]

\[ = -\text{tr} \left( P \left| A \right|^2 \right) - \text{Re}(\Gamma \gamma) \text{tr} \left( P \left| B \right|^2 \right) \]

\[ + \text{Re} \left[ \text{tr} \left( PB^*A \right) \right] + \text{Re} \left[ \text{tr} \left( \Gamma \text{tr}(PB^*A) \right) \right]. \]

Since

\[ \text{Re} \left[ \text{tr} \left( P \left( A^* - \bar{\gamma}B^* \right) (\Gamma B - A) \right) \right] = \text{Re} \left[ \Gamma \text{tr}(PB^*A) + \bar{\gamma} \text{tr}(PB^*A) \right] - \text{tr} \left( P \left| B \right|^2 \right) \text{Re}(\bar{\gamma}\Gamma) - \text{tr} \left( P \left| A \right|^2 \right), \]

then we get

\[ \frac{1}{4} |\Gamma - \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right) - \text{tr} \left( P \left| A - \frac{\gamma + \Gamma}{2} B \right|^2 \right) = \text{Re} \left[ \text{tr} \left( P \left( A^* - \bar{\gamma}B^* \right) (\Gamma B - A) \right) \right]. \]  

(36)

which proves the desired equivalence.

\[ \square \]

Corollary 3.5. Let, either \(P \in \mathcal{B}_+ (H)\), \(A, B \in \mathcal{B}_2 (H)\) or \(P \in \mathcal{B}_1^+ (H)\), \(A, B \in \mathcal{B}(H)\) and \(\gamma, \Gamma \in \mathbb{C}\). If the transform \(C_{\gamma, \Gamma} (A, B)\) is accretive, then \((A, B)\) satisfies the \(P-(\gamma, \Gamma)\)-trace property.

We have the following result:

Theorem 3.6. Let, either \(P \in \mathcal{B}_+ (H)\), \(A, B \in \mathcal{B}_2 (H)\) or \(P \in \mathcal{B}_1^+ (H)\), \(A, B \in \mathcal{B}(H)\) and \(\gamma, \Gamma \in \mathbb{C}\) with \(\Gamma + \gamma \neq 0\).

(i) If \((A, B)\) satisfies the \(P-(\gamma, \Gamma)\)-trace property, then we have

\[ \sqrt{\text{tr} \left( P \left| A \right|^2 \right) \text{tr} \left( P \left| B \right|^2 \right)} \leq \frac{\text{Re}(\gamma + \Gamma) \text{Re} \text{tr}(PB^*A) + \text{Im}(\gamma + \Gamma) \text{Im} \text{tr}(PB^*A)}{|\Gamma + \gamma|} \]

\[ + \frac{1}{4} |\Gamma - \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right) \]

\[ \leq |\text{tr} \left( PB^*A \right)| + \frac{1}{4} |\Gamma - \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right). \]  

(37)

(ii) If the transform \(C_{\gamma, \Gamma} (A, B)\) is accretive, then the inequality (37) also holds.

Proof. (i) If \((A, B)\) satisfies the \(P-(\gamma, \Gamma)\)-trace property, then

\[ \text{tr} \left( P \left| A - \frac{\gamma + \Gamma}{2} B \right|^2 \right) \leq \frac{1}{4} |\Gamma - \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right) \]

that is equivalent to

\[ \text{tr} \left( P \left| A \right|^2 \right) - \text{Re} \left[ (\bar{\gamma} + \Gamma) \text{tr}(PB^*A) \right] + \frac{1}{4} |\Gamma + \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right) \leq \frac{1}{4} |\Gamma - \gamma|^2 \text{tr} \left( P \left| B \right|^2 \right). \]
which implies that
\[
\text{tr} \left( P \left| A \right|^2 \right) + \frac{1}{4} \left| \Gamma + \gamma \right|^2 \text{tr} \left( P \left| B \right|^2 \right) \leq \text{Re} \left[ (\overline{\gamma} + \overline{\Gamma}) \text{tr} (PB^*A) \right] + \frac{1}{4} \left| \Gamma - \gamma \right|^2 \text{tr} \left( P \left| B \right|^2 \right).
\]
(38)

Making use of the elementary inequality
\[
2 \sqrt{pq} \leq p + q, \ p,q \geq 0,
\]
we also have
\[
\left| \Gamma + \gamma \right| \left[ \text{tr} \left( P \left| A \right|^2 \right) \text{tr} \left( P \left| B \right|^2 \right) \right]^{1/2} \leq \text{tr} \left( P \left| A \right|^2 \right) + \frac{1}{4} \left| \Gamma + \gamma \right|^2 \text{tr} \left( P \left| B \right|^2 \right).
\]
(39)

Utilising (38) and (39) we get
\[
\left| \Gamma + \gamma \right| \left[ \text{tr} \left( P \left| A \right|^2 \right) \text{tr} \left( P \left| B \right|^2 \right) \right]^{1/2} \leq \text{Re} \left[ (\overline{\gamma} + \overline{\Gamma}) \text{tr} (PB^*A) \right] + \frac{1}{4} \left| \Gamma - \gamma \right|^2 \text{tr} \left( P \left| B \right|^2 \right).
\]
(40)

Dividing by \( \left| \Gamma + \gamma \right| > 0 \) and observing that
\[
\text{Re} \left[ (\overline{\gamma} + \overline{\Gamma}) \text{tr} (PB^*A) \right] = \text{Re} (\gamma + \Gamma) \text{Re} \text{tr} (PB^*A) + \text{Im} (\gamma + \Gamma) \text{Im} \text{tr} (PB^*A)
\]
we get the first inequality in (37).

The second inequality in (37) is obvious by Schwarz inequality
\[
(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2), \ a,b,c,d \in \mathbb{R}.
\]

The (ii) is obvious from (i).

**Remark 3.7.** We observe that the inequality between the first and last term in (37) is equivalent to
\[
0 \leq \sqrt{\text{tr} \left( P \left| A \right|^2 \right) \text{tr} \left( P \left| B \right|^2 \right)} - \left| \text{tr} (PB^*A) \right| \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2 \text{tr} \left( P \left| B \right|^2 \right).
\]
(41)

**Corollary 3.8.** Let, either \( P \in B_+ (H), \ A \in B_2 (H) \) or \( P \in B_1^+ (H), \ A \in B (H) \) and \( \gamma, \Gamma \in \mathbb{C} \) with \( \gamma + \Gamma \neq 0 \).

(i) If \( A \) satisfies the \( P - \gamma, \Gamma \)-trace property, namely
\[
\text{Re} \left( \text{tr} \left( P \left( A^* - \overline{\gamma} 1_H \right) (\Gamma 1_H - A) \right) \right) \geq 0
\]
(42)
or, equivalently
\[
\text{tr} \left( P \left| A - \frac{\gamma + \Gamma}{2} 1_H \right|^2 \right) \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2 \text{tr} (P),
\]
(43)
then we have
\[
\sqrt{\frac{\text{tr} \left( P \left| A \right|^2 \right)}{\text{tr} (P)}} \leq \frac{\text{Re} (\gamma + \Gamma) \frac{\text{Re} \text{tr}(PA)}{\text{tr}(P)} + \text{Im} (\gamma + \Gamma) \frac{\text{Im} \text{tr}(PA)}{\text{tr}(P)}}{|\gamma + \Gamma|} + \frac{1}{4} \left| \Gamma - \gamma \right|^2 \frac{1}{|\gamma + \Gamma|}.
\]
(44)

(ii) If the transform \( C_{\gamma, \Gamma} (A) \) is accretive, then the inequality (37) also holds.

(iii) We have
\[
0 \leq \sqrt{\frac{\text{tr} \left( P \left| A \right|^2 \right)}{\text{tr} (P)}} - \left| \text{tr} (PA) \right| \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2 \frac{1}{|\gamma + \Gamma|}.
\]
(45)
Remark 3.9. The case of selfadjoint operators is as follows.

Let $A, B$ be selfadjoint operators and either $P \in \mathcal{B}_+(H), A, B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H), A, B \in \mathcal{B}(H)$ and $m, M \in \mathbb{R}$ with $m + M \neq 0$.

(i) If $(A, B)$ satisfies the $P$-$(m, M)$-trace property, then we have
\[
\sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} \leq \operatorname{Re} \operatorname{tr}(PBA) + \frac{(M - m)^2}{4|M + m|} \operatorname{tr}(PB^2)
\]
and
\[
0 \leq \sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} - \operatorname{Re} \operatorname{tr}(PBA) \leq \frac{(M - m)^2}{4|M + m|} \operatorname{tr}(PB^2).
\]

(ii) If the transform $C_{m, M} (A, B)$ is accretive, then the inequality (46) also holds.

(iii) If $(A - mB)(MB - A) \geq 0$, then (46) is valid.

We observe that the inequality (46) in the case when $M > m > 0$ is the operator trace inequality version of Shisha-Mond inequality (1) from Introduction.

Corollary 3.10. Let $A, B$ be selfadjoint operators and either $P \in \mathcal{B}_+(H), A, B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H), A, B \in \mathcal{B}(H)$ and $m, M \in \mathbb{R}$ with $m + M \neq 0$.

(i) If $(A, B)$ satisfies the $P$-$(m, M)$-trace property, then we have
\[
\left(\sqrt{\operatorname{tr}(PA^2) + \operatorname{tr}(PB^2)}\right) - \operatorname{tr}(P(A + B)^2) \leq \frac{(M - m)^2}{4|M + m|} \operatorname{tr}(PB^2)
\]
and
\[
\sqrt{\operatorname{tr}(PA^2) + \operatorname{tr}(PB^2)} - \sqrt{\operatorname{tr}(P(A + B)^2)} \leq \frac{\sqrt{2} |M - m|}{2\sqrt{|M + m|}} \sqrt{\operatorname{tr}(PB^2)}.
\]

Proof. Observe that
\[
\left(\sqrt{\operatorname{tr}(PA^2) + \operatorname{tr}(PB^2)}\right)^2 - \operatorname{tr}(P(A + B)^2) = 2\left(\sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} - \operatorname{Re} \operatorname{tr}(PBA)\right).
\]

Utilising (46) we deduce (47).

The inequality (48) follows from (47).

4 Trace inequalities of Grüss type

Let $P$ be a selfadjoint operator with $P \geq 0$. The functional $\langle \cdot, \cdot \rangle_{2, P}$ defined by
\[
\langle A, B \rangle_{2, P} := \operatorname{tr}(PB^*A) = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)
\]
is a nonnegative Hermitian form on $\mathcal{B}_2(H)$, i.e., $\langle \cdot, \cdot \rangle_{2, P}$ satisfies the properties:

(h) $\langle A, A \rangle_{2, P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

(hh) $\langle \cdot, \cdot \rangle_{2, P}$ is linear in the first variable;

(hhh) $\langle B, A \rangle_{2, P} = \langle A, B \rangle_{2, P}$ for any $A, B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations
\[
\|A\|_{2, P}^2 := \operatorname{tr}(PA^2) = \operatorname{tr}(APA^*) = \operatorname{tr}(|A|^2 P)
\]
and
\[
\langle A, B \rangle_{2, P} = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)
\]
for any $A, B \in B_2(H)$.

The same definitions can be considered if $P \in B_4^+(H)$ and $A, B \in B(H)$.

We have the following Grüss type inequality:

**Theorem 4.1.** Let, either $P \in B_4^+(H)$, $A, B, C \in B_2(H)$ or $P \in B_4^+(H)$, $A, B, C \in B(H)$ with $P|A|^2$, $P|B|^2$, $P|C|^2 \neq 0$ and $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ with $\gamma + \Gamma \neq 0, \delta + \Delta \neq 0$. If $(A, C)$ has the trace $P-(\lambda, \Gamma)$-property and $(B, C)$ has the trace $P-(\delta, \Delta)$-property, then

$$
\frac{\left| \text{tr} (PB^*A) - \text{tr} (PC^*A) \text{tr} (PB^*C) \right|^2}{\text{tr} (P|C|^2)^2} \leq \frac{1}{4} \frac{|\Gamma - \gamma|^2 |\Delta - \delta|^2}{|\Gamma + \gamma| |\Delta + \delta|} \left[ \frac{\text{tr} (P|A|^2) \text{tr} (P|B|^2)}{\text{tr} (P|C|^2)^2} \right].
$$

(49)

**Proof.** We prove in the case that $P \in B_4^+(H)$ and $A, B, C \in B_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $(\cdot, \cdot)_{2, p}$ we have

$$
|\langle A, B \rangle_{2, p}|^2 \leq \langle A, A \rangle_{2, p} \langle B, B \rangle_{2, p}
$$

for any $A, B \in B_2(H)$.

Let $C \in B_2(H), C \neq 0$. Define the mapping $[\cdot, \cdot]_{2, p, C} : B_2(H) \times B_2(H) \to \mathbb{C}$ by

$$
[A, B]_{2, p, C} := \langle A, B \rangle_{2, p} \|C\|_{2, p} - \langle A, C \rangle_{2, p} \langle C, B \rangle_{2, p}.
$$

Observe that $[\cdot, \cdot]_{2, p, C}$ is a nonnegative Hermitian form on $B_2(H)$ and by Schwarz inequality we also have

$$
\left| \langle A, B \rangle_{2, p} \|C\|_{2, p} - \langle A, C \rangle_{2, p} \langle C, B \rangle_{2, p} \right|^2
$$

$$
\leq \left[ \|A\|_{2, p} \|C\|_{2, p} - |\langle A, C \rangle_{2, p}|^2 \right] \left[ \|B\|_{2, p} \|C\|_{2, p} - |\langle B, C \rangle_{2, p}|^2 \right]
$$

for any $A, B \in B_2(H)$, namely

$$
\left| \text{tr} (PB^*A) - \text{tr} (PC^*A) \text{tr} (PB^*C) \right|^2 \leq \left[ \text{tr} (P|A|^2) \text{tr} (P|C|^2) - |\text{tr} (PC^*A)|^2 \right] \times \left[ \text{tr} (P|B|^2) \text{tr} (P|C|^2) - |\text{tr} (PB^*C)|^2 \right].
$$

(50)

where for the last term we used the equality $|\langle B, C \rangle_{2, p}|^2 = |\langle C, B \rangle_{2, p}|^2$.

Since $(A, C)$ has the trace $P-(\lambda, \Gamma)$-property and $(B, C)$ has the trace $P-(\delta, \Delta)$-property, then by (41) we have

$$
0 \leq \sqrt{\text{tr} (P|A|^2) \text{tr} (P|C|^2)} - |\text{tr} (PC^*A)| \leq \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \text{tr} (P|C|^2)
$$

and

$$
0 \leq \sqrt{\text{tr} (P|B|^2) \text{tr} (P|C|^2)} - |\text{tr} (PC^*B)| \leq \frac{1}{4} \frac{|\Delta - \delta|^2}{|\Delta + \delta|} \text{tr} (P|C|^2)
$$

which imply

$$
0 \leq \text{tr} (P|A|^2) \text{tr} (P|C|^2) - |\text{tr} (PC^*A)|^2 \leq \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \text{tr} (P|C|^2) \leq \frac{1}{4} \frac{|\Delta - \delta|^2}{|\Delta + \delta|} \text{tr} (P|C|^2)
$$

(51)
Corollary 4.2. Let, either $P \in \mathbb{B}_i^+(H)$, $A$, $B \in \mathbb{B}_2$ or $P \in \mathbb{B}_i^+(H)$, $A$, $B \in \mathbb{B}(H)$ with $P \geq 0$, $P \neq 0$ and $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ with $\gamma + \Gamma \neq 0, \delta + \Delta \neq 0$. If $A$ has the trace $P-(\lambda, \Gamma)$-property and $B$ has the trace $P-(\delta, \Delta)$-property, then

$$\left| \text{tr}(PB^*A) - \text{tr}(P(C^*)A) - \text{tr}(PB^*) \right| \leq \frac{1}{4} \left( |\Gamma - \gamma|^2 |\Delta - \delta|^2 \right) \frac{\text{tr}(P[A]^2)\text{tr}(P[B]^2)}{\left[ \text{tr}(P) \right]^2}. \tag{55}$$

The case of selfadjoint operators is useful for applications.

Remark 4.3. Assume that $A, B, C$ are selfadjoint operators. If, either $P \in \mathbb{B}_i^+(H)$, $A, B, C \in \mathbb{B}_2$ or $P \in \mathbb{B}_i^+(H)$, $A, B, C \in \mathbb{B}(H)$ with $P \geq 0$, $P \neq 0$ and $m, M, n, N \in \mathbb{R}$ with $m + M, n + N \neq 0$. If $(A, C)$ has the trace $P-(m, M)$-property and $(B, C)$ has the trace $P-(n, N)$-property, then

$$\left| \frac{\text{tr}(P[BA])}{\text{tr}(PC^2)} - \frac{\text{tr}(PCA)\text{tr}(PC^2)}{\text{tr}(PC^2)^2} \left[ \frac{\text{tr}(PA)^2}{\text{tr}(PC^2)^2} \right] \right| \leq \frac{1}{4} \left( M - m \right)^2 \left( N - n \right)^2 \frac{\left[ \text{tr}(P[A]^2)\text{tr}(P[B]^2) \right]}{\left[ \text{tr}(P) \right]^2}. \tag{56}$$

If $A$ has the trace $P-(k, K)$-property and $B$ has the trace $P-(l, L)$-property, then

$$\left| \frac{\text{tr}(P[BA])}{\text{tr}(P)} - \frac{\text{tr}(PA)\text{tr}(PB)}{\text{tr}(P)^2} \left[ \frac{\text{tr}(PA)^2}{\text{tr}(P)^2} \right] \right| \leq \frac{1}{4} \left( K - k \right)^2 \left( L - l \right)^2 \frac{\left[ \text{tr}(P[A]^2)\text{tr}(P[B]^2) \right]}{\left[ \text{tr}(P) \right]^2}. \tag{57}$$

where $k + K, l + L \neq 0$. 

\[ \frac{1}{2} |\Gamma - \gamma|^2 \text{tr}(P[C]^2) \text{tr}(P[C]^2) \]

and

$$0 \leq \text{tr}(P |B|^2) \text{tr}(P |C|^2) - |\text{tr}(PB^*C)|^2 \leq \frac{1}{4} |\Delta - \delta|^2 \left( \frac{\text{tr}(P |B|^2)\text{tr}(P |C|^2)}{\text{tr}(P |A|^2)} + |\text{tr}(PC^*B)| \right) \leq \frac{1}{4} |\Delta - \delta|^2 \text{tr}(P |C|^2) \frac{\text{tr}(P |B|^2)\text{tr}(P |C|^2)}{\text{tr}(P |A|^2)}.$$
5 Applications for convex functions

In the paper [34] we obtained amongst other the following reverse of the Jensen trace inequality:

Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a continuously differentiable convex function on $[m, M]$ and $P \in B_1(H) \setminus \{0\}, P \geq 0$, then we have

$$0 \leq \frac{\text{tr}( Pf(A))}{\text{tr}(P)} - f \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right) \quad (58)$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{\text{tr}(P)} \left[ \frac{P[A - \frac{u(A)}{n}1_H]}{u(P)} \right]$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{\text{tr}(P)} \left[ \frac{P[A - \frac{u(A)}{n}1_H]}{u(P)} \right]$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{\text{tr}(P)} \left[ \frac{P[A - \frac{u(A)}{n}1_H]}{u(P)} \right]$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{\text{tr}(P)} \left( \frac{u(A)}{n} \right)^2$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{\text{tr}(P)} \left( \frac{u(A)}{n} \right)^2$$

$$\leq \frac{1}{4} [f'(M) - f'(m)] (M - m).$$

Let $\mathcal{M}_n(C)$ be the space of all square matrices of order $n$ with complex elements and $A \in \mathcal{M}_n(C)$ be a Hermitian matrix such that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a continuously differentiable convex function on $[m, M]$, then by taking $P = I_n$ in (58) we get

$$0 \leq \frac{\text{tr}( f(A))}{n} - f \left( \frac{\text{tr}(A)}{n} \right) \quad (59)$$

$$\leq \frac{\text{tr}( f'(A) A)}{n} - \frac{\text{tr}(A)}{n}, \frac{\text{tr}( f'(A))}{n}$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{n} \left[ \frac{P[A - \frac{u(A)}{n}1_H]}{u(P)} \right]$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{n} \left[ \frac{P[A - \frac{u(A)}{n}1_H]}{u(P)} \right]$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{n} \left[ \frac{P[A - \frac{u(A)}{n}1_H]}{u(P)} \right]$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{n} \left[ \frac{P[A - \frac{u(A)}{n}1_H]}{u(P)} \right]$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{n} \left( \frac{u(A)}{n} \right)^2$$

$$\leq \frac{\frac{1}{2} [f'(M) - f'(m)]}{n} \left( \frac{u(A)}{n} \right)^2$$

$$\leq \frac{1}{4} [f'(M) - f'(m)] (M - m).$$

The following reverse inequality also holds:

**Proposition 5.1.** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m + M \neq 0$. If $f$ is a continuously differentiable convex function on $[m, M]$ with $f'(m) + f'(M) \neq 0$ and $P \in B_1(H) \setminus \{0\}, P \geq 0$, then we have

$$0 \leq \frac{\text{tr}( Pf(A))}{\text{tr}(P)} - f \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right) \quad (60)$$
We consider the convex function $f$ and assume that $Sp (A) \subseteq [m, M]$ for some scalars $m, M$ with $m + M \neq 0$. If $f$ is a continuously differentiable convex function on $[m, M]$ with $f'(m) + f'(M) \neq 0$ then taking $P = I_n$ in (60) we get

$$0 \leq \frac{\text{tr} (f (A))}{n} - f \left( \frac{\text{tr} (A)}{n} \right)$$

(61)

We consider the power function $f : (0, \infty) \to (0, \infty) \ni f (t) = t^r$ with $r \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1, \infty)$, $f$ is convex while for $r \in (0, 1)$, $f$ is concave.

Let $r \geq 1$ and $A$ be a self-adjoint operator on the Hilbert space $H$ and assume that $Sp (A) \subseteq [m, M]$ for some scalars $m, M$ with $0 < m < M$. If $P \in B_1^+ (H) \setminus \{0\}$, then

$$0 \leq \frac{\text{tr} (P f^r (A) A)}{\text{tr} (P)} - \frac{\text{tr} (P A)}{\text{tr} (P)} \left( \frac{\text{tr} (P f^r (A))}{\text{tr} (P)} \right)^r$$

(62)

Consider the convex function $f : \mathbb{R} \to (0, \infty) \ni f (t) = \exp t$ and let $A$ be a self-adjoint operator on the Hilbert space $H$ and assume that $Sp (A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $P \in B_1^+ (H) \setminus \{0\}$, then using (60) we have

$$0 \leq \frac{\text{tr} (P \exp A)}{\text{tr} (P)} - \exp \left( \frac{\text{tr} (P A)}{\text{tr} (P)} \right)$$

(63)

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