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Some Hermite-Hadamard type inequalities for operator convex functions and positive maps

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Abstract: In this paper we establish some inequalities of Hermite-Hadamard type for operator convex functions and positive maps. Applications for power function and logarithm are also provided.

Keywords: Jensen's inequality, Hermite-Hadamard inequality, Positive maps, Operator convex functions, Arithmetic mean-Geometric mean inequality

MSC: 47A63, 47A30, 15A60

1 Introduction

A real valued continuous function f on an interval I is said to be *operator convex* (*operator concave*) if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B) \quad (\text{OC})$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I . Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\text{Sp}(A), \text{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see for instance [12] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In [4], see also [5, p. 60], we established the following Hermite-Hadamard type inequality for operator convex functions:

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Theorem 1. Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I we have the inequality

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{2} \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right) \right] \\ &\leq \int_0^1 f((1-t)A + tB) dt \\ &\leq \frac{1}{2} \left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2} \right] \leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.1}$$

For recent related results on operator Hermite-Hadamard type inequalities, see [1]-[2], [5]-[10] and [13].

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H . We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H .

Let H, K be complex Hilbert spaces. Following [3] (see also [12, p. 18]) we can introduce the following definition:

Definition 1. A map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any $\lambda, \mu \in \mathbb{C}$ and $A, B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the order relation, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

The following Jensen’s type result is well known [3]:

Theorem 2 (Davis-Choi-Jensen’s Inequality). Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.2}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ in (1.2) we get

$$f\left(\Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H) \Psi(f(A)) \Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following Davis-Choi-Jensen’s inequality for general positive linear maps:

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H)\right) \Psi^{1/2}(1_H) \leq \Psi(f(A)). \tag{1.3}$$

In this paper, motivated by the above results, we establish some inequalities of Hermite-Hadamard type for operator convex functions and positive maps. Applications for power function and logarithm are also provided.

2 Refinements of HH-Inequality

Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I and two selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$. We know that Φ is continuous, see for instance [11, Proposition 2.8]. By taking the positive map Φ in (1.1) and using the continuity property of Φ , we have

$$\begin{aligned} \Phi\left(f\left(\frac{A+B}{2}\right)\right) &\leq \frac{1}{2} \left[\Phi\left(f\left(\frac{3A+B}{4}\right)\right) + \Phi\left(f\left(\frac{A+3B}{4}\right)\right) \right] \\ &\leq \int_0^1 \Phi(f((1-t)A + tB)) dt \\ &\leq \frac{1}{2} \left[\Phi\left(f\left(\frac{A+B}{2}\right)\right) + \frac{\Phi(f(A)) + \Phi(f(B))}{2} \right] \\ &\leq \frac{\Phi(f(A)) + \Phi(f(B))}{2}. \end{aligned} \quad (2.1)$$

If we write the inequality (2.1) for $\Phi(A)$ and $\Phi(B)$ then we also have

$$\begin{aligned} f\left(\frac{\Phi(A) + \Phi(B)}{2}\right) &\leq \frac{1}{2} \left[f\left(\frac{3\Phi(A) + \Phi(B)}{4}\right) + f\left(\frac{\Phi(A) + 3\Phi(B)}{4}\right) \right] \\ &\leq \int_0^1 f((1-t)\Phi(A) + t\Phi(B)) dt \\ &\leq \frac{1}{2} \left[f\left(\frac{\Phi(A) + \Phi(B)}{2}\right) + \frac{f(\Phi(A)) + f(\Phi(B))}{2} \right] \\ &\leq \frac{f(\Phi(A)) + f(\Phi(B))}{2}. \end{aligned} \quad (2.2)$$

It is then natural to ask how the following integrals

$$\int_0^1 \Phi(f((1-t)A + tB)) dt \quad \text{and} \quad \int_0^1 f((1-t)\Phi(A) + t\Phi(B)) dt$$

do compare?

The following simple result holds:

Theorem 3. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\int_0^1 f((1-t)\Phi(A) + t\Phi(B)) dt \leq \int_0^1 \Phi(f((1-t)A + tB)) dt. \quad (2.3)$$

Proof. By (1.2) we have

$$f((1-t)\Phi(A) + t\Phi(B)) = f(\Phi(((1-t)A + tB))) \leq \Phi(f((1-t)A + tB))$$

for any $t \in [0, 1]$.

By integrating this inequality on $[0, 1]$ and using the continuity property of Φ we get the desired result (2.3). \square

We define by $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ the convex cone of all linear, positive maps Ψ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, namely $\Psi(1_H)$ is positive invertible operator in K .

Corollary 1. Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I and selfadjoint operators A and B with spectra in I . If $\Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, then we have

$$\begin{aligned} \Psi^{1/2}(1_H) \left(\int_0^1 f \left(\Psi^{-1/2}(1_H) ((1-t)\Psi(A) + t\Psi(B)) \Psi^{-1/2}(1_H) \right) dt \right) \Psi^{1/2}(1_H) \\ \leq \int_0^1 \Psi(f((1-t)A + tB)) dt. \end{aligned} \tag{2.4}$$

Proof. If we write the inequality (2.3) for $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$, then we get

$$\begin{aligned} \int_0^1 f \left((1-t)\Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H) + t\Psi^{-1/2}(1_H) \Psi(B) \Psi^{-1/2}(1_H) \right) dt \\ \leq \int_0^1 \Psi^{-1/2}(1_H) \Psi(f((1-t)A + tB)) \Psi^{-1/2}(1_H) dt, \end{aligned}$$

that can be written as

$$\begin{aligned} \int_0^1 f \left(\Psi^{-1/2}(1_H) ((1-t)\Psi(A) + t\Psi(B)) \Psi^{-1/2}(1_H) \right) dt \\ \leq \Psi^{-1/2}(1_H) \left(\int_0^1 \Psi(f((1-t)A + tB)) dt \right) \Psi^{-1/2}(1_H). \end{aligned}$$

Finally, if we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$, then we get the desired result (2.4). □

The following representation result holds.

Lemma 1. Let $f : I \rightarrow \mathbb{C}$ be a continuous function on the interval I and two selfadjoint operators A and B with spectra in I . Then for any $\lambda \in [0, 1]$ we have the representation

$$\begin{aligned} \int_0^1 f((1-t)A + tB) dt = (1-\lambda) \int_0^1 f([(1-t)((1-\lambda)A + \lambda B) + tB]) dt \\ + \lambda \int_0^1 f([(1-t)A + t((1-\lambda)A + \lambda B)]) dt. \end{aligned} \tag{2.5}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (2.5) is obvious.

Let $\lambda \in (0, 1)$. Observe that

$$\int_0^1 f([(1-t)(\lambda B + (1-\lambda)A) + tB]) dt = \int_0^1 f([(1-t)\lambda + t]B + (1-t)(1-\lambda)A) dt$$

and

$$\int_0^1 f[t(\lambda B + (1-\lambda)A) + (1-t)A] dt = \int_0^1 f[t\lambda B + (1-\lambda t)A] dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1-u = (1-t)(1-\lambda)$ and $du = (1-\lambda) du$.

Then

$$\int_0^1 f([(1-t)\lambda + t]B + (1-t)(1-\lambda)A) dt = \frac{1}{1-\lambda} \int_\lambda^1 f[uB + (1-u)A] du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 f[t\lambda B + (1 - \lambda t)A] dt = \frac{1}{\lambda} \int_0^\lambda f[uB + (1 - u)A] du.$$

Therefore

$$\begin{aligned} (1 - \lambda) \int_0^1 f[(1 - t)(\lambda B + (1 - \lambda)A) + tB] dt + \lambda \int_0^1 f[t(\lambda B + (1 - \lambda)A) + (1 - t)A] dt \\ = \int_\lambda^1 f[uB + (1 - u)A] du + \int_0^\lambda f[uB + (1 - u)A] du \\ = \int_0^1 f[uB + (1 - u)A] du \end{aligned}$$

and the identity (2.5) is proved. \square

We have now the following generalization of (1.1):

Theorem 4. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0, 1]$ we have the inequalities*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq (1 - \lambda)f\left[\frac{(1 - \lambda)A + (1 + \lambda)B}{2}\right] + \lambda f\left[\frac{(2 - \lambda)A + \lambda B}{2}\right] \\ &\leq \int_0^1 f((1 - t)A + tB) dt \\ &\leq \frac{1}{2} [f((1 - \lambda)A + \lambda B) + (1 - \lambda)f(B) + \lambda f(A)] \\ &\leq \frac{f(A) + f(B)}{2}. \end{aligned} \tag{2.6}$$

Proof. Using the Hermite-Hadamard inequality (1.1) we have

$$\begin{aligned} f\left[\frac{(1 - \lambda)A + (1 + \lambda)B}{2}\right] &\leq \int_0^1 f[(1 - t)((1 - \lambda)A + \lambda B) + tB] dt \\ &\leq \frac{f((1 - \lambda)A + \lambda B) + f(B)}{2} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} f\left[\frac{(2 - \lambda)A + \lambda B}{2}\right] &\leq \int_0^1 f[(1 - t)A + t((1 - \lambda)A + \lambda B)] dt \\ &\leq \frac{f(A) + f((1 - \lambda)A + \lambda B)}{2} \end{aligned} \tag{2.8}$$

for any $\lambda \in [0, 1]$.

If we multiply inequality (2.7) by $1 - \lambda$ and (2.8) by λ , add the obtained inequalities and use representation (2.5), then we get

$$\begin{aligned} & (1 - \lambda)f \left[\frac{(1 - \lambda)A + (1 + \lambda)B}{2} \right] + \lambda f \left[\frac{(2 - \lambda)A + \lambda B}{2} \right] \\ & \leq \int_0^1 f((1 - t)A + tB) dt \\ & \leq (1 - \lambda) \frac{f((1 - \lambda)A + \lambda B) + f(B)}{2} + \lambda \frac{f(A) + f((1 - \lambda)A + \lambda B)}{2}, \end{aligned}$$

which proves the second and third inequalities in (2.6).

By the operator convexity of f we have

$$\begin{aligned} & (1 - \lambda)f \left[\frac{(1 - \lambda)A + (1 + \lambda)B}{2} \right] + \lambda f \left[\frac{(2 - \lambda)A + \lambda B}{2} \right] \\ & \geq f \left[(1 - \lambda) \frac{(1 - \lambda)A + (1 + \lambda)B}{2} + \lambda \frac{(2 - \lambda)A + \lambda B}{2} \right] = f \left(\frac{A + B}{2} \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} [f((1 - \lambda)A + \lambda B) + (1 - \lambda)f(B) + \lambda f(A)] \\ & \leq \frac{1}{2} [(1 - \lambda)f(A) + \lambda f(B) + (1 - \lambda)f(B) + \lambda f(A)] = \frac{f(A) + f(B)}{2} \end{aligned}$$

that prove the first and last inequality in (2.6). □

We have:

Corollary 2. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\begin{aligned} \Phi \left(f \left(\frac{A + B}{2} \right) \right) & \leq (1 - \lambda) \Phi \left(f \left[\frac{(1 - \lambda)A + (1 + \lambda)B}{2} \right] \right) + \lambda \Phi \left(f \left[\frac{(2 - \lambda)A + \lambda B}{2} \right] \right) \tag{2.9} \\ & \leq \int_0^1 \Phi(f((1 - t)A + tB)) dt \\ & \leq \frac{1}{2} [\Phi(f((1 - \lambda)A + \lambda B)) + (1 - \lambda) \Phi(f(B)) + \lambda \Phi(f(A))] \\ & \leq \frac{\Phi(f(A)) + \Phi(f(B))}{2} \end{aligned}$$

for any $\lambda \in [0, 1]$.

3 Bounds for HH-Difference

We consider the difference functional

$$J_n(\mathbf{p}; \mathbf{A}, f, I) := \sum_{j=1}^n p_j f(A_j) - P_n f \left(\frac{1}{P_n} \sum_{j=1}^n p_j A_j \right) \tag{3.1}$$

where $\mathbf{p} = (p_1, \dots, p_n)$, $p_j \geq 0$ with $j \in \{1, \dots, n\}$ and $P_n > 0$, $\mathbf{A} = (A_1, \dots, A_n)$ is an n -tuple of selfadjoint operators with $\text{Sp}(A_j) \subseteq I$ for $j \in \{1, \dots, n\}$ and $f : I \rightarrow \mathbb{R}$ is a operator convex function defined on the interval I .

We denote by \mathcal{P}_n^+ the set of all n -tuples $\mathbf{p} = (p_1, \dots, p_n)$, $p_j \geq 0$ with $j \in \{1, \dots, n\}$ and $P_n > 0$. For $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1, \dots, n\}$.

In [7] we established the following properties of the functional $J_n(\cdot; \mathbf{A}, f, I)$:

Theorem 5. Assume that $f : I \rightarrow \mathbb{R}$ is an operator convex function and $\mathbf{A} = (A_1, \dots, A_n)$ an n -tuple of selfadjoint operators with $\text{Sp}(A_j) \subseteq I$, then for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we have

$$J_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, f, I) \geq J_n(\mathbf{p}; \mathbf{A}, f, I) + J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \quad (3.2)$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$J_n(\mathbf{p}; \mathbf{A}, f, I) \geq J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \quad (3.3)$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a monotonic functional in the operator order.

The following boundedness property also holds:

Corollary 3. Assume that the function $f : I \rightarrow \mathbb{R}$ is operator convex and the n -tuple of selfadjoint operators (A_1, \dots, A_n) satisfies the condition $\text{Sp}(A_j) \subseteq I$ for any $j \in \{1, \dots, n\}$. If $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and there exists the positive constants m, M such that

$$m\mathbf{q} \leq \mathbf{p} \leq M\mathbf{q}, \quad (3.4)$$

then

$$mJ_n(\mathbf{q}; \mathbf{A}, f, I) \leq J_n(\mathbf{p}; \mathbf{A}, f, I) \leq MJ_n(\mathbf{q}; \mathbf{A}, f, I) \quad (3.5)$$

in the operator order.

We observe that if all $q_j > 0, j \in \{1, \dots, n\}$, then we have the inequality

$$\begin{aligned} \min_{j \in \{1, \dots, n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, I) &\leq J_n(\mathbf{p}; \mathbf{A}, f, I) \\ &\leq \max_{j \in \{1, \dots, n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, I) \end{aligned} \quad (3.6)$$

in the operator order.

In particular, by (3.6) for $n = 2, p_1 = 1 - p, p_2 = p, q_1 = 1 - q$ and $q_2 = q$ with $p \in [0, 1]$ and $q \in (0, 1)$ we get

$$\begin{aligned} \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] & \\ \leq [(1-p)f(A) + pf(B) - f((1-p)A + pB)] & \\ \leq \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] & \end{aligned} \quad (3.7)$$

for any selfadjoint operators A and B with spectra in I .

If we take $q = \frac{1}{2}$ in (1.1), then we get

$$\begin{aligned} 2 \min \{t, 1-t\} \left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) \right] &\leq [(1-t)f(A) + tf(B) - f((1-t)A + tB)] \\ &\leq 2 \max \{t, 1-t\} \left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) \right] \end{aligned} \quad (3.8)$$

for any selfadjoint operators A and B with spectra in I and $t \in [0, 1]$.

If we take in (3.7) the map Φ , then we have

$$\begin{aligned} \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))] & \\ \leq [(1-p)\Phi(f(A)) + p\Phi(f(B)) - \Phi(f((1-p)A + pB))] & \\ \leq \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))] & \end{aligned} \quad (3.9)$$

for any $\Phi \in \mathfrak{P}_N[\mathfrak{B}(H), \mathfrak{B}(K)]$.

The following result provides some upper and lower bounds for the *HH-difference*

$$\frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) dt.$$

Theorem 6. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\begin{aligned} \frac{1}{2} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] &\leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) dt \\ &\leq \frac{1}{2} \frac{q^2 - q + 1}{q(1-q)} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] \end{aligned} \tag{3.10}$$

for any $q \in (0, 1)$.

Proof. From (3.7) we have

$$\begin{aligned} \min \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] \\ \leq [(1-t)f(A) + tf(B) - f((1-t)A + tB)] \\ \leq \max \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] \end{aligned} \tag{3.11}$$

with $t \in [0, 1]$ and $q \in (0, 1)$.

If we integrate over $t \in [0, 1]$ the inequality (3.11), then we get

$$\begin{aligned} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] \int_0^1 \min \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} dt \\ \leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) dt \\ \leq [(1-q)f(A) + qf(B) - f((1-q)A + qB)] \int_0^1 \max \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} dt \end{aligned} \tag{3.12}$$

for any A, B with spectra in I and $q \in (0, 1)$.

Observe that

$$\frac{t}{q} - \frac{1-t}{1-q} = \frac{t-q}{q(1-q)}$$

showing that

$$\min \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} = \begin{cases} \frac{t}{q} & \text{if } 0 \leq t \leq q \leq 1 \\ \frac{1-t}{1-q} & \text{if } 0 \leq q \leq t \leq 1 \end{cases}$$

and

$$\max \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} = \begin{cases} \frac{1-t}{1-q} & \text{if } 0 \leq t \leq q \leq 1 \\ \frac{t}{q} & \text{if } 0 \leq q \leq t \leq 1. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 \min \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} dt &= \int_0^q \frac{t}{q} dt + \int_q^1 \frac{1-t}{1-q} dt \\ &= \frac{q^2}{2q} + \frac{1}{1-q} \left(1 - q - \left(\frac{1-q^2}{2} \right) \right) = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \max \left\{ \frac{t}{q}, \frac{1-t}{1-q} \right\} dt &= \int_0^q \frac{1-t}{1-q} dt + \int_q^1 \frac{t}{q} dt \\ &= \frac{1}{1-q} \left(q - \frac{q^2}{2} \right) + \frac{1-q^2}{2q} \\ &= \frac{q^2 - q + 1}{2q(1-q)} \end{aligned}$$

and by (3.12) we obtain the desired result (3.10). \square

Corollary 4. Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have

$$\begin{aligned} &\frac{1}{2} [(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))] \\ &\leq \frac{\Phi(f(A)) + \Phi(f(B))}{2} - \int_0^1 \Phi(f((1-t)A + tB)) dt \\ &\leq \frac{1}{2} \frac{q^2 - q + 1}{q(1-q)} [(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))]. \end{aligned} \quad (3.13)$$

We also have the following bounds for the other *HH-difference*

$$\int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right).$$

Theorem 7. Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I we have the inequality

$$\begin{aligned} &\frac{1}{2q(1-q)} \min\{1-q, q\} \left[\int_0^1 f((1-t)A + tB) dt - \frac{1}{1-2q} \int_q^{1-q} f((1-s)A + sB) ds \right] \\ &\leq \int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{2q(1-q)} \max\{1-q, q\} \left[\int_0^1 f((1-t)A + tB) dt - \frac{1}{1-2q} \int_q^{1-q} f((1-s)A + sB) ds \right] \end{aligned} \quad (3.14)$$

or, equivalently

$$\begin{aligned} &\frac{2q(1-q)}{\max\{1-q, q\}} \left[\int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right) \right] \\ &\leq \int_0^1 f((1-t)A + tB) dt - \frac{1}{1-2q} \int_q^{1-q} f((1-s)A + sB) ds \\ &\leq \frac{2q(1-q)}{\min\{1-q, q\}} \left[\int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right) \right] \end{aligned} \quad (3.15)$$

for any $q \in (0, 1)$, $q \neq \frac{1}{2}$.

Proof. If we take in (3.7) $p = \frac{1}{2}$, then we have

$$\begin{aligned} & \frac{1}{2q(1-q)} \min\{1-q, q\} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] \\ & \leq \frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2q(1-q)} \max\{1-q, q\} [(1-q)f(A) + qf(B) - f((1-q)A + qB)] \end{aligned} \tag{3.16}$$

for any A, B with spectra in I and $q \in (0, 1)$.

If we replace A by $(1-t)A + tB$ and B by $tA + (1-t)B$ in (3.16), then we get

$$\begin{aligned} & \frac{1}{2q(1-q)} \min\{1-q, q\} [(1-q)f((1-t)A + tB) + qf(tA + (1-t)B) \\ & - f((1-q)[(1-t)A + tB] + q[tA + (1-t)B])] \\ & \leq \frac{f((1-t)A + tB) + f(tA + (1-t)B)}{2} - f\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2q(1-q)} \max\{1-q, q\} [(1-q)f((1-t)A + tB) + qf(tA + (1-t)B) \\ & - f((1-q)[(1-t)A + tB] + q[tA + (1-t)B])] \end{aligned} \tag{3.17}$$

for any $A, B \in C, t \in [0, 1]$ and $q \in (0, 1)$.

If we take the integral over $t \in [0, 1]$ in (3.17) and take into account that

$$\int_0^1 f((1-t)A + tB) dt = \int_0^1 f(tA + (1-t)B) dt$$

we get

$$\begin{aligned} & \frac{1}{2q(1-q)} \min\{1-q, q\} \left[\int_0^1 f((1-t)A + tB) dt - \int_0^1 f((1-q)[(1-t)A + tB] + q[tA + (1-t)B]) dt \right] \\ & \leq \int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2q(1-q)} \max\{1-q, q\} \left[\int_0^1 f((1-t)A + tB) dt - \int_0^1 f((1-q)[(1-t)A + tB] + q[tA + (1-t)B]) dt \right] \end{aligned} \tag{3.18}$$

or any A, B with spectra in I and $q \in (0, 1)$.

Observe that for any A, B with spectra in $I, t \in [0, 1]$ and $q \in (0, 1)$ we have

$$(1-q)[(1-t)A + tB] + q[tA + (1-t)B] = [(1-q)(1-t) + qt]A + [(1-q)t + (1-t)q]B$$

and by putting $s := (1-q)t + (1-t)q$, for $q \neq \frac{1}{2}$ we have

$$[(1-q)(1-t) + qt]A + [(1-q)t + (1-t)q]B = (1-s)A + sB.$$

If $q \neq \frac{1}{2}$, then s is a change of variable, $ds = (1-2q) dt$ and we have for any A, B with spectra in I that

$$\int_0^1 f((1-q)[(1-t)A + tB] + q[tA + (1-t)B]) dt = \frac{1}{1-2q} \int_q^{1-q} f((1-s)A + sB) ds.$$

On making use of (3.18) we get the desired result (3.14). □

Corollary 5. Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have

$$\begin{aligned} & \frac{2q(1-q)}{\max\{1-q, q\}} \left[\int_0^1 \Phi(f((1-t)A + tB)) dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right) \right] \\ & \leq \int_0^1 \Phi(f((1-t)A + tB)) dt - \frac{1}{1-2q} \int_q^{1-q} \Phi(f((1-s)A + sB)) ds \\ & \leq \frac{2q(1-q)}{\min\{1-q, q\}} \left[\int_0^1 \Phi(f((1-t)A + tB)) dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right) \right] \end{aligned} \quad (3.19)$$

for any $q \in (0, 1)$, $q \neq \frac{1}{2}$.

Remark 1. If we take $q = \frac{1}{4}$ in (3.15) and (3.19), then we get

$$\begin{aligned} \frac{1}{2} \left[\int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right) \right] & \leq \int_0^1 f((1-t)A + tB) dt - 2 \int_{1/4}^{3/4} f((1-s)A + sB) ds \\ & \leq \frac{3}{2} \left[\int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right) \right] \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} \frac{1}{2} \left[\int_0^1 \Phi(f((1-t)A + tB)) dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right) \right] & \leq \int_0^1 \Phi(f((1-t)A + tB)) dt - 2 \int_{1/4}^{3/4} \Phi(f((1-s)A + sB)) ds \\ & \leq \frac{3}{2} \left[\int_0^1 \Phi(f((1-t)A + tB)) dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right) \right] \end{aligned} \quad (3.21)$$

for any A, B with spectra in I and $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

4 Some Examples

The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$.

If we write the inequality (2.3) for the power $1 \leq r \leq 2$ (or $-1 \leq r \leq 0$) we have

$$\int_0^1 ((1-t)\Phi(A) + t\Phi(B))^r dt \leq \int_0^1 \Phi(((1-t)A + tB)^r) dt, \quad (4.1)$$

where $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ ($A, B \in \mathcal{B}^{++}(H)$). In the case $0 \leq r \leq 1$ the inequalities reverse in (4.1).

If we write the inequality (2.9) for the power $1 \leq r \leq 2$ (or $-1 \leq r \leq 0$) we have

$$\begin{aligned} \Phi \left(\left(\frac{A+B}{2} \right)^r \right) &\leq (1-\lambda) \Phi \left(\left[\frac{(1-\lambda)A + (1+\lambda)B}{2} \right]^r \right) + \lambda \Phi \left(\left[\frac{(2-\lambda)A + \lambda B}{2} \right]^r \right) \\ &\leq \int_0^1 \Phi \left(((1-t)A + tB)^r \right) dt \\ &\leq \frac{1}{2} [\Phi \left(((1-\lambda)A + \lambda B)^r \right) + (1-\lambda) \Phi \left(B^r \right) + \lambda \Phi \left(A^r \right)] \\ &\leq \frac{\Phi \left(A^r \right) + \Phi \left(B^r \right)}{2}, \end{aligned} \tag{4.2}$$

where $\lambda \in [0, 1]$, $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ ($A, B \in \mathcal{B}^{++}(H)$). In the case $0 \leq r \leq 1$ the inequalities reverse in (4.2).

If we write the inequality (3.9) for the power $1 \leq r \leq 2$ (or $-1 \leq r \leq 0$) we get for $p \in [0, 1]$, $q \in (0, 1)$ that

$$\begin{aligned} \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} &[(1-q) \Phi \left(A^r \right) + q \Phi \left(B^r \right) - \Phi \left(((1-q)A + qB)^r \right)] \\ &\leq [(1-p) \Phi \left(A^r \right) + p \Phi \left(B^r \right) - \Phi \left(((1-p)A + pB)^r \right)] \\ &\leq \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q) \Phi \left(A^r \right) + q \Phi \left(B^r \right) - \Phi \left(((1-q)A + qB)^r \right)] \end{aligned} \tag{4.3}$$

where $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ ($A, B \in \mathcal{B}^{++}(H)$).

From (3.13) we have for $1 \leq r \leq 2$ (or $-1 \leq r \leq 0$) that

$$\begin{aligned} &\frac{1}{2} [(1-q) \Phi \left(A^r \right) + q \Phi \left(B^r \right) - \Phi \left(((1-q)A + qB)^r \right)] \\ &\leq \frac{\Phi \left(A^r \right) + \Phi \left(B^r \right)}{2} - \int_0^1 \Phi \left(((1-t)A + tB)^r \right) dt \\ &\leq \frac{1}{2} \frac{q^2 - q + 1}{q(1-q)} [(1-q) \Phi \left(A^r \right) + q \Phi \left(B^r \right) - \Phi \left(((1-q)A + qB)^r \right)] \end{aligned} \tag{4.4}$$

while from (3.19) we have that

$$\begin{aligned} &\frac{2q(1-q)}{\max \{1-q, q\}} \left[\int_0^1 \Phi \left(((1-t)A + tB)^r \right) dt - \Phi \left(\left(\frac{A+B}{2} \right)^r \right) \right] \\ &\leq \int_0^1 \Phi \left(((1-t)A + tB)^r \right) dt - \frac{1}{1-2q} \int_q^{1-q} \Phi \left(((1-s)A + sB)^r \right) ds \\ &\leq \frac{2q(1-q)}{\min \{1-q, q\}} \left[\int_0^1 \Phi \left(((1-t)A + tB)^r \right) dt - \Phi \left(\left(\frac{A+B}{2} \right)^r \right) \right], \end{aligned} \tag{4.5}$$

where $p \in [0, 1]$, $q \in (0, 1)$, $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ ($A, B \in \mathcal{B}^{++}(H)$).

The function $f(t) = -\ln t$ is operator convex on $(0, \infty)$. Then by (2.3) we have

$$\int_0^1 \ln \left((1-t) \Phi(A) + t \Phi(B) \right) dt \geq \int_0^1 \Phi \left(\ln \left((1-t)A + tB \right) \right) dt \tag{4.6}$$

while by (2.9) we have, for $\lambda \in [0, 1]$ that

$$\begin{aligned} \Phi \left(\ln \left(\frac{A+B}{2} \right) \right) &\geq (1-\lambda) \Phi \left(\ln \left[\frac{(1-\lambda)A + (1+\lambda)B}{2} \right] \right) + \lambda \Phi \left(\ln \left[\frac{(2-\lambda)A + \lambda B}{2} \right] \right) \\ &\geq \int_0^1 \Phi(\ln((1-t)A + tB)) dt \\ &\geq \frac{1}{2} [\Phi(\ln((1-\lambda)A + \lambda B)) + (1-\lambda) \Phi(\ln(B)) + \lambda \Phi(\ln(A))] \\ &\geq \frac{\Phi(\ln(A)) + \Phi(\ln(B))}{2}, \end{aligned} \quad (4.7)$$

where $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^{++}(H)$.

From (3.9) we have for $p \in [0, 1], q \in (0, 1)$ that

$$\begin{aligned} &\min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [\Phi(\ln((1-q)A + qB)) - (1-q) \Phi(\ln(A)) - q \Phi(\ln(B))] \\ &\leq [\Phi(\ln((1-p)A + pB)) - (1-p) \Phi(\ln(A)) - p \Phi(\ln(B))] \\ &\leq \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [\Phi(\ln((1-q)A + qB)) - (1-q) \Phi(\ln(A)) - q \Phi(\ln(B))], \end{aligned} \quad (4.8)$$

from (3.13) we have

$$\begin{aligned} &\frac{1}{2} [\Phi(\ln((1-q)A + qB)) - (1-q) \Phi(\ln(A)) - q \Phi(\ln(B))] \\ &\leq \int_0^1 \Phi(\ln((1-t)A + tB)) dt - \frac{\Phi(\ln(A)) + \Phi(\ln(B))}{2} \\ &\leq \frac{1}{2} \frac{q^2 - q + 1}{q(1-q)} [\Phi(\ln((1-q)A + qB)) - (1-q) \Phi(\ln(A)) - q \Phi(\ln(B))]. \end{aligned} \quad (4.9)$$

while from (3.19)

$$\begin{aligned} &\frac{2q(1-q)}{\max\{1-q, q\}} \left[\Phi \left(\ln \left(\frac{A+B}{2} \right) \right) - \int_0^1 \Phi(\ln((1-t)A + tB)) dt \right] \\ &\leq \frac{1}{1-2q} \int_q^{1-q} \Phi(\ln((1-s)A + sB)) ds - \int_0^1 \Phi(\ln((1-t)A + tB)) dt \\ &\leq \frac{2q(1-q)}{\min\{1-q, q\}} \left[\Phi \left(\ln \left(\frac{A+B}{2} \right) \right) - \int_0^1 \Phi(\ln((1-t)A + tB)) dt \right] \end{aligned} \quad (4.10)$$

where $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^{++}(H)$.

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