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# Applications of Kato's inequality for $n$-tuples of operators in Hilbert spaces, (II)

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**Abstract**
In this paper, by the use of famous Kato's inequality for bounded linear operators, we establish some new inequalities for $n$-tuples of operators and apply them to functions of normal operators defined by power series as well as to some norms and numerical radii that arise in multivariate operator theory. They provide a natural continuation of the results in previous paper with (I) in the title.

**MSC:** 47A50; 47A63

**Keywords:** bounded linear operators; functions of normal operators; inequalities for operators; norm and numerical radius inequalities; Kato's inequality

## 1 Introduction

In 1952, Kato [1] proved the following generalization of the Schwarz inequality:

$$\left|\langle Tx,y\rangle\right|^2 \le \left\langle \left(T^*T\right)^\alpha x,x\right\rangle\left\langle \left(TT^*\right)^{1-\alpha} y,y\right\rangle \tag{1.1}$$

for any $x,y\in H$, $\alpha\in[0,1]$, and $T$ is a bounded linear operator on $H$.

Utilizing the operator modulus notation, we can write (1.1) as follows:

$$\left|\langle Tx,y\rangle\right|^2 \le \left\langle |T|^{2\alpha} x,x\right\rangle\left\langle \left|T^*\right|^{2(1-\alpha)} y,y\right\rangle. \tag{1.2}$$

For results related to Kato's inequality, see [1–17] and [18].

In the recent paper [19], by employing Kato's inequality (1.2), Dragomir, Cho and Kim established the following results for sequences of bounded linear operators on complex Hilbert spaces.

**Theorem 1** *Let $(T_1,\dots,T_n)\in\mathcal{B}(H)\times\cdots\times\mathcal{B}(H):=\mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$ and $(p_1,\dots,p_n)\in\mathbb{R}_+^{*n}$ be an n-tuple of nonnegative weights not all of them equal to zero, then*

$$\sum_{j=1}^n p_j\left|\langle T_jx,y\rangle\right| \le \left\langle \sum_{j=1}^n p_j\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}}{2}\right)x,x\right\rangle^{1/2} \times\left\langle \sum_{j=1}^n p_j\left(\frac{|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)y,y\right\rangle^{1/2} \tag{1.3}$$

*for any $\alpha\in[0,1]$ and any $x,y\in H$.*

**Theorem 2** *With the assumptions in Theorem* 1, *we have*

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right|^2 \\
&\quad\le \frac{1}{2}\sum_{j=1}^{n} p_j \left(\|T_j x\|^{2\alpha}\left\|T_j^* y\right\|^{2(1-\alpha)} + \left\|T_j^* y\right\|^{2\alpha}\|T_j x\|^{2(1-\alpha)}\right) \\
&\quad\le \frac{1}{2}\left[\left(\sum_{j=1}^{n} p_j\|T_j x\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j\left\|T_j^* y\right\|^2\right)^{1-\alpha} + \left(\sum_{j=1}^{n} p_j\|T_j x\|^2\right)^{1-\alpha}\left(\sum_{j=1}^{n} p_j\left\|T_j^* y\right\|^2\right)^{\alpha}\right] \\
&\quad\le \frac{1}{2}\sum_{j=1}^{n} p_j \left(\|T_j x\|^2 + \left\|T_j^* y\right\|^2\right) \qquad (1.4)
\end{aligned}$$

*for any* $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and* $\alpha \in [0,1]$.

For various related results, see the papers [20–22] and [23–27].

Motivated by the above results, we establish in this paper more inequalities for $n$-tuples of bounded linear operators that can be obtained from Kato's result (1.2) and apply them to functions of normal operators defined by power series as well as to some norms and numerical radii that can be associated with these $n$-tuples of bounded linear operators on Hilbert spaces. The paper is a natural continuation of [19].

## 2 Some inequalities for *n*-tuples of operators

The following result holds.

**Theorem 3** *Let* $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$ *be an n-tuple of bounded linear operators on the Hilbert space* $(H; \langle\cdot,\cdot\rangle)$ *and* $(p_1, \ldots, p_n) \in \mathbb{R}_+^{*n}$ *be an n-tuple of nonnegative weights not all of them equal to zero, then*

$$\begin{aligned}
\left|\left\langle \sum_{j=1}^{n} p_j\left(\frac{T_j + T_j^*}{2}\right)x, y\right\rangle\right| &\le \sum_{j=1}^{n} p_j\left|\left\langle \frac{T_j + T_j^*}{2}x, y\right\rangle\right| \\
&\le \sum_{j=1}^{n} p_j\left[\frac{|\langle T_j x, y\rangle| + |\langle T_j^* x, y\rangle|}{2}\right] \\
&\le \left\langle \sum_{j=1}^{n} p_j\left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right]x, x\right\rangle^{1/2} \\
&\quad\times \left\langle \sum_{j=1}^{n} p_j\left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right]y, y\right\rangle^{1/2} \qquad (2.1)
\end{aligned}$$

*for any* $\alpha \in [0,1]$ *and, in particular, for* $\alpha = \frac{1}{2}$

$$\begin{aligned}
\left|\left\langle \sum_{j=1}^{n} p_j\left(\frac{T_j + T_j^*}{2}\right)x, y\right\rangle\right| &\le \sum_{j=1}^{n} p_j\left|\left\langle \frac{T_j + T_j^*}{2}x, y\right\rangle\right| \\
&\le \sum_{j=1}^{n} p_j\left[\frac{|\langle T_j x, y\rangle| + |\langle T_j^* x, y\rangle|}{2}\right]
\end{aligned}$$

$$\le \left\langle \sum_{j=1}^{n} p_j \left[ \frac{|T_j| + |T_j^*|}{2} \right] x, x \right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n} p_j \left[ \frac{|T_j| + |T_j^*|}{2} \right] y, y \right\rangle^{1/2} \tag{2.2}$$

*for any* $x, y \in H$.

*Proof* The first two inequalities are obvious by the properties of the modulus.

Utilizing Kato's inequality, we have

$$\left|\langle T_j x, y\rangle\right| \le \langle |T_j|^{2\alpha} x, x\rangle^{1/2} \langle \left|T_j^*\right|^{2(1-\alpha)} y, y\rangle^{1/2} \tag{2.3}$$

and, by replacing $x$ with $y$, we have

$$\left|\langle T_j y, x\rangle\right| \le \langle |T_j|^{2\alpha} y, y\rangle^{1/2} \langle \left|T_j^*\right|^{2(1-\alpha)} x, x\rangle^{1/2},$$

*i.e.*,

$$\left|\langle T_j^* x, y\rangle\right| \le \langle \left|T_j^*\right|^{2(1-\alpha)} x, x\rangle^{1/2} \langle |T_j|^{2\alpha} y, y\rangle^{1/2} \tag{2.4}$$

for any $j \in \{1, \dots, n\}$ and $x, y \in H$.

Adding inequalities (2.3) and (2.4) and utilizing the elementary inequality

$$ab + cd \le \left(a^2 + c^2\right)^{1/2} \left(b^2 + d^2\right)^{1/2}, \quad a, b, c, d \ge 0,$$

we get

$$\begin{aligned} \left|\langle T_j x, y\rangle\right| + \left|\langle T_j^* x, y\rangle\right| &\le \langle |T_j|^{2\alpha} x, x\rangle^{1/2} \langle \left|T_j^*\right|^{2(1-\alpha)} y, y\rangle^{1/2} \\ &\quad + \langle \left|T_j^*\right|^{2(1-\alpha)} x, x\rangle^{1/2} \langle |T_j|^{2\alpha} y, y\rangle^{1/2} \\ &\le \langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] x, x\rangle^{1/2} \\ &\quad \times \langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] y, y\rangle^{1/2} \end{aligned} \tag{2.5}$$

for any $j \in \{1, \dots, n\}$ and $x, y \in H$.

Multiplying inequalities (2.5) by $p_j \ge 0$ and then summing over $j$ from 1 to $n$ and utilizing the weighted Cauchy-Buniakowski-Schwarz inequality, we have

$$\begin{aligned} &\sum_{j=1}^{n} p_j \left[ \left|\langle T_j x, y\rangle\right| + \left|\langle T_j^* x, y\rangle\right| \right] \\ &\quad \le \sum_{j=1}^{n} p_j \langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] x, x\rangle^{1/2} \langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] y, y\rangle^{1/2} \\ &\quad \le \left\langle \sum_{j=1}^{n} p_j \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] y, y \right\rangle^{1/2} \end{aligned} \tag{2.6}$$

for $x, y \in H$, which is equivalent to the third inequality in (2.1). □

**Remark 1** The particular case $y = x$ is of interest for providing numerical radii inequalities and can be stated as follows:

$$\begin{aligned} \left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{T_j + T_j^*}{2}\right) x, x \right\rangle\right| &\le \sum_{j=1}^{n} p_j \left|\left\langle \frac{T_j + T_j^*}{2} x, x \right\rangle\right| \\ &\le \sum_{j=1}^{n} p_j \left|\langle T_j x, x\rangle\right| \\ &\le \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right] x, x \right\rangle \end{aligned} \tag{2.7}$$

for any $\alpha \in [0,1]$ and, for $\alpha = \frac{1}{2}$,

$$\begin{aligned} \left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{T_j + T_j^*}{2}\right) x, x \right\rangle\right| &\le \sum_{j=1}^{n} p_j \left|\left\langle \frac{T_j + T_j^*}{2} x, x \right\rangle\right| \\ &\le \sum_{j=1}^{n} p_j \left|\langle T_j x, x\rangle\right| \\ &\le \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] x, x \right\rangle \end{aligned} \tag{2.8}$$

for any $x \in H$.

The case of unitary vectors provides more refinements as follows.

**Remark 2** With the assumptions in Theorem 3, we have

$$\begin{aligned} &\sum_{j=1}^{n} p_j \left[\frac{|\langle T_j x, y\rangle| + |\langle T_j^* x, y\rangle|}{2}\right] \\ &\quad\le \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right] x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right] y, y \right\rangle^{1/2} \\ &\quad\le \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right]\right)^{1/2} x, x \right\rangle \\ &\qquad\times \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right]\right)^{1/2} y, y \right\rangle \\ &\quad\le \frac{1}{2}\left[\left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right]\right)^{1/2} x, x \right\rangle^2 \right. \\ &\qquad \left. + \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right]\right)^{1/2} y, y \right\rangle^2\right] \\ &\quad\le \frac{1}{2}\left[\left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right] x, x \right\rangle + \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right] y, y \right\rangle\right] \end{aligned} \tag{2.9}$$

for any $\alpha \in [0,1]$ and, in particular,

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \left[\frac{|\langle T_j x, y\rangle| + |\langle T_j^* x, y\rangle|}{2}\right] \\
&\quad \leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] y, y \right\rangle^{1/2} \\
&\quad \leq \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right]\right)^{1/2} x, x \right\rangle \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right]\right)^{1/2} y, y \right\rangle \\
&\quad \leq \frac{1}{2} \left[\left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right]\right)^{1/2} x, x \right\rangle^2 + \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right]\right)^{1/2} y, y \right\rangle^2\right] \\
&\quad \leq \frac{1}{2} \left[\left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] x, x \right\rangle + \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] y, y \right\rangle\right] \qquad (2.10)
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

The proofs follow by utilizing the Hölder-McCarthy inequalities $\langle P^r x, x\rangle \leq \langle Px, x\rangle^r$ and $\langle Px, x\rangle^s \leq \langle P^s x, x\rangle$ that hold for the positive operator $P$, for $r \in (0,1)$, $s \in [1,\infty)$ and $x \in H$ with $\|x\| = 1$. The details are omitted.

In order to employ the above result in obtaining some inequalities for functions of normal operators defined by power series, we need the following version of (2.1).

**Remark 3** If we write inequality (2.1) for the normal operators $N_j$, $j \in \{1, \ldots, n\}$, then we get

$$\begin{aligned}
\left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{N_j + N_j^*}{2}\right) x, y \right\rangle\right| &\leq \sum_{j=1}^{n} p_j \left|\left\langle \frac{N_j + N_j^*}{2} x, y \right\rangle\right| \\
&\leq \sum_{j=1}^{n} p_j \left[\frac{|\langle N_j x, y\rangle| + |\langle N_j^* x, y\rangle|}{2}\right] \\
&\leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|N_j|^{2\alpha} + |N_j|^{2(1-\alpha)}}{2}\right] x, x \right\rangle^{1/2} \\
&\quad \times \left\langle \sum_{j=1}^{n} p_j \left[\frac{|N_j|^{2\alpha} + |N_j|^{2(1-\alpha)}}{2}\right] y, y \right\rangle^{1/2} \qquad (2.11)
\end{aligned}$$

for any $\alpha \in [0,1]$ and, in particular, for $\alpha = \frac{1}{2}$

$$\begin{aligned}
\left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{N_j + N_j^*}{2}\right) x, y \right\rangle\right| &\leq \sum_{j=1}^{n} p_j \left|\left\langle \frac{N_j + N_j^*}{2} x, y \right\rangle\right| \leq \sum_{j=1}^{n} p_j \left[\frac{|\langle N_j x, y\rangle| + |\langle N_j^* x, y\rangle|}{2}\right] \\
&\leq \left\langle \sum_{j=1}^{n} p_j |N_j| x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |N_j| y, y \right\rangle^{1/2} \qquad (2.12)
\end{aligned}$$

for any $x, y \in H$.

The following results involving quadratics also hold.

**Theorem 4** *Let* $(T_1,\ldots,T_n)\in\mathcal{B}^{(n)}(H)$ *be an n-tuple of bounded linear operators on the Hilbert space* $(H;\langle\cdot,\cdot\rangle)$ *and* $(p_1,\ldots,p_n)\in\mathbb{R}_+^{*n}$ *be an n-tuple of nonnegative weights not all of them equal to zero, then*

$$\begin{aligned}
&\sum_{j=1}^{n}p_j\bigl[\left|\langle T_jx,y\rangle\right|^2+\left|\langle T_j^*x,y\rangle\right|^2\bigr]\\
&\quad\le\sum_{j=1}^{n}p_j\bigl[\|T_jx\|^{2\alpha}\|T_j^*y\|^{2(1-\alpha)}+\|T_jy\|^{2\alpha}\|T_j^*x\|^{2(1-\alpha)}\bigr]\\
&\quad\le\left(\sum_{j=1}^{n}p_j\|T_jx\|^2\right)^{\alpha}\left(\sum_{j=1}^{n}p_j\|T_j^*y\|^2\right)^{1-\alpha}+\left(\sum_{j=1}^{n}p_j\|T_jy\|^2\right)^{\alpha}\left(\sum_{j=1}^{n}p_j\|T_j^*x\|^2\right)^{1-\alpha}\\
&\quad\le\left(\sum_{j=1}^{n}p_j\bigl[\|T_jx\|^2+\|T_jy\|^2\bigr]\right)^{\alpha}\left(\sum_{j=1}^{n}p_j\bigl[\|T_j^*y\|^2+\|T_j^*x\|^2\bigr]\right)^{1-\alpha}
\end{aligned}\tag{2.13}$$

*for any* $x,y\in H$ *with* $\|x\|=\|y\|=1$ *and* $\alpha\in[0,1]$.

*Proof* We must prove the inequalities only in the case $\alpha\in(0,1)$, since the case $\alpha=0$ or $\alpha=1$ follows directly from the corresponding case of Kato's inequality.

Utilizing Kato's inequality, we have

$$\left|\langle T_jx,y\rangle\right|^2\le\langle|T_j|^{2\alpha}x,x\rangle\langle|T_j^*|^{2(1-\alpha)}y,y\rangle\tag{2.14}$$

and, by replacing $x$ with $y$, we have

$$\left|\langle T_j^*x,y\rangle\right|^2\le\langle|T_j^*|^{2(1-\alpha)}x,x\rangle\langle|T_j|^{2\alpha}y,y\rangle\tag{2.15}$$

for any $j\in\{1,\ldots,n\}$ and $x,y\in H$.

By the Hölder-McCarthy inequality $\langle P^rx,x\rangle\le\langle Px,x\rangle^r$ for $r\in(0,1)$ and $x\in H$ with $\|x\|=1$, we also have

$$\langle|T_j|^{2\alpha}x,x\rangle\langle|T_j^*|^{2(1-\alpha)}y,y\rangle\le\langle|T_j|^2x,x\rangle^{\alpha}\langle|T_j^*|^2y,y\rangle^{1-\alpha}\tag{2.16}$$

and

$$\langle|T_j^*|^{2(1-\alpha)}x,x\rangle\langle|T_j|^{2\alpha}y,y\rangle\le\langle|T_j|^2y,y\rangle^{\alpha}\langle|T_j^*|^2x,x\rangle^{1-\alpha}\tag{2.17}$$

for any $j\in\{1,\ldots,n\}$ and $x,y\in H$ with $\|x\|=\|y\|=1$.

We then obtain by summation

$$\begin{aligned}
&\left|\langle T_jx,y\rangle\right|^2+\left|\langle T_j^*x,y\rangle\right|^2\\
&\quad\le\langle|T_j|^2x,x\rangle^{\alpha}\langle|T_j^*|^2y,y\rangle^{1-\alpha}+\langle|T_j|^2y,y\rangle^{\alpha}\langle|T_j^*|^2x,x\rangle^{1-\alpha}
\end{aligned}\tag{2.18}$$

for any $j\in\{1,\ldots,n\}$ and $x,y\in H$ with $\|x\|=\|y\|=1$.

Now, if we multiply (2.18) with $p_j \geq 0$, sum over $j$ from 1 to $n$, we get

$$\sum_{j=1}^{n} p_j\left[\left|\langle T_j x, y\rangle\right|^2 + \left|\left\langle T_j^* x, y\right\rangle\right|^2\right] \leq \sum_{j=1}^{n} p_j\left\langle |T_j|^2 x, x\right\rangle^{\alpha}\left\langle \left|T_j^*\right|^2 y, y\right\rangle^{1-\alpha} + \sum_{j=1}^{n} p_j\left\langle |T_j|^2 y, y\right\rangle^{\alpha}\left\langle \left|T_j^*\right|^2 x, x\right\rangle^{1-\alpha} \tag{2.19}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in (0,1)$.

Since $\langle |T_j|^2 x, x\rangle = \|T_j x\|^2$, $\langle |T_j^*|^2 y, y\rangle = \|T_j^* y\|^2$, $\langle |T_j|^2 y, y\rangle = \|T_j y\|^2$ and $\langle |T_j^*|^2 x, x\rangle = \|T_j^* x\|^2$ $j \in \{1, \ldots, n\}$, then we get from (2.19) the first part of (2.13).

Now, on making use of the weighted Hölder discrete inequality

$$\sum_{j=1}^{n} p_j a_j b_j \leq \left(\sum_{j=1}^{n} p_j a_j^p\right)^{1/p}\left(\sum_{j=1}^{n} p_j b_j^q\right)^{1/q}, \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

where $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}_+^n$, we also have

$$\sum_{j=1}^{n} p_j \|T_j x\|^{2\alpha}\left\|T_j^* y\right\|^{2(1-\alpha)} \leq \left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^{2(1-\alpha)}\right)^{1-\alpha}$$

and

$$\sum_{j=1}^{n} p_j \|T_j y\|^{2\alpha}\left\|T_j^* x\right\|^{2(1-\alpha)} \leq \left(\sum_{j=1}^{n} p_j \|T_j y\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* x\right\|^2\right)^{1-\alpha}.$$

Summing these two inequalities, we deduce the second inequality in (2.13).

Finally, on utilizing the Hölder inequality

$$ab + cd \leq \left(a^p + c^p\right)^{1/p}\left(b^q + d^q\right)^{1/q}, \quad a, b, c, d \geq 0,$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2\right)^{1-\alpha} + \left(\sum_{j=1}^{n} p_j \|T_j y\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* x\right\|^2\right)^{1-\alpha}$$
$$\leq \left(\sum_{j=1}^{n} p_j \|T_j x\|^2 + \sum_{j=1}^{n} p_j \|T_j y\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2 + \sum_{j=1}^{n} p_j \left\|T_j^* x\right\|^2\right)^{1-\alpha}$$

and the proof is completed. □

**Remark 4** Utilizing the elementary inequality for complex numbers

$$\left|\frac{z + w}{2}\right|^2 \leq \frac{|z|^2 + |w|^2}{2}, \quad z, w \in \mathbb{C},$$

we have

$$\sum_{j=1}^{n} p_j\left[\left|\left\langle\left(\frac{T_j + T_j^*}{2}\right)x, y\right\rangle\right|^2\right] \le \sum_{j=1}^{n} p_j\left[\frac{|\langle T_j x, y\rangle|^2 + |\langle T_j^* x, y\rangle|^2}{2}\right] \tag{2.20}$$

and by the weighted arithmetic mean-geometric mean inequality

$$a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha)b, \quad a, b \ge 0, \alpha \in [0,1],$$

we also have

$$\begin{aligned}
&\left(\sum_{j=1}^{n} p_j\left[\|T_j x\|^2 + \|T_j y\|^2\right]\right)^{\alpha}\left(\sum_{j=1}^{n} p_j\left[\left\|T_j^* y\right\|^2 + \left\|T_j^* x\right\|^2\right]\right)^{1-\alpha} \\
&\quad \le \alpha \sum_{j=1}^{n} p_j\left[\|T_j x\|^2 + \|T_j y\|^2\right] + (1-\alpha)\sum_{j=1}^{n} p_j\left[\left\|T_j^* y\right\|^2 + \left\|T_j^* x\right\|^2\right].
\end{aligned} \tag{2.21}$$

If we choose $\alpha = \frac{1}{2}$ and use (2.4), (2.20) and (2.22), we derive

$$\begin{aligned}
&\sum_{j=1}^{n} p_j\left[\left|\left\langle\left(\frac{T_j + T_j^*}{2}\right)x, y\right\rangle\right|^2\right] \\
&\quad \le \sum_{j=1}^{n} p_j\left[\frac{|\langle T_j x, y\rangle|^2 + |\langle T_j^* x, y\rangle|^2}{2}\right] \le \frac{1}{2}\sum_{j=1}^{n} p_j\left[\|T_j x\|\left\|T_j^* y\right\| + \|T_j y\|\left\|T_j^* x\right\|\right] \\
&\quad \le \frac{1}{2}\left(\sum_{j=1}^{n} p_j\|T_j x\|^2\right)^{1/2}\left(\sum_{j=1}^{n} p_j\left\|T_j^* y\right\|\right)^{1/2} + \frac{1}{2}\left(\sum_{j=1}^{n} p_j\|T_j y\|^2\right)^{1/2}\left(\sum_{j=1}^{n} p_j\left\|T_j^* x\right\|^2\right)^{1/2} \\
&\quad \le \left(\sum_{j=1}^{n} p_j\left[\frac{\|T_j x\|^2 + \|T_j y\|^2}{2}\right]\right)^{1/2}\left(\sum_{j=1}^{n} p_j\left[\frac{\|T_j^* y\|^2 + \|T_j^* x\|^2}{2}\right]\right)^{1/2} \\
&\quad \le \sum_{j=1}^{n} p_j\left[\frac{\|T_j x\|^2 + \|T_j y\|^2 + \|T_j^* y\|^2 + \|T_j^* x\|^2}{4}\right] \\
&\quad = \frac{1}{2}\left[\sum_{j=1}^{n} p_j\left\langle\frac{|T_j|^2 + |T_j^*|^2}{2}x, x\right\rangle + \sum_{j=1}^{n} p_j\left\langle\frac{|T_j|^2 + |T_j^*|^2}{2}y, y\right\rangle\right]
\end{aligned} \tag{2.22}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

**Remark 5** The case of normal operators $N_j$, $j \in \{1, \dots, n\}$, is of interest for functions of operators and may be stated as follows:

$$\begin{aligned}
&\sum_{j=1}^{n} p_j\left[\left|\left\langle\left(\frac{N_j + N_j^*}{2}\right)x, y\right\rangle\right|^2\right] \\
&\quad \le \sum_{j=1}^{n} p_j\left[\frac{|\langle N_j x, y\rangle|^2 + |\langle N_j^* x, y\rangle|^2}{2}\right]
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\sum_{j=1}^{n} p_j\bigl[\|N_jx\|^{2\alpha}\|N_jy\|^{2(1-\alpha)} + \|N_jy\|^{2\alpha}\|N_jx\|^{2(1-\alpha)}\bigr] \\
&\le \frac{1}{2}\left(\sum_{j=1}^{n} p_j\|N_jx\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j\|N_jy\|^2\right)^{1-\alpha} + \frac{1}{2}\left(\sum_{j=1}^{n} p_j\|N_jy\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j\|N_jx\|^2\right)^{1-\alpha} \\
&\le \frac{1}{2}\sum_{j=1}^{n} p_j\bigl[\|N_jx\|^2 + \|N_jy\|^2\bigr] \qquad (2.23)
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

## 3 Inequalities for functions of normal operators

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \ge 0$, then $f_A = f$.

As some natural examples that are useful for applications, we can point out that if

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln\frac{1}{1+z}, \quad z \in D(0,1); \\
g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \quad z \in \mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \quad z \in \mathbb{C}; \\
l(z) &= \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \quad z \in D(0,1);
\end{aligned} \qquad (3.1)$$

then the corresponding functions constructed by the use of absolute values of the coefficients are

$$\begin{aligned}
f_A(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln\frac{1}{1-z}, \quad z \in D(0,1); \\
g_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbb{C}; \\
h_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \quad z \in \mathbb{C}; \\
l_A(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in D(0,1).
\end{aligned} \qquad (3.2)$$

The following result is a functional inequality for normal operators that can be obtained from (2.1).

**Theorem 5** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $N$ is a normal operator on* *the Hilbert space $H$ and for $\alpha \in [0,1]$, we have that $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then we have the inequalities*

$$\left|\left\langle\left(\frac{f(N)+f(N^*)}{2}\right)x,y\right\rangle\right| \le \left\langle\left(\frac{f_A(|N|^{2\alpha})+f(|N|^{2(1-\alpha)})}{2}\right)x,x\right\rangle^{1/2} \times \left\langle\left(\frac{f_A(|N|^{2\alpha})+f(|N|^{2(1-\alpha)})}{2}\right)y,y\right\rangle^{1/2} \tag{3.3}$$

*for any $x,y \in H$.*

*Proof* If $N$ is a normal operator, then for any $j \in \mathbb{N}$, we have that

$$|N^j|^2 = (N^*N)^j = |N|^{2j}.$$

Utilizing inequality (2.11), we have

$$\begin{aligned}
&\left|\left\langle\sum_{j=0}^{n} a_j\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right| \\
&\quad\le \sum_{j=0}^{n}|a|_j\left|\left\langle\frac{N^j+(N^*)^j}{2}x,y\right\rangle\right| \\
&\quad\le \sum_{j=0}^{n}|a|_j\left[\frac{|\langle N^jx,y\rangle|+|\langle (N^*)^jx,y\rangle|}{2}\right] \\
&\quad\le \left\langle\sum_{j=0}^{n}|a|_j\left[\frac{(|N|^{2\alpha})^j+(|N|^{2(1-\alpha)})^j}{2}\right]x,x\right\rangle^{1/2} \\
&\qquad\times\left\langle\sum_{j=1}^{n}|a|_0\left[\frac{(|N|^{2\alpha})^j+(|N|^{2(1-\alpha)})^j}{2}\right]y,y\right\rangle^{1/2}
\end{aligned} \tag{3.4}$$

for any $\alpha \in [0,1]$, $n \in \mathbb{N}$ and any $x,y \in H$. Since $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then it follows that the series $\sum_{j=0}^{\infty}|a_j|(|N|^{2\alpha})^j$ and $\sum_{j=0}^{\infty}|a_j|(|N|^{2(1-\alpha)})^j$ are absolute convergent in $\mathcal{B}(H)$, and by taking the limit over $n \to \infty$ in (3.4), we deduce the desired result (3.3). □

**Remark 6** With the assumptions in Theorem 5, if we take the supremum over $y \in H$, $\|y\| = 1$, then we get the vector inequality

$$\left\|\left(\frac{f(N)+f(N^*)}{2}\right)x\right\| \le \frac{1}{2}\langle(f_A(|N|^{2\alpha})+f(|N|^{2(1-\alpha)}))x,x\rangle^{1/2} \times \|f_A(|N|^{2\alpha})+f(|N|^{2(1-\alpha)})\| \tag{3.5}$$

for any $x \in H$, which in its turn produces the norm inequality

$$\left\|\frac{f(N)+f(N^*)}{2}\right\| \le \frac{1}{2}\|f_A(|N|^{2\alpha})+f(|N|^{2(1-\alpha)})\| \tag{3.6}$$

for any $\alpha \in [0,1]$.

Moreover, if we take $y = x$ in (3.3), then we have

$$\left|\left\langle\frac{f(N)+f(N^*)}{2}x,x\right\rangle\right| \le \frac{1}{2}\left\langle\left[f_A\left(|N|^{2\alpha}\right)+f\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle \tag{3.7}$$

for any $x \in H$, which, by taking the supremum over $x \in H$, $\|x\| = 1$, generates the numerical radius inequality

$$w\left(\frac{f(N)+f(N^*)}{2}\right) \le \frac{1}{2}w\left[f_A\left(|N|^{2\alpha}\right)+f\left(|N|^{2(1-\alpha)}\right)\right] \tag{3.8}$$

for any $\alpha \in [0,1]$.

Making use of the examples in (3.1) and (3.2), we can state the vector inequalities

$$\begin{aligned}&\left|\left\langle\left[\frac{\ln(1_H+N)^{-1}+\ln(1_H+N^*)^{-1}}{2}\right]x,y\right\rangle\right|\\&\quad\le\frac{1}{2}\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]x,x\right\rangle^{1/2}\\&\qquad\times\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]y,y\right\rangle^{1/2}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\left|\left\langle\left[\frac{(1_H+N)^{-1}+(1_H+N^*)^{-1}}{2}\right]x,y\right\rangle\right|\\&\quad\le\frac{1}{2}\left\langle\left[\left(1_H-|N|^{2\alpha}\right)^{-1}+\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]x,x\right\rangle^{1/2}\\&\qquad\times\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]y,y\right\rangle^{1/2}\end{aligned} \tag{3.10}$$

for any $x, y \in H$ and $\|N\| < 1$.

We also have the inequalities

$$\begin{aligned}&\left|\left\langle\left[\frac{\sin(N)+\sin(N^*)}{2}\right]x,y\right\rangle\right|\\&\quad\le\frac{1}{2}\left\langle\left[\sinh\left(|N|^{2\alpha}\right)+\sinh\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\&\qquad\times\left\langle\left[\sinh\left(|N|^{2\alpha}\right)+\sinh\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\left|\left\langle\left[\frac{\cos(N)+\cos(N^*)}{2}\right]x,y\right\rangle\right|\\&\quad\le\frac{1}{2}\left\langle\left[\cosh\left(|N|^{2\alpha}\right)+\cosh\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\&\qquad\times\left\langle\left[\cosh\left(|N|^{2\alpha}\right)+\cosh\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned} \tag{3.12}$$

for any $x, y \in H$ and $N$, a normal operator.

If we utilize the following function as power series representations with nonnegative coefficients:

$$\begin{aligned}
&\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \quad z\in D(0,1);\\
&\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!}z^{2n+1}, \quad z\in D(0,1);\\
&\tanh^{-1}(z)=\sum_{n=0}^{\infty}\frac{1}{2n+1}z^{2n-1}, \quad z\in D(0,1);\\
&{}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n, \quad \alpha,\beta,\gamma>0, z\in D(0,1);
\end{aligned}\tag{3.13}$$

where $\Gamma$ is the *gamma function*, then we can state the following vector inequalities:

$$\begin{aligned}
&\left|\left\langle\left[\frac{\exp(N)+\exp(N^*)}{2}\right]x,y\right\rangle\right|\\
&\quad\le\frac{1}{2}\left\langle\left[\exp\left(|N|^{2\alpha}\right)+\exp\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\
&\qquad\times\left\langle\left[\exp\left(|N|^{2\alpha}\right)+\exp\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}
\end{aligned}\tag{3.14}$$

for any $x,y\in H$ and $N$, a normal operator.

If $\|N\|<1$, then we also have the inequalities

$$\begin{aligned}
&\left|\left\langle\left[\frac{\ln(\frac{1_H+N}{1_H-N})+\ln(\frac{1_H+N^*}{1_H-N^*})}{2}\right]x,y\right\rangle\right|\\
&\quad\le\frac{1}{2}\left\langle\left[\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)+\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right]x,x\right\rangle^{1/2}\\
&\qquad\times\left\langle\left[\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)+\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right]y,y\right\rangle^{1/2},
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&\left|\left\langle\left[\frac{\tanh^{-1}(N)+\tanh^{-1}(N^*)}{2}\right]x,y\right\rangle\right|\\
&\quad\le\frac{1}{2}\left\langle\left[\tanh^{-1}\left(|N|^{2\alpha}\right)+\tanh^{-1}\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\
&\qquad\times\left\langle\left[\tanh^{-1}\left(|N|^{2\alpha}\right)+\tanh^{-1}\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
&\left|\left\langle\left[\frac{{}_2F_1(\alpha,\beta,\gamma,N)+{}_2F_1(\alpha,\beta,\gamma,N^*)}{2}\right]x,y\right\rangle\right|\\
&\quad\le\frac{1}{2}\left\langle\left[{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2\alpha}\right)+{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\
&\qquad\times\left\langle\left[{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2\alpha}\right)+{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}
\end{aligned}\tag{3.17}$$

for any $x,y\in H$.

From a different perspective, we also have the following.

**Theorem 6** *With the assumption of Theorem 5 and if $N$ is a normal operator on the Hilbert space $H$ and $z \in \mathbb{C}$ such that $\|N\|^2, |z|^2 < R$, then we have the inequalities*

$$\begin{aligned} &\left|\left\langle\left(\frac{f(zN)+f(zN^*)}{2}\right)x,y\right\rangle\right|^2 \\ &\quad\le \frac{1}{2}f_A\left(|z|^2\right)\left[\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{\alpha}\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{1-\alpha}+\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{\alpha}\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{1-\alpha}\right] \\ &\quad\le \frac{1}{2}f_A\left(|z|^2\right)\left[\left\langle f_A\left(|N|^2\right)x,x\right\rangle+\left\langle f_A\left(|N|^2\right)y,y\right\rangle\right] \end{aligned} \tag{3.18}$$

*for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.*

*In particular, for $\alpha = \frac{1}{2}$, we have*

$$\begin{aligned} \left|\left\langle\left(\frac{f(zN)+f(zN^*)}{2}\right)x,y\right\rangle\right|^2 &\le f_A\left(|z|^2\right)\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{1/2}\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{1/2} \\ &\le \frac{1}{2}f_A\left(|z|^2\right)\left[\left\langle f_A\left(|N|^2\right)x,x\right\rangle+\left\langle f_A\left(|N|^2\right)y,y\right\rangle\right] \end{aligned} \tag{3.19}$$

*for any $x, y \in H$ with $\|x\| = \|y\| = 1$.*

*Proof* If we use the third and fourth inequalities in (2.23), we have

$$\begin{aligned} &\sum_{j=0}^{n}|a_j|\left[\left|\left\langle\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2\right] \\ &\quad\le \frac{1}{2}\left(\sum_{j=0}^{n}|a_j|\left\|N^jx\right\|^2\right)^{\alpha}\left(\sum_{j=0}^{n}|a_j|\left\|N^jy\right\|^2\right)^{1-\alpha} \\ &\qquad+\frac{1}{2}\left(\sum_{j=0}^{n}|a_j|\left\|N^jy\right\|^2\right)^{\alpha}\left(\sum_{j=0}^{n}|a_j|\left\|N^jx\right\|^2\right)^{1-\alpha} \\ &\quad\le \frac{1}{2}\sum_{j=0}^{n}|a_j|\left[\left\|N^jx\right\|^2+\left\|N^jy\right\|^2\right] \end{aligned} \tag{3.20}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Since $N$ is a normal operator on the Hilbert space $H$, then

$$\left\|N^jx\right\|^2 = \left\langle\left|N^j\right|^2x,x\right\rangle = \left\langle|N|^{2j}x,x\right\rangle$$

for any $j \in \{0,\dots,n\}$ and for any $x \in H$ with $\|x\| = 1$.

Then from (3.20) we get

$$\begin{aligned} &\sum_{j=0}^{n}|a_j|\left[\left|\left\langle\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2\right] \\ &\quad\le \frac{1}{2}\left(\left\langle\sum_{j=0}^{n}|a_j||N|^{2j}x,x\right\rangle\right)^{\alpha}\left(\left\langle\sum_{j=0}^{n}|a_j||N|^{2j}y,y\right\rangle\right)^{1-\alpha} \end{aligned}$$

$$+\frac{1}{2}\left(\left\langle\sum_{j=0}^{n}|a_j||N|^{2j}y,y\right\rangle\right)^{\alpha}\left(\left\langle\sum_{j=0}^{n}|a_j||N|^{2j}x,x\right\rangle\right)^{1-\alpha}$$
$$\le\frac{1}{2}\left[\left\langle\sum_{j=0}^{n}|a_j||N|^{2j}x,x\right\rangle+\left\langle\sum_{j=0}^{n}|a_j||N|^{2j}y,y\right\rangle\right] \tag{3.21}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and $\alpha\in[0,1]$.

By the weighted Cauchy-Buniakowski-Schwarz inequality, we also have

$$\left|\left\langle\sum_{j=0}^{n}a_jz^j\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2$$
$$\le\sum_{j=0}^{n}|a_j||z|^{2j}\sum_{j=0}^{n}|a_j|\left[\left|\left\langle\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2\right] \tag{3.22}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Now, since the series $\sum_{j=0}^{\infty}a_jz^jN^j$, $\sum_{j=0}^{\infty}a_jz^j(N^*)^j$, $\sum_{j=0}^{\infty}|a_j||z|^{2j}$, $\sum_{j=0}^{\infty}|a_j||N|^{2j}$ are convergent, then by (3.21) and (3.22) on letting $n\to\infty$, we deduce the desired result (3.18). □

Similar inequalities for some particular functions of interest can be stated. However, the details are left to the interested reader.

## 4 Applications to the Euclidian norm

In [28], the author introduced the following norm on the Cartesian product $\mathcal{B}^{(n)}(H):=\mathcal{B}(H)\times\cdots\times\mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space $H$:

$$\|(T_1,\dots,T_n)\|_e:=\sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n}\|\lambda_1T_1+\cdots+\lambda_nT_n\|, \tag{4.1}$$

where $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$ and $\mathbb{B}_n:=\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n|\sum_{j=1}^{n}|\lambda_j|^2\le1\}$ is the Euclidean closed ball in $\mathbb{C}^n$.

It is clear that $\|\cdot\|_e$ is a norm on $B^{(n)}(H)$ and for any $(T_1,\dots,T_n)\in B^{(n)}(H)$, we have

$$\|(T_1,\dots,T_n)\|_e=\|(T_1^*,\dots,T_n^*)\|_e,$$

where $T_j^*$ is the adjoint operator of $T_j$, $j\in\{1,\dots,n\}$. We call this the *Euclidian norm* of an $n$-tuple of operators $(T_1,\dots,T_n)\in B^{(n)}(H)$.

It has been shown in [28] that the following basic inequality for the Euclidian norm holds true:

$$\frac{1}{\sqrt{n}}\left\|\sum_{j=1}^{n}|T_j^*|^2\right\|^{\frac{1}{2}}\le\|(T_1,\dots,T_n)\|_e\le\left\|\sum_{j=1}^{n}|T_j^*|^2\right\|^{\frac{1}{2}} \tag{4.2}$$

for any $n$-tuple $(T_1,\dots,T_n)\in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [28], the author introduced the *Euclidean operator radius* of an $n$-tuple of operators $(T_1,\ldots,T_n)$ by

$$w_e(T_1,\ldots,T_n) := \sup_{\|x\|=1}\left(\sum_{j=1}^{n}\left|\langle T_j x,x\rangle\right|^2\right)^{\frac{1}{2}} \tag{4.3}$$

and proved that $w_e(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality

$$\frac{1}{2}\left\|(T_1,\ldots,T_n)\right\|_e \le w_e(T_1,\ldots,T_n) \le \left\|(T_1,\ldots,T_n)\right\|_e \tag{4.4}$$

for each $n$-tuple $(T_1,\ldots,T_n)\in B^{(n)}(H)$.

As pointed out in [28], the Euclidean numerical radius also satisfies the double inequality

$$\frac{1}{2\sqrt{n}}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{\frac{1}{2}} \le w_e(T_1,\ldots,T_n) \le \left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{\frac{1}{2}} \tag{4.5}$$

for any $(T_1,\ldots,T_n)\in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

In [29], by utilizing the concept of *hypo-Euclidean norm* on $H^n$, we obtained the following representation for the Euclidian norm.

**Proposition 1** *For any* $(T_1,\ldots,T_n)\in B^{(n)}(H)$, *we have*

$$\left\|(T_1,\ldots,T_n)\right\|_e = \sup_{\|y\|=1,\|x\|=1}\left(\sum_{j=1}^{n}\left|\langle T_j y,x\rangle\right|^2\right)^{\frac{1}{2}}. \tag{4.6}$$

**Theorem 7** *For any* $(T_1,\ldots,T_n)\in B^{(n)}(H)$, *we have*

$$\begin{aligned}\left\|\left(\frac{T_1+T_1^*}{2},\ldots,\frac{T_n+T_n^*}{2}\right)\right\|_e^2 &\le \left\|\sum_{j=1}^{n}|T_j|^2\right\|^{\alpha}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1-\alpha}\\ &\le \alpha\left\|\sum_{j=1}^{n}|T_j|^2\right\| + (1-\alpha)\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\end{aligned} \tag{4.7}$$

*and*

$$\begin{aligned}&w_e^2\left(\frac{T_1+T_1^*}{2},\ldots,\frac{T_n+T_n^*}{2}\right)\\ &\quad\le \sup_{\|x\|=1}\left[\left\langle\sum_{j=1}^{n}|T_j|^2 x,x\right\rangle^{\alpha}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2 x,x\right\rangle^{1-\alpha}\right]\\ &\quad\le \begin{cases}\|\sum_{j=1}^{n}|T_j|^2\|^{\alpha}\|\sum_{j=1}^{n}|T_j^*|^2\|^{1-\alpha},\\ \|\alpha\sum_{j=1}^{n}|T_j|^2 + (1-\alpha)\sum_{j=1}^{n}|T_j^*|^2\|\end{cases}\\ &\quad\le \alpha\left\|\sum_{j=1}^{n}|T_j|^2\right\| + (1-\alpha)\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\end{aligned} \tag{4.8}$$

*for any* $\alpha\in[0,1]$.

*Proof* Making use of inequalities (2.13) and (2.20), we have

$$\begin{aligned}
&\sum_{j=1}^{n}\left[\left|\left\langle\left(\frac{T_j+T_j^*}{2}\right)x,y\right\rangle\right|^2\right] \\
&\quad\le \frac{1}{2}\left\langle\sum_{j=1}^{n}|T_j|^2x,x\right\rangle^{\alpha}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2y,y\right\rangle^{1-\alpha} \\
&\qquad+\frac{1}{2}\left\langle\sum_{j=1}^{n}|T_j|^2y,y\right\rangle^{\alpha}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2x,x\right\rangle^{1-\alpha}
\end{aligned} \tag{4.9}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and $\alpha\in[0,1]$.

Taking the supremum over $\|x\|=\|y\|=1$ in (4.9), we get

$$\begin{aligned}
&\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_e^2 \\
&\quad\le \frac{1}{2}\sup_{\|x\|=1}\left\langle\sum_{j=1}^{n}|T_j|^2x,x\right\rangle^{\alpha}\sup_{\|y\|=1}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2y,y\right\rangle^{1-\alpha} \\
&\qquad+\frac{1}{2}\sup_{\|y\|=1}\left\langle\sum_{j=1}^{n}|T_j|^2y,y\right\rangle^{\alpha}\sup_{\|x\|=1}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2x,x\right\rangle^{1-\alpha} \\
&\quad=\left\|\sum_{j=1}^{n}|T_j|^2\right\|^{\alpha}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1-\alpha}
\end{aligned}$$

and inequality (4.7) is proved.

Now, if we take $y=x$ in (4.9), we get

$$\begin{aligned}
&\sum_{j=1}^{n}\left[\left|\left\langle\left(\frac{T_j+T_j^*}{2}\right)x,x\right\rangle\right|^2\right] \\
&\quad\le \left\langle\sum_{j=1}^{n}|T_j|^2x,x\right\rangle^{\alpha}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2x,x\right\rangle^{1-\alpha} \\
&\quad\le \left\langle\left[\alpha\sum_{j=1}^{n}|T_j|^2+(1-\alpha)\sum_{j=1}^{n}\left|T_j^*\right|^2\right]x,x\right\rangle
\end{aligned} \tag{4.10}$$

for any $x\in H$ with $\|x\|=1$ and $\alpha\in[0,1]$.

Taking the supremum over $\|x\|=1$ in (4.10), we get the desired result. □

**Remark 7** In the particular case $\alpha=\frac{1}{2}$, we get

$$\begin{aligned}
\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_e^2 &\le \left\|\sum_{j=1}^{n}|T_j|^2\right\|^{1/2}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1/2} \\
&\le \frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|^2\right\|+\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\right]
\end{aligned} \tag{4.11}$$

and

$$
\begin{aligned}
& w_e^2\left(\frac{T_1+T_1^*}{2},\ldots,\frac{T_n+T_n^*}{2}\right) \\
& \quad \le \sup_{\|x\|=1}\left[\left\langle\sum_{j=1}^n |T_j|^2 x,x\right\rangle^{1/2}\left\langle\sum_{j=1}^n \left|T_j^*\right|^2 x,x\right\rangle^{1/2}\right] \qquad (4.12)\\
& \quad \le \begin{cases} \|\sum_{j=1}^n |T_j|^2\|^{1/2}\|\sum_{j=1}^n |T_j^*|^2\|^{1/2}, \\ \|\sum_{j=1}^n \frac{|T_j|^2+|T_j^*|^2}{2}\| \end{cases}\\
& \quad \le \frac{1}{2}\left[\left\|\sum_{j=1}^n |T_j|^2\right\|+\left\|\sum_{j=1}^n \left|T_j^*\right|^2\right\|\right]. \qquad (4.13)
\end{aligned}
$$

## 5 Applications for *s*-1-norm and *s*-1-numerical radius

Following [30], we consider the *s-p-norm* of the $n$-tuple of operators $(T_1,\ldots,T_n)\in B^{(n)}(H)$ given by

$$\left\|(T_1,\ldots,T_n)\right\|_{s,p} := \sup_{\|y\|=1,\|x\|=1}\left[\left(\sum_{j=1}^n \left|\langle T_j y,x\rangle\right|^p\right)^{\frac{1}{p}}\right]. \qquad (5.1)$$

For $p=2$ we get

$$\left\|(T_1,\ldots,T_n)\right\|_{s,2} = \left\|(T_1,\ldots,T_n)\right\|_e.$$

We are interested in this section in the case $p=1$, namely on the $s$-1-norm defined by

$$\left\|(T_1,\ldots,T_n)\right\|_{s,1} := \sup_{\|y\|=1,\|x\|=1}\sum_{j=1}^n \left|\langle T_j y,x\rangle\right|.$$

Since for any $x,y\in H$ we have $\sum_{j=1}^n |\langle T_j y,x\rangle| \ge |\langle\sum_{j=1}^n T_j y,x\rangle|$, then by the properties of the supremum, we get the basic inequality

$$\left\|\sum_{j=1}^n T_j\right\| \le \left\|(T_1,\ldots,T_n)\right\|_{s,1} \le \sum_{j=1}^n \|T_j\|. \qquad (5.2)$$

Similarly, we can also consider the *s-p-numerical radius* of the $n$-tuple of operators $(T_1,\ldots,T_n)\in B^{(n)}(H)$ defined by [30]

$$w_{s,p}(T_1,\ldots,T_n) := \sup_{\|x\|=1}\left[\left(\sum_{j=1}^n \left|\langle T_j x,x\rangle\right|^p\right)^{\frac{1}{p}}\right], \qquad (5.3)$$

which for $p=2$ reduces to the Euclidean operator radius introduced previously. We observe that the $s$-$p$-numerical radius is also a norm on $B^{(n)}(H)$ for $p\ge 1$, and for $p=1$ it satisfies the basic inequality

$$w\left(\sum_{j=1}^n T_j\right) \le w_{s,1}(T_1,\ldots,T_n) \le \sum_{j=1}^n w(T_j). \qquad (5.4)$$

**Theorem 8** *For any* $(T_1,\dots,T_n)\in B^{(n)}(H)$, *we have*

$$\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_{s,1}\le\left\|\sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]\right\|\le\frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|^{2\alpha}\right\|+\left\|\sum_{j=1}^{n}|T_j^*|^{2(1-\alpha)}\right\|\right]\tag{5.5}$$

*and*

$$w_{s,1}\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\le w_{s,1}(T_1,\dots,T_n)\le\left\|\sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]\right\|\le\frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|^{2\alpha}\right\|+\left\|\sum_{j=1}^{n}|T_j^*|^{2(1-\alpha)}\right\|\right]\tag{5.6}$$

*for any* $\alpha\in[0,1]$.

*Proof* Utilizing inequality (2.1), we have

$$\sum_{j=1}^{n}\left|\left\langle\frac{T_j+T_j^*}{2}x,y\right\rangle\right|\le\left\langle\sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]x,x\right\rangle^{1/2}\times\left\langle\sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]y,y\right\rangle^{1/2}\tag{5.7}$$

for any $x,y\in H$ and $\alpha\in[0,1]$.

Taking the supremum in (5.7) over $\|x\|=\|y\|=1$, we get the first inequality in (5.5).

The second part follows by the triangle inequality.

By inequality (2.7) we have

$$\sum_{j=1}^{n}\left|\left\langle\frac{T_j+T_j^*}{2}x,x\right\rangle\right|\le\sum_{j=1}^{n}\left|\langle T_jx,x\rangle\right|\le\left\langle\sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]x,x\right\rangle$$

for any $x\in H$.

Taking the supremum over $\|x\|=1$, we deduce the desired result (5.6). □

**Remark 8** The case $\alpha=\frac{1}{2}$ produces the following chains of inequalities:

$$\left\|\sum_{j=1}^{n}\left(\frac{T_j+T_j^*}{2}\right)\right\|\le\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_{s,1}\le\left\|\sum_{j=1}^{n}\left(\frac{|T_j|+|T_j^*|}{2}\right)\right\|\le\frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|\right\|+\left\|\sum_{j=1}^{n}|T_j^*|\right\|\right]\tag{5.8}$$

and

$$
\begin{aligned}
w\left(\sum_{j=1}^{n}\left(\frac{T_j + T_j^*}{2}\right)\right) &\le w_{s,1}\left(\frac{T_1 + T_1^*}{2}, \dots, \frac{T_n + T_n^*}{2}\right) \\
&\le w_{s,1}(T_1, \dots, T_n) \\
&\le \left\| \sum_{j=1}^{n}\left(\frac{|T_j| + |T_j^*|}{2}\right)\right\| \\
&\le \frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|\right\| + \left\|\sum_{j=1}^{n}|T_j^*|\right\|\right].
\end{aligned}
\tag{5.9}
$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
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