# *Bounds for the deviation of a function from a generalised chord generated by its extremities with applications*

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# BOUNDS FOR THE DEVIATION OF A FUNCTION FROM A GENERALISED CHORD GENERATED BY ITS EXTREMITIES WITH APPLICATIONS

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Abstract. Bounds for the deviation of a real-valued function $f$ defined on a compact interval $[a,b]$ to the generalised chord

$$\frac{v(b)-v(t)}{v(b)-v(a)} \cdot f(a) + \frac{v(t)-v(a)}{v(b)-v(a)} \cdot f(b),$$

where $v : [a,b] \to \mathbb{R}$ and $v(a) \neq v(b)$, that connects its end points $(a, f(a))$ and $(b, f(b))$ are given. Applications for normalised positive linear functionals are provided as well.

## 1. Introduction

Consider a function $f : [a,b] \to \mathbb{R}$ and assume that it is bounded on $[a,b]$. Denote by $\Phi_f(t)$ the error in approximating the function $f$ by its (straight line) chord $d_f$ which connects the points $(a, f(a))$ and $(b, f(b))$, i.e.,

$$\Phi_f(t) := \frac{b-t}{b-a} \cdot f(a) + \frac{t-a}{b-a} f(b) - f(t), \qquad t \in [a,b]. \tag{1.1}$$

In the recent paper [3], sharp error estimates for $\Phi_f(t)$ under various assumptions on the function $f$ have been derived. We recall here some of them.

If there exist the constants $-\infty < m < M < \infty$ such that $m \leq f(t) \leq M$ for each $t \in [a,b]$, then $|\Phi_f(t)| \leq M - m$. The multiplication constant 1 in front of $(M-m)$ cannot be replaced by a smaller quantity. If $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then

$$\begin{aligned} 0 \leq \Phi_f(t) &\leq \frac{1}{b-a}(t-a)(b-t)\left[f'_-(b) - f'_+(a)\right] \qquad (1.2) \\ &\leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right], \end{aligned}$$

for any $t \in [a,b]$. In the case where the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $\frac{1}{4}$ are sharp.

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If $f:[a,b]\to\mathbb{R}$ is a function of bounded variation, then

$$|\Phi_f(t)| \le \frac{b-t}{b-a}\cdot\bigvee_a^t(f)+\frac{t-a}{b-a}\bigvee_t^b(f) \tag{1.3}$$
$$\le \begin{cases} \left[\frac{1}{2}+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f); \\ \left[\left(\frac{b-t}{b-a}\right)^p+\left(\frac{t-a}{b-a}\right)^p\right]^{\frac{1}{p}}\left[\left(\bigvee_a^t(f)\right)^q+\left(\bigvee_t^b(f)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^t(f)-\bigvee_t^b(f)\right|. \end{cases}$$

The first inequality in (1.3) is sharp. The constant $\frac{1}{2}$ is best possible in the first and third branches.

In particular, if $f$ is $L-$Lipschitzian on $[a,b]$, i.e., $|f(t)-f(s)|\le L|t-s|$ for any $t,s\in[a,b]$, then

$$|\Phi_f(t)| \le \frac{2(b-t)(t-a)}{b-a}L \le \frac{1}{2}(b-a)L, \tag{1.4}$$

for any $t\in[a,b]$. The constants 2 and $\frac{1}{2}$ are best possible.

For extensions to $n$- time differentiable functions see [4].

In this paper we consider a natural generalisation of the above problem by introducing the error function for the approximation of $f(t)$ with $\frac{v(b)-v(t)}{v(b)-v(a)}\cdot f(a)+\frac{v(t)-v(a)}{v(b)-v(a)}\cdot f(b)$, where $v:[a,b]\to\mathbb{R}$ is another function with the property that $v(a)\ne v(b)$. Error bounds for different pairs of functions $(f,v)$ are derived. Applications in obtaining error bounds in approximating the quantity $A(f\circ u)$ by the generalised trapezoid formula

$$\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(a)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(b),$$

where $A$ is a normalised linear functional are also given.

## 2. Bounds for $\Phi_{f,v}$ when $f,v$ are of Bounded Variation

For a function $p:[a,b]\to\mathbb{R}$ we define the kernel $Q_p:[a,b]^2\to\mathbb{R}$ by

$$Q_p(t,s):=\begin{cases} p(t)-p(b) & \text{if } a\le s\le t\le b, \\ p(t)-p(a) & \text{if } a\le t<s\le b. \end{cases} \tag{2.1}$$

With this notation we have the following representation of the function $\Phi_{f,v}$, where

$$\Phi_{f,v}(t)=\frac{v(t)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-v(t)}{v(b)-v(a)}\cdot f(a)-f(t)$$

with $t\in[a,b]$.

**Lemma 2.1.** *If $f, v : [a,b] \to \mathbb{R}$ are bounded functions on $[a,b]$, then*

$$\begin{aligned} \Phi_{f,v}(t) &= \frac{1}{v(b)-v(a)} \int_a^b Q_v(t,s)\, df(s) \\ &= \frac{1}{v(b)-v(a)} \int_a^b Q_{-f}(t,s)\, dv(s) \end{aligned} \tag{2.2}$$

*provided $v(b) \neq v(a)$, where the integrals are taken in the Riemann-Stieltjes sense.*

*Proof.* We have

$$\begin{aligned} \Phi_{f,v}(t) &= \frac{[v(t)-v(b)][f(t)-f(a)] + [v(t)-v(a)][f(b)-f(t)]}{v(b)-v(a)} \\ &= \frac{[v(t)-v(b)] \int_a^t df(s) + [v(t)-v(a)] \int_t^b df(s)}{v(b)-v(a)} \\ &= \frac{1}{v(b)-v(a)} \int_a^b Q_v(t,s)\, df(s). \end{aligned} \tag{2.3}$$

Also, by rearranging the terms in the first equality, we also have

$$\begin{aligned} \Phi_{f,v}(t) &= \frac{[f(a)-f(t)] \int_t^b dv(s) + [f(b)-f(t)] \int_a^t dv(s)}{v(b)-v(a)} \\ &= \frac{1}{v(b)-v(a)} \int_a^b Q_{-f}(t,s)\, dv(s) \end{aligned} \tag{2.4}$$

and the representation (2.2) is proved. $\square$

The following estimation result can be stated.

**Theorem 2.2.** *Assume that $f, v : [a,b] \to \mathbb{R}$ are bounded and $v(a) \neq v(b)$.*

(i) *If $f$ is of bounded variation on $[a,b]$, then*

$$|\Phi_{f,v}(t)| \leq \left| \frac{v(b)-v(t)}{v(b)-v(a)} \right| \cdot \bigvee_a^t (f) + \left| \frac{v(t)-v(a)}{v(b)-v(a)} \right| \cdot \bigvee_t^b (f) \tag{2.5}$$

$$\leq \begin{cases} \max\left\{ \left| \frac{v(b)-v(t)}{v(b)-v(a)} \right|, \left| \frac{v(t)-v(a)}{v(b)-v(a)} \right| \right\} \bigvee_a^b (f); \\ \left[ \left| \frac{v(b)-v(t)}{v(b)-v(a)} \right|^p + \left| \frac{v(t)-v(a)}{v(b)-v(a)} \right|^p \right]^{\frac{1}{p}} \left\{ \left[ \bigvee_a^t (f) \right]^q + \left[ \bigvee_t^b (f) \right]^q \right\}^{\frac{1}{q}}, \\ \qquad\qquad\qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|v(b)-v(t)| + |v(t)-v(a)|}{|v(b)-v(a)|} \left\{ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^t (f) - \bigvee_t^b (f) \right| \right\}. \end{cases}$$

(ii) *If $v$ is of bounded variation on $[a,b]$, then*

$$|\Phi_{f,v}(t)| \leq \left|\frac{f(b)-f(t)}{v(b)-v(a)}\right| \cdot \bigvee_a^t (v) + \left|\frac{f(t)-f(a)}{v(b)-v(a)}\right| \cdot \bigvee_t^b (v) \tag{2.6}$$

$$\leq \begin{cases} \max\left\{\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|, \left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|\right\} \bigvee_a^b (v); \\ \left[\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|^p + \left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|^p\right]^{\frac{1}{p}} \left\{\left[\bigvee_a^t (f)\right]^q + \left[\bigvee_t^b (f)\right]^q\right\}^{\frac{1}{q}}, \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{|f(b)-f(t)|+|f(t)-f(a)|}{|v(b)-v(a)|} \left\{\frac{1}{2}\bigvee_a^b (v) + \frac{1}{2}\left|\bigvee_a^t (v) - \bigvee_t^b (v)\right|\right\}. \end{cases}$$

*Proof.* Utilising the equality (2.3) and taking the modulus, we have successively:

$$\begin{aligned} |\Phi_{f,v}(t)| &\leq \left|\frac{v(b)-v(t)}{v(b)-v(a)}\right| \cdot \left|\int_a^t df(s)\right| + \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right| \cdot \left|\int_t^b df(s)\right| \\ &\leq \left|\frac{v(b)-v(t)}{v(b)-v(a)}\right| \cdot \bigvee_a^t (f) + \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right| \cdot \bigvee_t^b (f) \end{aligned}$$

$$\leq \begin{cases} \max\left\{\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|, \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|\right\} \bigvee_a^b (f); \\ \left[\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|^p + \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|^p\right]^{\frac{1}{p}} \left\{\left[\bigvee_a^t (f)\right]^q + \left[\bigvee_t^b (f)\right]^q\right\}^{\frac{1}{q}}, \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{|v(b)-v(t)|+|v(t)-v(a)|}{|v(b)-v(a)|} \left\{\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^t (f) - \bigvee_t^b (f)\right|\right\}, \end{cases}$$

where for the last inequality we have used the Hölder inequality.

The inequality (2.6) goes likewise by utilising the equality (2.4). □

**Remark.** *Since $v(a) \neq v(b)$, we can assume without loss the generality that $v(a) < v(b)$. Now, if we assume that*

$$v(a) \leq v(t) \leq v(b) \quad \text{for any} \quad t \in (a,b), \tag{2.7}$$

*then from the first branch of (2.5) we get the inequality*

$$|\Phi_{f,v}(t)| \leq \left[\frac{1}{2} + \frac{\left|v(t) - \frac{v(a)+v(b)}{2}\right|}{v(b)-v(a)}\right] \bigvee_a^b (f), \qquad t \in [a,b]. \tag{2.8}$$

*The constant $\frac{1}{2}$ is sharp in (2.8).*

To prove the sharpness of the constant we take in (2.8) $v(t) = t$ and then choose $t = \frac{a+b}{2}$. This produces the result:

$$\left|f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right| \leq \frac{1}{2}\bigvee_a^b (f), \tag{2.9}$$

which is sharp since for $f(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$ we obtain in both sides of (2.9) the same quantity $\frac{b-a}{2}$.

**Remark.** *We also remark that, if $v$ satisfies (2.7), then from the last inequality in (2.5) we get*

$$|\Phi_{f,v}(t)| \leq \frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^t(f) - \bigvee_t^b(f)\right|, \qquad t \in [a,b] \tag{2.10}$$

*for which the first constant $\frac{1}{2}$ is also best possible.*

**Remark.** *If $f$ satisfies the property that $f(a) \leq f(t) \leq f(b)$ for any $t \in [a,b]$, then from the first inequality in (2.6) we get*

$$|\Phi_{f,v}(t)| \leq \left[\frac{1}{2}\cdot\frac{f(b)-f(a)}{|v(b)-v(a)|} + \left|\frac{f(t)-\frac{f(a)+f(b)}{2}}{v(b)-v(a)}\right|\right]\bigvee_a^b(f), \qquad t \in [a,b]. \tag{2.11}$$

*With the same assumptions for $f$ we have from the second inequality in (2.6) that*

$$|\Phi_{f,v}(t)| \leq \frac{f(b)-f(a)}{|v(b)-v(a)|}\left\{\frac{1}{2}\bigvee_a^b(v) + \frac{1}{2}\left|\bigvee_a^t(v) - \bigvee_t^b(v)\right|\right\}, \qquad t \in [a,b]. \tag{2.12}$$

*The first constant $\frac{1}{2}$ in (2.12) is best possible.*

Indeed, if we choose $v(t) = t$ and then $t = \frac{a+b}{2}$ in (2.12), we have

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{2}[f(b)-f(a)]. \tag{2.13}$$

Now, for $f : [a,b] \to \mathbb{R}$, $f(t) = 0$ if $t \in [a,b]$ and $f(b) = k > 0$, we obtain on both sides the same quantity $\frac{k}{2}$.

## 3. Bounds for $\Phi_{f,v}$ when $v(a) < v(t) < v(b)$ $(f(a) < f(t) < f(b))$

The following result may be stated as well.

**Theorem 3.1.** *Assume that $f, v : [a,b] \to \mathbb{R}$ are bounded and $v(a) \neq v(b)$.*

(i) *If $v(a) < v(t) < v(b)$ for any $t \in (a,b)$, then*

$$|\Phi_{f,v}(t)| \leq \frac{1}{4}[v(b)-v(a)]\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{v(b)-v(t)}\right|\right], \quad t \in [a,b]. \tag{3.1}$$

*The constant $\frac{1}{4}$ is best possible.*

(ii) *If $f(a) < f(t) < f(b)$ for $t \in (a,b)$, then*

$$|\Phi_{f,v}(t)| \leq \frac{1}{4}\frac{[f(b)-f(a)]^2}{|v(b)-v(a)|}\left[\left|\frac{v(t)-v(a)}{f(t)-f(a)}\right| + \left|\frac{v(b)-v(t)}{f(b)-f(t)}\right|\right], \ t \in [a,b]. \tag{3.2}$$

*Proof.* (i) From the first equality in (2.3), we have:

$$\begin{aligned}|\Phi_{f,v}(t)| &\leq \frac{|[v(b)-v(t)][v(t)-v(a)]|}{|v(b)-v(a)|}\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{v(b)-v(t)}\right|\right] \\ &= \frac{[v(b)-v(t)][v(t)-v(a)]}{|v(b)-v(a)|}\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{v(b)-v(t)}\right|\right] \\ &\leq \frac{1}{4}[v(b)-v(a)]\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{v(b)-v(t)}\right|\right]\end{aligned}$$

since, for any $t \in (a,b)$,

$$[v(b)-v(t)][v(t)-v(a)] \leq \frac{1}{4}[v(b)-v(a)]^2.$$

For the best constant, choose $v(t) = t$ and then $t = \frac{a+b}{2}$ in (3.1) to obtain

$$\left| \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{2}\left[ \left| f\left(\frac{a+b}{2}\right) - f(a) \right| + \left| f(b) - f\left(\frac{a+b}{2}\right) \right| \right]. \quad (3.3)$$

If we consider the function $f : [a,b] \to \mathbb{R}$,

$$f(t) = \begin{cases} 0 & \text{if } t \in [a,b) \\ k & \text{if } t = b, \ k > 0, \end{cases}$$

then (3.3) becomes an equality with both terms $\frac{k}{2}$.

(ii) The proof goes likewise and the details are omitted. □

**Remark.**

(a) *Under the assumptions of (i) of Theorem 3.1 and if there exist $L_a > 0$, $L_b > 0$, $\alpha, \beta \geq 0$ such that*

$$\left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| \leq L_a (t-a)^{\alpha}, \quad \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \leq L_b (b-t)^{\beta}, \quad t \in (a,b), \quad (3.4)$$

*then we have the inequality:*

$$|\Phi_{f,v}(t)| \leq \frac{1}{4}[v(b) - v(a)]\left[L_a (t-a)^{\alpha} + L_b (b-t)^{\beta}\right], \qquad t \in (a,b). \quad (3.5)$$

(aa) *Under the assumptions of (ii) of Theorem 3.1 and if there exist the constants $H_a, H_b > 0$ and $\gamma, \delta \geq 0$ such that*

$$\left| \frac{v(t) - v(a)}{f(t) - f(a)} \right| \leq H_a (t-a)^{\gamma}, \quad \left| \frac{v(b) - v(t)}{f(b) - f(t)} \right| \leq H_b (b-t)^{\delta}, \quad t \in (a,b), \quad (3.6)$$

*then we have the inequality:*

$$|\Phi_{f,v}(t)| \leq \frac{1}{4} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|}\left[H_a (t-a)^{\gamma} + H_b (b-t)^{\delta}\right], \qquad t \in (a,b). \quad (3.7)$$

The following corollary provides some uniform bounds in the case where the functions are differentiable.

**Corollary 3.2.** *Assume that $f, v : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on $(a,b)$ with $v(a) \neq v(b)$.*

(i) *If $v(a) < v(t) < v(b)$ and $v'(t) \neq 0$ for $t \in (a,b)$, then*

$$|\Phi_{f,v}(t)| \leq \frac{1}{2} \cdot [v(b) - v(a)] \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|, \qquad t \in (a,b). \quad (3.8)$$

(ii) *If $f(a) < f(t) < f(b)$ and $f'(t) \neq 0$ for $t \in (a,b)$, then*

$$|\Phi_{f,v}(t)| \leq \frac{1}{2} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|} \sup_{s \in (a,b)} \left| \frac{v'(s)}{f'(s)} \right|, \qquad t \in (a,b). \quad (3.9)$$

*Proof.* (i) Applying Cauchy's mean value theorem, we deduce that for any $t \in (a,b)$ there exists an $s$ between $t$ and $a$ such that

$$\frac{f(t) - f(a)}{v(t) - v(a)} = \frac{f'(s)}{v'(s)}.$$

Therefore,

$$\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| \le \sup_{s\in(a,b)}\left|\frac{f'(s)}{v'(s)}\right|, \qquad t\in(a,b)$$

and in a similar manner,

$$\left|\frac{f(b)-f(t)}{v(b)-v(t)}\right| \le \sup_{s\in(a,b)}\left|\frac{f'(s)}{v'(s)}\right|, \qquad t\in(a,b).$$

Utilising the inequality (2.13) we deduce (3.8).

The proof of (ii) goes likewise and we omit the details. □

## 4. Bounds for $\Phi_{f,v}$ when $f, v$ are Lipschitzian

We can state the following result.

**Theorem 4.1.** *Assume that $f, v : [a,b] \to \mathbb{R}$ are bounded functions on $[a,b]$ and $v(a) \neq v(b)$.*

(i) *If there exist constants $M_a, M_b > 0$ and $\alpha, \beta > 0$ such that $|f(t) - f(a)| \le M_a (t-a)^{\alpha}$, $|f(b) - f(t)| \le M_b (b-t)^{\beta}$ for any $t \in [a,b]$ and $v : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$, then*

$$|\Phi_{f,v}(t)| \le M_a \left|\frac{v(b)-v(t)}{f(b)-f(t)}\right| (t-a)^{\alpha} + M_b \left|\frac{v(t)-v(a)}{f(t)-f(a)}\right| (b-t)^{\beta} \tag{4.1}$$

*for any $t \in [a,b]$.*

(ii) *If there exist constants $N_a, N_b > 0$, $\gamma, \delta > 0$ such that $|v(t) - v(a)| \le N_a (t-a)^{\gamma}$, $|v(b) - v(t)| \le N_b (b-t)^{\delta}$ for any $t \in [a,b]$, then*

$$|\Phi_{f,v}(t)| \le N_b \left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| (b-t)^{\delta} + N_a \left|\frac{f(b)-f(t)}{v(b)-v(t)}\right| (t-a)^{\gamma} \tag{4.2}$$

*for any $t \in [a,b]$.*

*Proof.* Utilising the representation (2.3) we have:

$$|\Phi_{f,v}(t)| \le \frac{|f(t)-f(a)|\,|v(b)-v(t)| + |v(t)-v(a)|\,|f(b)-f(t)|}{|v(b)-v(a)|}$$

for any $t \in [a,b]$, which clearly produces the desired inequalities (4.1) and (4.2). □

We notice that, if more information is provided for $f$ and $v$, then more specific bounds can be obtained. For instance, if $f$ is as in (i) of Theorem 4.1 and $v(a) < v(t) < v(b)$ for each $t \in (a,b)$, then we get from (4.1) the following inequality:

$$|\Phi_{f,v}(t)| \le \left[\frac{1}{2} + \left|\frac{v(t) - \frac{v(a)+v(b)}{2}}{v(b)-v(a)}\right|\right] \left[M_a (t-a)^{\alpha} + M_b (b-t)^{\beta}\right] \tag{4.3}$$

for any $t \in [a,b]$.

Similarly, if $v$ satisfies condition (ii) of Theorem 4.1 and $f(a) < f(t) < f(b)$ for each $t \in (a,b)$, then

$$|\Phi_{f,v}(t)| \le \left[\frac{1}{2} \cdot \frac{f(b)-f(a)}{|v(b)-v(a)|} + \left|\frac{f(t) - \frac{f(a)+f(b)}{2}}{v(b)-v(a)}\right|\right] \times \left[N_b (b-t)^{\delta} + N_a (t-a)^{\gamma}\right] \tag{4.4}$$

for any $t \in [a,b]$.

If $f$ is $M-$Lipschitzian, then from (4.1) we get

$$
\begin{aligned}
|\Phi_{f,v}(t)| &\leq M\left[\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|(t-a)+\left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|(b-t)\right] \\
&\leq M\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\left[\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|+\left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|\right],
\end{aligned} \tag{4.5}
$$

for any $t\in[a,b]$.

Also, if $v$ is $N-$Lipschitzian, then from (4.1) we get

$$
\begin{aligned}
|\Phi_{f,v}(t)| &\leq N\left[\left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|(b-t)+\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|(t-a)\right] \\
&\leq N\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\left[\left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|+\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|\right]
\end{aligned} \tag{4.6}
$$

for any $t\in[a,b]$.

Moreover, if $f$ is $M-$Lipschitzian and $v(a)<v(t)<v(b)$ for any $t\in[a,b]$, then from (4.5) we get the simpler inequality:

$$
|\Phi_{f,v}(t)|\leq M\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right] \tag{4.7}
$$

for any $t\in[a,b]$.

If $v$ is $N-$Lipschitzian and $f(a)<f(t)<f(b)$, $v(a)<v(b)$, then from (4.6) we also have:

$$
|\Phi_{f,v}(t)|\leq N\cdot\frac{f(b)-f(a)}{v(b)-v(a)}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right], \tag{4.8}
$$

for each $t\in[a,b]$.

## 5. Applications for Positive Linear Functionals

Let $L$ be a linear class of real-valued functions $g:E\to\mathbb{R}$ having the properties

(L1) $f,g\in L$ imply $(\alpha f+\beta g)\in L$ for all $\alpha,\beta\in\mathbb{R}$;
(L2) $\mathbf{1}\in L$, i.e., if $f_0(t)=1$, $t\in E$, then $f_0\in L$.

An isotonic linear functional $A:L\to\mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ for all $f,g\in L$ and $\alpha,\beta\in\mathbb{R}$;
(A2) If $f\in L$ and $f\geq 0$, then $A(f)\geq 0$;
(A3) The mapping $A$ is *normalised* if $A(\mathbf{1})=1$.

For a function $u:E\to[a,b]$, we consider the function

$$
\Phi_{f,v}(u):=\frac{v\circ u-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-v\circ u}{v(b)-v(a)}\cdot f(a)-f\circ u
$$

and assume throughout this section that $\Phi_{f,v}(u)\in L$.

It is obvious that for a normalised linear functional $A:L\to\mathbb{R}$ we have

$$
A(\Phi_{f,v}(u))=\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(a)-A(f\circ u)
$$

and the inequalities in the previous section can be utilised to provide various upper bounds for the quantity

$$
|A(\Phi_{f,v}(u))|.
$$

For the sake of brevity we give here only some bounds that are simple and perhaps more useful for applications.

**Proposition 5.1.** *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$ and $v(a) < v(b)$, $v(a) \le v(t) \le v(b)$ for each $t \in [a,b]$. If $u \in L$ so that $\Phi_{f,v}(u) \in L$ and $A : L \to \mathbb{R}$ is a normalised positive linear functional on $L$, then:*

$$\left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \le \left[ \frac{1}{2} + \frac{1}{v(b) - v(a)} A\left( \left| v \circ u - \frac{v(a) + v(b)}{2} \cdot \mathbf{1} \right| \right) \right] \bigvee_a^b (f). \quad (5.1)$$

*Proof.* Utilising the inequality (2.8) and the properties of the functional $A$, we have

$$\begin{aligned} |A(\Phi_{f,v}(u))| &\le A(|\Phi_{f,v}(u)|) \\ &\le A\left[ \left( \frac{1}{2} + \left| \frac{v \circ u - \frac{v(a)+v(b)}{2}}{v(b) - v(a)} \right| \right) \bigvee_a^b (f) \right] \\ &= \bigvee_a^b (f) \left[ \frac{1}{2} + \frac{1}{v(b) - v(a)} A\left( \left| v \circ u - \frac{v(a) + v(b)}{2} \cdot \mathbf{1} \right| \right) \right] \end{aligned}$$

and the inequality (5.1) is proved. □

**Proposition 5.2.** *Let $f, v : [a,b] \to \mathbb{R}$ be bounded and $v(a) \ne v(b)$. Also, assume that $u \in L$ such that $\Phi_{f,v}(u) \in L$ and $A : L \to \mathbb{R}$ is a normalised positive linear functional on $L$.*

(i) *If $v(a) < v(t) < v(b)$ for each $t \in [a,b]$, then*

$$\left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \le \frac{1}{4} [v(b) - v(a)] \left[ A\left( \left| \frac{f - f(a) \cdot \mathbf{1}}{v - v(a) \cdot \mathbf{1}} \right| \right) + A\left( \left| \frac{f(b) \cdot \mathbf{1} - f}{v(b) \cdot \mathbf{1} - v} \right| \right) \right], \quad (5.2)$$

*provided* $\frac{f - f(a) \cdot \mathbf{1}}{v - v(a) \cdot \mathbf{1}}, \frac{f(b) \cdot \mathbf{1} - f}{v(b) \cdot \mathbf{1} - v} \in L$;

(ii) *If $f(0) < f(t) < f(b)$ for $t \in (a,b)$, then*

$$\left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \le \frac{1}{4} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|} \left[ A\left( \left| \frac{v - v(a) \cdot \mathbf{1}}{f - f(a) \cdot \mathbf{1}} \right| \right) + A\left( \left| \frac{v(b) \cdot \mathbf{1} - v}{f(b) \cdot \mathbf{1} - f} \right| \right) \right], \quad (5.3)$$

*provided* $\frac{v - v(a) \cdot \mathbf{1}}{f - f(a) \cdot \mathbf{1}}, \frac{v(b) \cdot \mathbf{1} - v}{f(b) \cdot \mathbf{1} - f} \in L$.

Utilising Corollary 3.2 we can state the following result that can be utilised for applications.

**Proposition 5.3.** *Let $f, v : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$. Also, assume that $u \in L$ such that $\Phi_{f,v}(u) \in L$ and $A : L \to \mathbb{R}$ is a normalised positive functional on $L$.*

(i) *If $v$ is strictly monotonic on $[a,b]$, then*

$$\left|\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(a)-A(f\circ u)\right| \leq \frac{1}{2}\left|v(b)-v(a)\right|\sup_{s\in(a,b)}\left|\frac{f'(s)}{v'(s)}\right|. \quad (5.4)$$

(ii) *If $f$ is strictly monotonic on $[a,b]$, then*

$$\left|\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(a)-A(f\circ u)\right| \leq \frac{1}{2}\cdot\frac{\left[f(b)-f(a)\right]^2}{\left|v(b)-v(a)\right|}\sup_{s\in(a,b)}\left|\frac{v'(s)}{f'(s)}\right|, \quad (5.5)$$

*provided $v(a)\neq v(b)$.*

For other inequalities for isotonic linear functionals, see the papers [1], [2], [6] and the books [5] and [7].

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