Summing Series Using Residues

by

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Contents

0.1 Abstract.

This thesis deals with the problem of representation of series in closed form, mainly by the use of residue theory. Forced differential-difference equations of arbitrary order are considered from which infinite sums of the form

$$S_1(R,k,b,a,t) = \sum_{n=0}^{\infty} \binom{n+R-1}{n} \frac{b^{nk} e^{-b(t-an)} (t-an)^{nk+Rk-1}}{(nk+Rk-1)!},$$

with arbitrary parameters (R,k,b,a,t), are generated. For the most basic case of $(R,k) = (1,1)$ the infinite sum $S_1(1,1,b,a,t)$, in different form, has been considered by various mathematicians including Euler, Jensen and Pólya and Szegö. Their methods of representing $S_1(1,1,b,a,t)$ in closed form are different than those developed by the author; moreover the author demonstrates that $S_1(1,1,b,a,t)$ has many applications in a wide area of study including teletraffic theory, neutron behaviour, renewal processes and grazing systems. The author proves that for the general case, $S_1(R,k,b,a,t)$ may be represented in closed form which depends on k dominant zeros of an associated transcendental characteristic function.

In a similar vein, arbitrary order forced difference-delay equations are considered, from which infinite sums of the form

$$S_2(R,k,b,a,n) = \sum_{r=0}^{\infty} \binom{r+R-1}{r} \binom{n-akr}{kr+Rr-1} b^{n-akr-Rk+1},$$

with arbitrary parameters (R,k,b,a,n) are generated. It is shown that the finite form, $S_{2F}(R,k,b,a,n)$, of $S_2(R,k,b,a,n)$ is associated with Fibonacci and other related polynomials. Many functional forms of $S_{2F}(1,1,b,1,n)$ are also proved. For some special cases of the finite form, $S_{2F}(R,k,b,a,n)$ may be represented as an identity and residue theory, together with automated techniques and recurrences are employed in their proof. By the use of residue theory and induction the author proves that the infinite sum $S_2(R,k,b,a,n)$ may be represented in closed form, which depends on k zeros of an associated polynomial characteristic function. Moreover, $S_2(R,k,b,a,n)$ may be represented in hypergeometric form which in some particular instances reduce to known identities incorporating, for example, Kummer's form.

0.2 Statement.

To the best of my knowledge and belief this thesis does not contain any material previously published or written by another author, except where due reference is made in the text. This thesis has not been submitted for any other degree or diploma, at any tertiary institution.

Anthony Sofo.

[redacted]

0.3 Acknowledgments.

It is indeed a pleasure to acknowledge the assistance received from my supervisors, Associate Professor Pietro Cerone and Dr. Don Watson. I've been fortunate in being able to spend many hours of constructive discourse with them.

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0.4 Dedication.

A mia madre; lei non sa leggere né scrivere, ma la sua saggezza è infinitamente più grande della mia.

0.5 List of Tables.

0.6 List of Figures.

0.7 Summary.

This thesis deals with the problem of representation of series in closed form, mainly by the use of residue theory. Chapter one is a brief overview of some methods; residue theory, recurrences and automated procedures, that are usefully employed in this thesis. Some results given by various authors are generalized and extended. Chapter two develops the techniques, mainly residue theory, that are useful in this thesis. An identity is proved, which has previously been given by Euler and others using different methods than the authors. It it also shown that the particular identity has applications in a wide area of study. Chapter three is concerned with a proof of Bürmann's theorem and the application of the theorem to the identity obtained in chapter two. Some particular finite sums are generated in chapter four and it is proved that they may be represented in polynomial forms, moreover they are gainfully utilized in chapter five. Forced differential-delay equations of arbitrary order are considered, in chapter five, from which infinite sums of the form

$$S_1(R,k,b,a,t) = \sum_{n=0}^{\infty} \binom{n+R-1}{n} \frac{b^{nk} e^{-b(t-an)} (t-an)^{nk+Rk-1}}{(nk+Rk-1)!},$$

with arbitrary parameters (R,k,b,a,t), are generated. For the specific case of $(R,k) = (1,1)$, $S_1(1,1,b,a,t)$ reduces to the identity of chapter two. In the general case, the author proves that $S_1(R,k,b,a,t)$ may be represented in closed form which depends on k dominant zeros of an associated transcendental characteristic function.

Chapter six deals with a first order difference-delay equation, and by the use of residue theory generate infinite series which may be represented in closed form and which depend on a dominant zero of an associated characteristic function. The finite version of this series is related to Fibonacci and other special polynomials. Many functional forms of the finite series are also proved. Chapter seven deals with a generalization of a finite version of the sum obtained in chapter six, and many identities are proved by the use of recurrences and residue theory. Forced difference-delay equations of arbitrary order are considered, in chapter eight, from which infinite sums of the form

$$S_2(R,k,b,a,n) = \sum_{r=0}^{\infty} \binom{r+R-1}{r} \binom{n-akr}{kr+Rr-1} b^{n-akr-Rk+1},$$

with arbitrary parameters (R,k,b,a,n), are generated. For the special case of $(R,k) = (1,1)$, $S_2(1,1,b,a,n)$ reduces to the identity obtained in chapter six. By the use of residue theory and induction the author proves that, in general, $S_2(R,k,b,a,n)$ may be represented in closed form, which depends on k dominant zeros of an associated polynomial characteristic function. It is also shown that the infinite sums $S_2(R,k,b,a,n)$ may be represented in hypergeometric form and in some particular instances of parameter values, Kummer and other identities may be recovered.

Chapter 1

A review of methods for closed form summation

This chapter consists of two sections. The first section 1.1, is a brief overview of some methods, basically ones dealing with residue theory, which are useful for the summation of series and their representation in closed form. Some results given by various authors are generalized and extended.

In the second section 1.2, a particular tree search sum, with some variations is considered. A number of techniques are utilized, recurrences and automated procedures, that are useful in determining its closed form representation. Some related results, which the author believes to be new, are also presented.

1.1 Some Methods

1.1.1 Introduction.

Identities play an important role in mathematics and have been a source of inspiration and sweat for many mathematicians over a long period of time. Jacques Bernoulli (1654-1705), a contemporary of Newton (1642-1722), and Leibniz (1646-1716) discovered the sum of several infinite series in closed form, but did not succeed in finding, in closed form, the sum of the reciprocals of the squares

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

"If somebody should succeed", wrote Bernoulli, "in finding what till now withstood our efforts and communicate it to us, we shall be obliged to him". The problem came to the attention of Euler (1707-1783). He found various expressions for the desired sum, definite integrals and other representations, none of which satisfied him. He used the integral representation to compute the sum, S, numerically to seven places, yet this is only an approximate value, his goal was to find an exact value. Euler succeeded, eventually in writing

$$S = \frac{\pi^2}{6}. \tag{1.1}$$

Euler [43], moreover wrote *"There are many properties of numbers with which we are well acquainted, but which we are not yet able to prove; only observations have led us to their knowledge. Hence we see that in the theory of numbers, which is still very imperfect, we can place our highest hopes in observations; they will lead us continually to new properties which we shall endeavour to prove afterwards. The kind of knowledge which is supported by observations and is not yet proved must be carefully distinguished from truth; it is gained by induction as we usually say. Yet we have seen cases in which mere induction led to error. Therefore, we shall take great care not to accept as true such properties of numbers which we have discovered by observation and which are supported by induction alone. Indeed, we shall use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful".* One may imagine the excitement and sense of achievement when Pythagoras (c.580 B.C.-c.500 B.C.) first wrote that for a right angle triangle

$$a^2 = b^2 + c^2,$$

Koecher [69] gave

$$\zeta(5) = \frac{5}{2}\sum_{j=1}^{\infty} \frac{(-1)^j}{j^3 \binom{2j}{j}} \sum_{k=1}^{j-1} \frac{1}{k^2} - 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j^5 \binom{2j}{j}},$$

Ramanujan, see Berggren [8], evaluated

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{j=0}^{\infty} \frac{(4j)!(1103 + 26390j)}{(j!)^4 396^{4j}},$$

Bailey [4] and coworkers wrote, (originally given by Plouffe [75])

$$\pi = \sum_{j=0}^{\infty} \frac{1}{16^j}\left[\frac{4}{8j+1} - \frac{2}{8j+4} - \frac{1}{8j+5} - \frac{1}{8j+6}\right],$$

Amdeberhan and Zeilberger [2] published

$$\zeta(3) = \sum_{j=0}^{\infty} \frac{(-1)^j (j!)^{10}(205j^2 + 250j + 77)}{64((2j+1)!)^5},$$

and Clausen [29] gave the result

$$\left\{ {}_2F_1 \left[\begin{array}{c} a, b \\ a+b+1/2 \end{array} \middle| x \right] \right\}^2 = {}_3F_2 \left[\begin{array}{c} 2a, 2b, a+b \\ a+b+1/2, 2a+2b \end{array} \middle| x \right],$$

where ${}_pF_q$ is the hypergeometric function. Some of the main techniques, dealing mainly with residue theory, in the investigation of the representation of series in closed form are now discussed.

1.1.2 Contour Integration.

Residue theory and contour integration can be gainfully employed to express certain sums in closed form. From

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_j \mathrm{Res}_j (\pi \csc \pi z\, f(z))$$

where Res_j are the residues at the poles of $f(z)$, we may obtain some classical results, namely

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2a^2}\left[\pi a \coth \pi a - 1\right] \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)^2} = \frac{1}{4a^2}\left[\frac{\pi}{a}\coth \pi a + (\pi \mathrm{cosech} \pi a)^2 - \frac{2}{a^2}\right]. \tag{1.3}$$

The residue evaluation of the integral

$$\frac{1}{2\pi i}\oint \frac{\pi}{\sin \pi z\,(z^2+1)} dz$$

leads to the alternating sign identity

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right)$$

and also we may obtain

$$\sum_{n=1}^{\infty} \frac{1}{\sinh^2 n\pi} = \frac{\pi-3}{6\pi}.$$

Flajolet and Salvy [45] apply contour integral methods to obtain some Euler sums, in particular they recover the alternating term identity, without the use of residue theory

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{5\pi^5}{1536}.$$

For another function of the form $f(z) = \frac{\pi \coth \pi z}{z^q}$ they recover some results, one of which was originally given by Ramanujan

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56700}.$$

The strength of the Flajolet Salvy paper is that it exposues a method and shows many connections of identities with the logarithmic derivative of the Gamma function $\psi(z)$, the zeta function, harmonic numbers and double infinite sums. The results (1.2) and (1.3) are also obtained and extended by Cerone [23] using different methods. The method is described as follows.

1.1.3 Cerone's method and extension.

Cerone [23] considers an integral equation of the form

$$B(t) = \frac{\phi(x+t)}{l(x)} + \int_0^t B(t-u)\,\phi(u)\,du \tag{1.4}$$

where $B(t)$ is a single sex deterministic model representing births at time t, $\phi(t)$ is a net maternity function which is of compact support and $l(x)$ is the survivor function which gives the probability of surviving to age x of a newborn. The Inverse Laplace transform of (1.4) is

$$B_x(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt} V(p,x)}{1 - \Phi(p)} dp \tag{1.5}$$

where

$$V(p,x) = \frac{e^{px} \int_x^{\infty} e^{-pu} du}{l(x)} = \frac{v(p,x)}{l(x)}$$

and $\Phi(p)$ is the Laplace transform of $\phi(x)$. Assuming that $\Phi(p) = 1$ has simple roots, p_j, which are the only poles in (1.5) then

$$B_x(t) = \sum_j \frac{V(p_j, x)\, e^{p_j t}}{\mu_j}, t > 0$$

where

$$\mu_j = -\left[\frac{d\Phi(p)}{dp}\right]_{p=p_j} = \int_0^\infty e^{-up_j}\phi(u)\,du. \tag{1.6}$$

By allowing $\phi(t)$ to be exponentially constrained Cerone shows that

$$S_0 = \sum_j \frac{1}{\mu_j} = \frac{\phi(0+)}{2},$$

$$S_1 = \sum_j \frac{1}{p_j\mu_j} = \frac{M_0+1}{2(M_0-1)} \tag{1.7}$$

and, in general

$$S_n = \sum_j \frac{1}{p_j^n\mu_j}$$

satisfies the recurrence relation

$$(1-M_0)S_n = \sum_{k=2}^{n-1} \frac{(-1)^{n+k}M_{n-k}S_k}{(n-k)!} + \frac{(-1)^n M_{n-1}}{(n-1)!(1-M_0)}, n = 2,3,4,..., \tag{1.8}$$

where

$$M_n = \int_0^\infty u^n\phi(u)\,du < \infty$$

are the n^{th} moments of $\phi(t)$. Now, in particular if $\phi(x) = c\delta(x-b)$ with c, b constants and $\delta(x)$ is the Dirac delta function, then $\Phi(p) = ce^{-bp}$ and $M_n = cb^n$. The roots of the characteristic equation $\Phi(p) = 1$ are given explicitly as

$$p_j = \frac{\ln c - 2\pi ij}{b}, j = 0, \pm1, \pm2, ...$$

and, from (1.6) $\mu_j = b$. Using (1.7) with $a = \frac{\ln c}{2\pi}$ gives, after some simplification, the result (1.2). Other identities, similar to (1.2), may be evaluated from (1.8) for n=2,3,4,..., or indeed by differentiating (1.2) with respect to the parameter a. The following two points are certainly worthy of mention. Firstly, replacing a with ia in (1.2) gives the result

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a^2} \left[1 - \pi a \cot \pi a\right]. \tag{1.9}$$

By considering a partial fraction decomposition, such as

$$\sum_{n=1}^{\infty} \frac{1}{n^4 - a^4} = \frac{1}{2a^2} \left[\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} - \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}\right]$$

we may obtain, by the use of (1.2) and (1.9), other identities of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^4 - a^4} = \frac{1}{4a^4} \left[2 - \pi a \left(\cot \pi a + \coth \pi a\right)\right]. \tag{1.10}$$

Taking the limit as $a \to 0$, in (1.10), confirms the result

$$\sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{90}.$$

Secondly, (1.2) may also be integrated with respect to the parameter a. From (1.2)

$$\sum_{n=1}^{\infty} \left(\frac{2a^2/n^2}{1 + a^2/n^2}\right) = \frac{d}{da} \left[\ln \left\{\frac{\sinh a\pi}{a\pi}\right\}\right]. \tag{1.11}$$

Integrating both sides of (1.11) with respect to the parameter a and interchanging sum and integral results in

$$\sum_{n=1}^{\infty} \ln \left(1 + \frac{a^2}{n^2}\right) = \ln \left(\frac{\sinh a\pi}{a\pi}\right), \tag{1.12}$$

where the constant of integration in (1.12) is identically zero. The identity (1.12) is also obtained by Wheelon [91] using a different technique. Notice that the left hand side of (1.12) may be rewritten such that

$$\prod_{n=1}^{\infty} \left(1 + \frac{a^2}{n^2}\right) = \frac{\sinh a\pi}{a\pi}.$$

The summation of zeros of other transcendental functions have also been considered by several other authors. For instance, Lord Raleigh [79], obtained $\sum_{j=1}^{\infty} m_j^{-4} = \frac{1}{12}$ and $\sum_{j=1}^{\infty} m_j^{-8} = \frac{33}{35(12)^2}$

where the $m_j^{'s}$ are the zeros of the frequency function

$$g(m) = \cos m_j \cosh m_j + 1 \tag{1.13}$$

A Taylor series expansion of (1.13) is

$$g(m) = 2 - \frac{m^4}{6} + \frac{2m^8}{7!} - \frac{16m^{12}}{3.11!} + \frac{16m^{16}}{15!} - \frac{16^2 m^{20}}{5.19!} + \frac{16^3 m^{24}}{24!} - \ldots$$

and since $g(m)$ is an even function in m then $-m_j$ and $\pm i m_j$ are also zeros of (1.13). If we write

$$S(a) = \sum_{j=1}^{\infty} m_j^{-a} = \frac{1}{2\pi i} \oint \frac{g'(z) z^{-a} dz}{g(z)}, a > 1 \tag{1.14}$$

and choosing $a = 4$ and $a = 8$ in (1.14) we recover the two results of Lord Raleigh. From residue calculations and (1.14) we may also give, for example $\sum_{j=1}^{\infty} m_j^{-12} = \frac{2641}{34650(12)^2}$ and $\sum_{j=1}^{\infty} m_j^{-16} = \frac{12343}{2002000(12)^2}$. Another operational technique for summing series is that which is described by Wheelon and is worthy of a mention here, since we can generalize some of his results and also make a connection with the polygamma functions, $\psi(x)$.

1.1.4 Wheelon's results.

Wheelon's method is based on the parametric representation of the general term of a series, so as to produce either the geometric or exponential series inside one or more integral signs. The fundamental operation is contained in the summation of both sides of a Laplace transform pair with respect to a transform variable which is interpreted as the dummy index of summation. This operation exhibits the desired sum as an integral of the geometric or exponential series each of which may be summed in closed form. Consider the Laplace transform of a function $f(x)$,

$$F(p) = \int_0^{\infty} e^{-xp} f(x) \, dx$$

and if we identify the transform variable p with a dummy index of summation n, we can write

$$\sum F(n) = \int_0^\infty \sum \left(e^{-x}\right)^n f(x)\, dx. \tag{1.15}$$

As an illustration choosing $f(x) = x$ in (1.15), leads to Euler's result (1.1). An obvious extension is that (1.15) may be generalized to

$$\sum \frac{F(n)}{(n+a)^k} = \frac{1}{(k-1)!} \int_0^\infty x^{k-1} e^{-ax} \sum \left(e^{-x}\right)^n f(x)\, dx.$$

The integral representation of (1.15) may be so chosen to allow for denominators with rational and irrational algebraic functions and linear factors, and the numerator may be so chosen to allow for algebraic, exponential, trigonometric, inverse trigonometric, logarithmic, Bessel and Legendre functions. The convolution theorem may be beneficially exploited, so that we may write, for $j \geq 2$

$$\frac{1}{\alpha_1 \alpha_2 \alpha_3 ... \alpha_j} = (j-1)! \underbrace{\int_0^1 du \int_0^u dv ... \int_0^v}_{(j-1)\ times} \frac{dw}{\left[\alpha_1 (1-u) + \alpha_2 (u-v) + \alpha_3 (v-w) ... + \alpha_j w\right]^j}$$

and using the relation

$$\frac{1}{\lambda_j} = \frac{1}{(j-1)!} \int_0^\infty s^{j-1} e^{-\lambda s} ds$$

allows a generalization of Wheelon's result as

$$S(a,j) = \sum_{n=0}^{\infty} \frac{1}{\prod_{k=1}^{j} (an+k)} = \frac{1}{(j-1)!} \int_{x=0}^{1} \frac{(1-x)^{j-1}}{1-x^a} dx \tag{1.16}$$

$$= \frac{1}{j!}\, {}_{j+1}F_j \left[\left. \begin{array}{c} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, ..., \frac{j}{a} \\ \frac{1+a}{a}, \frac{2+a}{a}, \frac{3+a}{a}, ..., \frac{j+a}{a} \end{array} \right| 1 \right] \tag{1.17}$$

for $a \in \Re$ and $j = 2, 3, 4, \ldots$. From(1.16) we can see that

$$j \int_{x=0}^{1} \frac{(1-x)^{j-1}}{1-x^a} dx = {}_{j+1}F_j \left[\begin{array}{c} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{j}{a} \\ \frac{1+a}{a}, \frac{2+a}{a}, \frac{3+a}{a}, \ldots \frac{j+a}{a} \end{array} \middle| 1 \right].$$

Also, for a and j integers ≥ 1 we have, because of symmetry and known properties of the hypergeometric function

$${}_{j+1}F_j \left[\begin{array}{c} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{j}{a} \\ \frac{1+a}{a}, \frac{2+a}{a}, \frac{3+a}{a}, \ldots, \frac{j+a}{a} \end{array} \middle| 1 \right] = {}_{a+1}F_a \left[\begin{array}{c} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a} \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a} \end{array} \middle| 1 \right].$$

For specific values of a and j various listings of (1.16) occur in the works of Jolley [64], Hansen [54] and Gradshteyn and Ryzhik [47]. We may also obtain some other interesting cases as follows. From (1.16)

$$S(1,j) = \frac{1}{(j-1)(j-1)!} = \frac{1}{j!} {}_{j+1}F_j \left[\begin{array}{c} 1, 1, 2, 3, \ldots, j \\ 2, 3, 4, \ldots, j+1 \end{array} \middle| 1 \right]$$

and we have the identity, from Gauss's ${}_2F_1$ summation

$${}_2F_1 \left[\begin{array}{c} 1, 1 \\ j+1 \end{array} \middle| 1 \right] = \frac{j}{j-1}; \quad j \geq 2.$$

For $a = 2$, and from(1.16)

$$S(2,j) = \frac{1}{(j-1)!} \left[2^{j-2} \ln 2 + \sum_{r=1}^{j-2} (-1)^r \binom{j-2}{r} \frac{2^{j-2-r}(2^r-1)}{r} \right]$$

$$= \frac{1}{j!} {}_{j+1}F_j \left[\begin{array}{c} 1, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{j}{2} \\ \frac{3}{2}, 2, \ldots, \frac{2+j}{2} \end{array} \middle| 1 \right], \text{ and hence}$$

$${}_3F_2 \left[\begin{array}{c} \frac{1}{2}, 1, 1 \\ \frac{1+j}{2}, \frac{2+j}{2} \end{array} \middle| 1 \right] = \frac{j2^j}{4} \left[\ln 2 + \sum_{r=1}^{j-2} (-1)^r \binom{j-2}{r} \frac{1-2^{-r}}{r} \right].$$

Other specific values of (1.16) may be obtained as follows

$$S(6,12) = \frac{1}{15\,(12!)}\left[61440\ln 2 + 10935\ln 3 + 1225\pi\sqrt{3} - 61251\right]$$

$$= \frac{1}{12!}\,{}_7F_6\left[\begin{array}{c} 1,\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},1 \\ \frac{13}{6},\frac{7}{3},\frac{15}{6},\frac{8}{3},\frac{17}{6},3 \end{array}\middle| 1\right]$$

and hence

$$15\,{}_7F_6\left[\begin{array}{c} 1,\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},1 \\ \frac{13}{6},\frac{7}{3},\frac{15}{6},\frac{8}{3},\frac{17}{6},3 \end{array}\middle| 1\right] = 61440\ln 2 + 10935\ln 3 + 1225\pi\sqrt{3} - 61251.$$

Non integer values of a may also be considered and hence (1.16) may be related to the polygamma functions. The following two examples are given; $S\left(\frac{1}{2},8\right) = \frac{135728}{91(11!)}$ and from (1.17) we have

$${}_9F_8\left[\begin{array}{c} 1,2,4,6,8,10,12,14,16 \\ 3,5,7,9,11,13,15,17 \end{array}\middle| 1\right] = \frac{67864}{45045},$$

also

$$S\left(\frac{3}{2},5\right) = \frac{1}{2800}\left(3455 - 560\pi\sqrt{3} - 700\ln 3 + 126\,{}_3F_2\left[\begin{array}{c} \frac{5}{3},2,\frac{10}{3} \\ \frac{11}{3},\frac{13}{3} \end{array}\middle| 1\right] - 36\,{}_3F_2\left[\begin{array}{c} 2,\frac{7}{3},\frac{8}{3} \\ \frac{11}{3},\frac{13}{3} \end{array}\middle| 1\right]\right)$$

and again from (1.17) we have the identity

$$\frac{70}{3}\,{}_6F_5\left[\begin{array}{c} 1,\frac{2}{3},\frac{4}{3},2,\frac{8}{3},\frac{10}{3} \\ \frac{5}{3},\frac{7}{3},3,\frac{11}{3},\frac{13}{3} \end{array}\middle| 1\right] + 36\,{}_3F_2\left[\begin{array}{c} 2,\frac{7}{3},\frac{8}{3} \\ \frac{11}{3},\frac{13}{3} \end{array}\middle| 1\right] - 126\,{}_3F_2\left[\begin{array}{c} \frac{5}{3},2,\frac{10}{3} \\ \frac{11}{3},\frac{13}{3} \end{array}\middle| 1\right]$$

$$= 3455 - 560\pi\sqrt{3} - 700\ln 3.$$

Numerical estimates of the integral (1.16) may be determined for those values of a and j which do not permit an analytical solution of the integral;

$$S(.1,9) \sim .00001315 = \frac{1}{9!}\,{}_{10}F_9\left[\begin{array}{c} 1,10,20,30,40,50,60,70,80,90 \\ 11,21,31,41,51,61,71,81,91 \end{array}\middle| 1\right].$$

Using this procedure Wheelon also sums the double infinite series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(4n-1)^{2m+1}}=\frac{\pi}{8}-\frac{1}{2}\ln 2, \text{ and}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{nm\,(-1)^{n+m}}{(n+m)^2}=\frac{1}{6}\left(\ln 2-\frac{1}{4}\right)$$

both of which agree with the results obtained by Bromwich [19]. A similar summation procedure, to that given by Wheelon, has been developed by MacFarlane [71], which depends upon the properties of the Fourier-Mellin transformation. From Wheelon's work, we may now see a connection with (1.16) and the polygamma functions, $\psi(x)$. From (1.16), let $j=2$ and $a=\frac{1}{2k}$, $k\in N$ in which case we may write, by partial fraction decomposition

$$S_k=\sum_{n=0}^{\infty}\frac{4k^2}{(n+2k)(n+4k)}=2k\sum_{r=0}^{2k-1}\frac{1}{r+2k}$$

from which, we obtain the very slow converging series

$$S_k=2k\left\{\psi(4k)-\psi(2k)\right\}.$$

If

$$S=\sum_{n=1}^{\infty}\frac{1}{n^2(4n^2+1)}$$

we may use partial fraction decomposition with polygamma functions, so that

$$\begin{aligned} S &= 8\psi(1)-8\psi\left(\frac{5}{4}\right)+\psi'(1)+\psi'\left(\frac{5}{4}\right) \\ &= 2+\zeta(2)-\pi\coth\frac{\pi}{2}. \end{aligned}$$

A great deal of exciting work has also recently been carried out by Borwein and his coworkers [13] on symbolically discovered identities with special and other functions. Flajolet and Salvy [45], by the use of residue theory also obtain identities involving special functions. Other transform techniques also provide a rich source of possibilities for investigating sums which may be represented in closed form; Z transform techniques are widely used and a general method may be seen in the books of Jury [66] and Vich [89].

1.1.5 Hypergeometric functions.

Binomial sums and hypergeometric functions are intrinsically related. It is of fundamental importance that binomial sums can be generally written as a terminating hypergeometric series, see Roy [80]. The book, A=B, by Petkovšek, Wilf and Zeilberger [74] expertly expounds the theory of hypergeometric closed form representation of binomial sums. The following is therefore a brief description of the hypergeometric function and some of it's prominent properties. The books of Bailey [6], Slater [82] and Gaspar and Rahman [46] cover all of the material presented here.

If the ratio of two consecutive terms T_{k+1}/T_k , in a series, is a constant, then we have a geometric series. A hypergeometric series arises when the ratio is a rational function of a positive integer k,

$$\frac{T_{k+1}}{T_k} = \frac{(a_1+k)\dots(a_p+k)\,z}{(b_1+k)\dots(b_q+k)\,(1+k)} \tag{1.18}$$

where $a_1, ..., a_p; b_1, ..., b_q$ and z are complex and $T_0 = 1$. Pochhammer's function is defined as

$$\left.\begin{aligned} (a)_0 &= 1 \\ (a)_k &= a\,(a+1)\dots(a+k-1) = \tfrac{\Gamma(a+k)}{\Gamma(a)} \end{aligned}\right\} \tag{1.19}$$

and hence a hypergeometric series may be written as

$${}_pF_q\left[\begin{array}{c|c} a_1, a_2, ..., a_p & z \\ b_1, b_2, ..., b_q & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k\, z^k}{(b_1)_k \dots (b_q)_k\, k!}. \tag{1.20}$$

The hypergeometric series (1.20) is symmetric both in its upper parameters $a_1, ..., a_p$ and its lower parameters $b_1, ..., b_q$. In general it is required that $b_1, ..., b_q \notin 0, -1, -2, ...$, since otherwise the denominators in the series will eventually become zero. If for some $j, a_j = -n$ then all terms with $k > n$ will vanish, so that the series will terminate. In the non-terminating case, the ratio test yields the radius of convergence, which is infinite for $p < q+1, 1$ for $p = q+1$ and 0 for $p > q+1$. Moreover, if $p = q+1$ then there will be absolute convergence for $|z| = 1$ if $\mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$. Hypergeometric functions play an important role in many fields of pure and applied mathematics as well as in science. The excellent survey paper of Andrews [3] puts basic hypergeometric functions in an applicable setting. More recently hypergeometric functions led to the solution of the long standing problem of the Bieberbach conjecture by deBranges [39]; which shows that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a normalized univalent analytic function in the unit disc, then for each $n \geq 2$ one has $|a_n| \leq n$. Some elementary cases of hypergeometric series are

$$ {}_0F_0\left[\begin{matrix} - \\ - \end{matrix}\middle| z\right] = e^z, \text{ and} $$

$$ {}_1F_0\left[\begin{matrix} -a \\ - \end{matrix}\middle| z\right] = (1-z)^a . $$

Bessel functions may be expressed in the form

$$ \left(\frac{z}{2}\right)^{\alpha} {}_0F_1\left[\begin{matrix} - \\ \alpha+1 \end{matrix}\middle| -\frac{z^2}{4}\right] = \Gamma(\alpha+1)\, J_{\alpha}(z) $$

and the ${}_2F_1$ series is the classical Gauss series with Gegenbauer, Chebyshev, Legendre and Jacobi polynomials as terminating cases. It is well known in the theory of hypergeometric functions that the confluent ${}_1F_1$ function can be obtained from the Gaussian ${}_2F_1$ function by a limit process called confluence. The hypergeometric function

$$ {}_{q+1}F_q\left[\begin{matrix} a_0, a_1, ..., a_q \\ b_1, b_2, ..., b_q \end{matrix}\middle| z\right] \tag{1.21} $$

is called $k-$ balanced if $z = 1$ and $k + a_0 + a_1 + ... + a_q = b_1 + b_2 + ... + b_q$; or just balanced if $k = 1$; well-poised if $1 + a_0 = a_1 + b_1 = ... = a_q + b_q$, and very well-poised if it is well-poised and $a_1 = 1 + \frac{a_0}{2}$. There are a number of cases where (1.21) with argument $z = \pm 1$ can be evaluated in closed form as a quotient of products of Gamma functions. Five of these cases are:

1. the Gauss summation formula

2. Kummer summation formula

3. the balanced Pfaff-Saalschütz summation formula

4. the well-poised Dixon summation formula, and

5. the 2-balanced and very well-poised Dougall summation formula.

The Gauss summation formula is a limit of the Pfaff-Saalschütz summation formula, Kummer's formula is a limit of Dixon's formula and may also be obtained from Dougall's formula. The Pfaff-Saalschütz summation formula can be explicitly written as

$$\sum_{k=0}^{n} \frac{(a)_k\,(b)_k\,(-n)_k}{(c)_k\,(1+a+b-c-n)_k\,k!} = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n},$$

in particular if $c = a + b + 1$ we have

$$\sum_{k=0}^{n} \frac{(a)_k\,(b)_k}{(1+a+b)_k\,k!} = \frac{(1+a)_n\,(1+b)_n}{(1+a+b)_n\,n!}.$$

Hypergeometric sums are often met in the form of combinatorial sums with binomial coefficients. Evidently, one hypergeometric sum may have many representations as a sum with binomial coefficients. Saalschütz's summation, for example, may be written as

$$\sum_{k=0}^{n} \frac{\binom{a+k-1}{k}\binom{c-a-b+n-k-1}{c-a-b-1}}{\binom{c+k-1}{c-b}} = \frac{(c-b)\binom{c-a+n-1}{n}}{b\binom{c+n-1}{b}}.$$

In the next section, rather than detail the theory and practice of summation of binomial series in closed form, we will consider a particular sum, with some variations, and investigate its solution through various procedures, including the automated approaches described by Petkovšek, Wilf and Zeilberger [74].

1.2 A tree search sum and some relations.

1.2.1 Binomial summation.

The sum, with some variations and relations, which we shall explore in detail, arises in the work of Jonassen and Knuth [65] in an algorithm known as tree search and insertion. In particular the sum is

$$f_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{2}\right)^k \binom{2k}{k}. \tag{1.22}$$

We shall explore (1.22) and survey several methods of finding a closed form solution. We shall compare the analytical techniques of Riordan, Jonassen and Knuth, Gessel, Rousseau, the hypergeometric connection, the generatingfunctionology method of Wilf and the automated approaches of Sister Celine, Zeilberger and the WZ pairs method.

1.2.2 Riordan.

Under the heading of Inverse Relations, Riordan [81] considers the identities

$$f_{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \left(\frac{-1}{2}\right)^k \binom{2k}{k} = 2^{-2n} \binom{2n}{n} \text{ and} \tag{1.23}$$

$$f_{2n+1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} \left(\frac{-1}{2}\right)^k \binom{2k}{k} = 0. \tag{1.24}$$

Riordan analyses (1.23) and (1.24) by recurrences. Writing $g_n = 2^{-n} \binom{2n}{n}$, then

$$f_{2n} = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k g_k = 2^{-n} g_n \text{ and}$$

$$f_{2n+1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k g_k = 0.$$

Now $g_n = \left(2 - \frac{1}{n}\right) g_{n-1}$ and also

$$
\begin{aligned}
f_{2n} &= f_{n-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k+1} g_{k+1}, \\
n\left(f_n + f_{n-1}\right) &= \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^k g_k := h_n.
\end{aligned}
$$

Hence

$$n\left(f_n + f_{n-1}\right) = h_{n-1} + f_{n-1} = nf_{n-1} + (n-1) f_{n-2}, \text{ and therefore}$$

$$nf_n = (n-1) f_{n-2}, f_0 = 1, f_1 = 0. \tag{1.25}$$

From (1.25), we have $f_{2n+1} = 0$ and

$$
\begin{aligned}
f_{2n} &= \frac{2n-1}{2n}\frac{2n-3}{2n-2}\frac{2n-5}{2n-4}\cdots\frac{1}{2} = (-1)^n \binom{-1/2}{n} = \binom{n-1/2}{n} \qquad (1.26) \\
&= 2^{-2n}\binom{2n}{n} = 2^{-n} g_n = \frac{(2n)!}{2^{2n} n!^2} = \prod_{j=0}^{n-1}\left(1 - \frac{1}{2n-2j}\right).
\end{aligned}
$$

Riordan expands on these ideas and obtains the additional identities

$$\sum_{k=0}^{2n-1} \binom{2n}{k+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k} = \frac{n}{2^{2n+1}} \binom{2n}{n} \quad \text{and}$$

$$\sum_{k=0}^{2n} \binom{2n+1}{k+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k} = \frac{2n+1}{2^{2n}} \binom{2n}{n}.$$

Riordan attributes the identities (1.23) and (1.24) to Reed Dawson. Another interesting identity related to (1.23) and which may be evaluated by inverse pair relations is

$$\sum_{k=0}^{n} \frac{(-4)^k \binom{n}{k}}{\binom{2k}{k}} = \frac{1}{1-2n}. \tag{1.27}$$

1.2.3 Method of Jonassen and Knuth.

Jonassen and Knuth [65] consider (1.22) and by algebraic manipulations obtain the recurrence (1.25) as follows. From (1.22)

$$\begin{aligned} f_n &= f_{n-1} + \sum_{k=0}^{n} \binom{n-1}{k-1} \left(\frac{-1}{2}\right)^k \binom{2k}{k} \\ &= f_{n-1} - \sum_{k=0}^{n} \binom{n}{k+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k} \frac{2k+1}{n} \\ &= f_{n-1} + \sum_{k=0}^{n} \binom{n}{k+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k} \left\{\frac{1}{n} - \frac{2k+2}{n}\right\} \\ &= f_{n-1} - 2f_{n-1} + \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k}, \end{aligned}$$

hence

$$n\left(f_n + f_{n-1}\right) = \sum_{k=0}^{n} \binom{n}{k+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k}. \tag{1.28}$$

Replacing n with $n-1$ in (1.28) we get

$$(n-1)\left(f_{n-1} + f_{n-2}\right) = \sum_{k=0}^{n} \binom{n}{k+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k} - f_{n-1}. \tag{1.29}$$

Subtracting (1.29) from (1.28) we obtain the recurrence relation (1.25) and hence identity (1.26) follows.

1.2.4 Method of Gessel.

This method is given on page 3 of the Greene and Knuth [50] book and is described as follows. Replace k with $n-k$, that is change the order of summation, in (1.22) such that

$$f_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{2}\right)^{n-k} \binom{2n-2k}{n-k}.$$

Let $[x^n] f(x)$ denote the coefficient of x^n in $f(x)$, hence

$$[x^n](1-2x)^n = \binom{n}{k}(-2)^k$$

$$\left[y^{n-k}\right](1+y)^{2n-2k} = \binom{2n-2k}{n-k}(-2)^k = [y^n]\, y^k (1+y)^{2n-2k}$$

and therefore

$$f_n = \left(\frac{-1}{2}\right)^n [y^n]\, y^k (1+y)^{2n} \sum_{k=0}^{n} \left[x^k\right](1-2x)^n \left(\frac{y}{(1+y)^2}\right)^k .$$

But since

$$\sum_{k=0}^{n} \left[x^k\right] f(x)\, g(y)^k = f(g(y))$$

when $f(x)$ is analytic, then

$$\begin{aligned} f_n &= (-2)^{-n} [y^n](1+y)^{2n} \left(1 - \frac{2y}{(1+y)^2}\right)^n \\ & \quad (-2)^{-n} [y^n]\left(1+y^2\right)^n , \end{aligned}$$

and the solution follows

$$f_n = \begin{cases} 2^{-n} \dbinom{n}{n/2} , \text{ for } n \text{ even} \\ 0 \text{ , for } n \text{ odd.} \end{cases} \tag{1.30}$$

1.2.5 Method of Rousseau.

This method is also described in the book of Greene and Knuth [50] and essentially it identifies the coefficient in a polynomial expansion. From

$$[x^0]\left(x + \frac{1}{x}\right)^{2k} = \binom{2k}{k}$$

$$\left(1-\frac{\left(x+\frac{1}{x}\right)^2}{2}\right)^n=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{-1}{2}\right)^k\left(x+\frac{1}{x}\right)^{2k} \text{ and}$$

$$f_n=\left[x^0\right]\left(x^2+\frac{1}{x^2}\right)^k=\left[x^0\right]\left(1-\frac{\left(x+\frac{1}{x}\right)^2}{2}\right)^n, \text{ hence}$$

$$f_n=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{-1}{2}\right)^k\binom{2k}{k}.$$

1.2.6 Hypergeometric form.

Here we consider a slightly more general version of (1.22) in terms of hypergeometric notation. Let

$$f_n(a,b)=\sum_{k=0}^{n}\binom{n}{k}(-a)^k\binom{bk}{k}=\sum_{k=0}^{n}T_k \tag{1.31}$$

for a real and b integer. The ratio of consecutive terms is

$$\frac{T_{k+1}}{T_k}=\frac{ab^b(k-n)}{(b-1)^{b-1}(k+1)^2}\frac{\prod_{j=1}^{b-1}\left(k+\frac{b-j}{b}\right)}{\prod_{j=2}^{b-1}\left(k+\frac{b-j}{b-1}\right)}, \tag{1.32}$$

$T_0=1$, and hence from (1.32)

$$f_n(a,b)={}_bF_{b-1}\left[\begin{matrix}\frac{b-1}{b},\frac{b-2}{b},\frac{b-3}{b},\ldots,\frac{1}{b},-n\\ 1,\frac{b-2}{b-1},\frac{b-3}{b-1},\ldots,\frac{1}{b-1}\end{matrix}\middle|\frac{ab^b}{(b-1)^{b-1}}\right], \tag{1.33}$$

moreover, for the relatively simple case of $b=1$

$$f_n(a,1)={}_1F_0\left[\begin{matrix}-n\\ -\end{matrix}\middle|a\right]=(1-a)^n.$$

Now, we concentrate on the case of $b=2$; from (1.33)

$$f_n(a,2)={}_2F_1\left[\begin{matrix}\frac{1}{2},-n\\ 1\end{matrix}\middle|4a\right] \tag{1.34}$$

and a recurrence relation for (1.34) obtained from the Zb algorithm in Mathematica, is

$$\left.\begin{array}{c}(n+2)\,f_{n+2}+(2n+3)\,(2a-1)\,f_{n+1}+(n+1)\,(1-4a)\,f_n=0,\\ f_0\,(a,2)=1,\,f_1\,(a,2)=1-2a.\end{array}\right\} \tag{1.35}$$

We can see from (1.35) that for two special cases of $a=1/2$ and $a=1/4$ the recurrence relation (1.35) becomes manageable. From (1.34) let $a=1/2$ such that

$$f_n\left(\frac{1}{2},2\right)={}_2F_1\left[\begin{array}{c}\frac{1}{2},-n\\ 1\end{array}\middle|\,2\right] \tag{1.36}$$

and replacing k with $n-k$ we have

$$f_n\left(\frac{1}{2},2\right)=T_0\,{}_2F_1\left[\begin{array}{c}-n,-n\\ \frac{1}{2}-n\end{array}\middle|\,\frac{1}{2}\right],T_0=\left(\frac{-1}{2}\right)^n\binom{2n}{n}. \tag{1.37}$$

There is an identity, due to Gauss, see Graham, Knuth and Patashnik [49], which states

$${}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_2\\ \alpha_1+\alpha_2+1\end{array}\middle|\,1\right]={}_2F_1\left[\begin{array}{c}2\alpha_1,2\alpha_2\\ \alpha_1+\alpha_2+\frac{1}{2}\end{array}\middle|\,\frac{1}{2}\right], \tag{1.38}$$

hence from (1.38) and (1.37)

$$f_n=\left(\frac{-1}{2}\right)^n\binom{2n}{n}\,{}_2F_1\left[\begin{array}{c}\frac{-n}{2},\frac{-n}{2}\\ \frac{1}{2}-n\end{array}\middle|\,1\right]. \tag{1.39}$$

Similarly by Pfaff's reflection law

$${}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_3-\alpha_2\\ \alpha_3\end{array}\middle|\,\frac{1}{2}\right]=2^{\alpha_1}\,{}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_2\\ \alpha_3\end{array}\middle|\,-1\right]$$

we have from (1.37)

$${}_2F_1\left[\begin{array}{c}-n,-n\\ \frac{1}{2}-n\end{array}\middle|\,\frac{1}{2}\right]=2^{-n}\,{}_2F_1\left[\begin{array}{c}-n,\frac{1}{2}\\ \frac{1}{2}-n\end{array}\middle|\,-1\right].$$

Using the classical Gauss formula

$$ {}_2F_1\left[\begin{array}{c} \alpha_1, \alpha_2 \\ \alpha_3 \end{array} \middle| 1\right] = \frac{\Gamma(\alpha_3)\,\Gamma(\alpha_3-\alpha_1-\alpha_2)}{\Gamma(\alpha_3-\alpha_2)\,\Gamma(\alpha_3-\alpha_1)} $$

we obtain from (1.39)

$$ f_n = \left(\frac{-1}{2}\right)^n \binom{2n}{n} \frac{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{2}-\frac{n}{2}\right)} \tag{1.40} $$

such that when n is odd $f_n = 0$ and when n is even

$$ f_{2n} = \binom{4n}{2n} \frac{1}{4^n} \left(\frac{(2n)!}{2^n n!}\right)^2 \frac{4^n\,(2n)!}{(4n)!} = 2^{-2n} \binom{2n}{n}. $$

Also, from (1.35) for $a = 1/2$ we have that $(n+2)\,f_{n+2} - (n+1)\,f_n = 0$ which is identical to (1.25) and hence the Reed Dawson identity follows. For $a = 1/4$, from (1.34)

$$ f_n\left(\frac{1}{4}, 2\right) = {}_2F_1\left[\begin{array}{c} \frac{1}{2}, -n \\ 1 \end{array} \middle| 1\right], $$

and from (1.35) $(n+2)\,f_{n+2} - \frac{1}{2}(2n+3)\,f_{n+1} = 0$, hence

$$ f_n = 2^{-n} \prod_{j=0}^{n-1} \left(\frac{2j+1}{j+1}\right) = \frac{\Gamma\left(\frac{1}{2}+n\right)}{n!\sqrt{\pi}} = 2^{-2n} \binom{2n}{n}, $$

also from (1.31) $f_{2n}(1/2, 2) = f_n(1/4, 2)$. For $b = 3$, a recurrence relation, using the *Zb* algorithm in Mathematica, $f_n(a, 3) = f_n$, of (1.31) is

$$ \left.\begin{array}{c} 2(n+3)(2n+5)\,f_{n+3} + \left(n^2(27a-12) + n(135a-56) + 168a - 66\right) f_{n+2} + \\ 2(n+2)(3n(2-9a) + 11 - 54a)\,f_{n+1} + (27a-4)(n+1)(n+2)\,f_n = 0, \\ f_0 = 1, f_1 = 1-3a, f_2 = 1-3a+15a^2 \end{array}\right\} \tag{1.41} $$

The recurrence (1.41) does not lend itself to easy closed form evaluations for any special values of a. Returning, briefly, to the identity (1.27), we begin with the general form

$$g_n(a,b) = \sum_{k=0}^{n} \frac{\binom{n}{k}(-a)^k}{\binom{bk}{k}} \tag{1.42}$$

and in hypergeometric notation

$$g_n(a,b) = {}_bF_{b-1}\left[\begin{array}{c} 1, \frac{b-2}{b-1}, \frac{b-3}{b-1}, \ldots, \frac{1}{b-1}, -n \\ \frac{b-1}{b}, \frac{b-2}{b}, \frac{b-3}{b}, \ldots, \frac{1}{b} \end{array} \middle| \frac{a(b-1)^{b-1}}{b^b}\right].$$

For $b = 1$, $g_n(a,1) = f_n(a,1) = (1-a)^2$. For $b = 2$,

$$g_n(a,2) = {}_2F_1\left[\begin{array}{c} 1, -n \\ \frac{1}{2} \end{array} \middle| \frac{a}{4}\right]$$

which has a recurrence relation

$$2(2n+1)g_{n+1} + (n+1)(a-4)g_n + 2 = 0, g_0 = 1.$$

In the specific case of $a = 4$, we obtain the identity (1.27), evaluated by Riordan, and it may be easily verified, utilizing the procedure described by Petkovšek et al. [74], by the rational certificate function

$$R(n,k) = \frac{k(1-2k)}{(n+1-k)(2n-1)}.$$

For $b = 3$, a recurrence relation of (1.42), using the Zb algorithm in Mathematica, is

$$\left.\begin{array}{c} 3(3n+4)(3n+5)g_{n+2} - 2(n+2)(n(27-2a)+27-3a)g_{n+1} + \\ (4a-27)(n+1)(n+2)g_n - 6 = 0, g_0 = 1, g_1 = 1 - \frac{1}{2}a \end{array}\right\},$$

and again it does not lend itself to easy closed form evaluations for any special values of a.

1.2.7 Snake oil method.

This method is described on page 126 of the book by Wilf [93]. Let

$$f_n(y) = \sum_{k=0}^{n} \binom{n}{k} y^k \binom{2k}{k} \tag{1.43}$$

and define $F(x,y) = \sum_{n\geq 0} f_n(y)\, x^n$. Now replace for $f_n(y)$ and interchange the order of summation, such that

$$\begin{aligned} F(x,y) &= \sum_{k\geq 0} \binom{2k}{k} y^k \sum_{n\geq 0} \binom{n}{k} x^k \\ &= \frac{1}{1-x} \sum_{k\geq 0} \binom{2k}{k} \left(\frac{xy}{1-x}\right)^k. \end{aligned} \tag{1.44}$$

Utilizing the identity $\sum_{k\geq 0} \binom{2k}{k} z^k = \frac{1}{\sqrt{1-4z}}$ it follows from(1.44) that

$$F(x,y) = \frac{1}{(1-x)\sqrt{1-\frac{4xy}{1-x}}} = \frac{1}{\sqrt{(1-x)(1-x(1+4y))}}.$$

If $y = -1/2, F(x,-1/2) = \frac{1}{\sqrt{1-x^2}}$ and the Reed Dawson identity follows. If $y = -1/4$, $F(x,-1/4) = \frac{1}{\sqrt{1-x}} = \sum_{m\geq 0} \binom{2m}{m} \left(\frac{x}{2}\right)^{2m}$ and hence,

$$\sum_{k\geq 0} \binom{n}{k} \left(\frac{-1}{4}\right)^k \binom{2k}{k} = 2^{-2n} \binom{2n}{n} = \frac{\Gamma(n+1/2)}{n!\sqrt{\pi}}.$$

We can generalize (1.43) a little by considering

$$f_n(b,c) = \sum_{k=0}^{n} \binom{n}{k} b^k \binom{c}{k} \tag{1.45}$$

and define

$$F(z,c,b)=\sum_{n\geq 0} f_n(b,c)\,z^n. \tag{1.46}$$

Putting (1.45) into (1.46) and interchanging the order of summation, we have

$$\begin{aligned}
F(z,b,c) &= \sum_{k\geq 0}\binom{c}{k} b^k \sum_{n\geq 0}\binom{n}{k} z^n \\
&= \frac{1}{1-z}\sum_{k\geq 0}\binom{c}{k}\left(\frac{zb}{1-z}\right)^k \\
&= \frac{(1+z(b-1))^c}{(1-z)^{c+1}}.
\end{aligned}$$

For c integer (1.45) will always have a closed form solution, For example, with $c=3$, we have

$$\sum_{k=0}^{n}\binom{n}{k} b^k \binom{3}{k} = \frac{1}{6}\left(b^3n^3+3b^2n^2(3-b)+bn\left(18-9b+2b^2\right)+6\right).$$

If $c=-1/2$ and $b=2$, we get $F(z,-1/2,2)=\left(1-z^2\right)^{-1/2}$ and from the relationship

$$\begin{aligned}
\binom{-1/2}{k}(-4)^k &= \binom{2k}{k} = (-1)^k\binom{k-1/2}{k}(-4)^k \\
&= \frac{4^k}{\sqrt{\pi}}\frac{\Gamma(k+1/2)}{\Gamma(k+1)} = \frac{2^k}{k!}\prod_{j=0}^{k-1}(2j+1)
\end{aligned}$$

the Reed Dawson identity follows. If $b=1$,

$$F(z,-1/2,1)=\frac{1}{\sqrt{1-z}}=\sum_{k\geq 0}\binom{2k}{k}\left(\frac{z}{4}\right)^k,$$

which corresponds to the Vandermonde identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{c}{k}=\binom{n+c}{c}.$$

1.2.8 Some relations.

The related sum

$$S_n(p,q) = \sum_{r=0}^{qn} (-1)^r \binom{qn}{r}^p \tag{1.47}$$

for p and q integers is an interesting one, and is briefly considered here. For $q = 1$ and $p = 1$, (1.47) is identical to (1.31) for $a = 1$ and $b = 1$. From (1.47) we have

$$S_n(p,q) = {}_pF_{p-1}\left[\begin{matrix} -qn, -qn, -qn, ..., -qn \\ 1, 1, 1, ..., 1 \end{matrix} \middle| (-1)^{p+1}\right], \tag{1.48}$$

and some special cases, from (1.48), are

$$S_n(1,q) = {}_1F_0\left[\begin{matrix} -qn \\ - \end{matrix} \middle| 1\right] = \begin{cases} 0 \; if \; qn \in Z^+ \\ 1 \; if \; qn = 0 \end{cases}$$

and

$$S_n(p,2) = {}_pF_{p-1}\left[\begin{matrix} -2n, -2n, -2n, ..., -2n \\ 1, 1, 1, ..., 1 \end{matrix} \middle| (-1)^{p+1}\right]. \tag{1.49}$$

It is known that $S_n(2,2) = (-1)^n \binom{2n}{n}$, $S_n(3,2) = (-1)^n \binom{3n}{n}\binom{2n}{n}$ and therefore $S_n(3,2) = \binom{3n}{n} S_n(2,2)$; however for $p \geqslant 4$, deBruijn [38] showed that (1.49) cannot be expressed as a ratio of products of factorials, and Graham et al. [48] also showed this by an application of the multidimensional saddle point method. We can deduce, from (1.48) the identity

$$S_n(2,q) = {}_2F_1\left[\begin{matrix} -qn, -qn \\ 1 \end{matrix} \middle| -1\right] = \frac{2^{qn+1}}{B\left(\frac{2+qn}{2}, \frac{1-qn}{2}\right)} \tag{1.50}$$

where $B(x,y)$ is the Beta function. From (1.47) and (1.48) we may also deduce that

$$S_{2n+1}(p,1) = \sum_{r=0}^{2n+1} (-1)^r \binom{2n+1}{r}^p = {}_pF_{p-1}\left[\begin{matrix} -2n+1, ..., -2n+1 \\ 1, ..., 1 \end{matrix} \middle| (-1)^{p+1}\right] = 0,$$

$$S_{2n}(p,1)=\sum_{r=0}^{2n}(-1)^r\binom{2n}{r}^p=(-1)^n\binom{2n}{n}^p+2\sum_{r=0}^{n-1}(-1)^r\binom{2n}{r}^p$$

and utilizing (1.50), gives the new result

$$\sum_{r=0}^{n}(-1)^r\binom{2n}{r}^2=\frac{2^{2n-1}\sqrt{\pi}}{n!\Gamma\left(\frac{1-2n}{2}\right)}+\frac{(-1)^n}{2}\binom{2n}{n}^2.$$

The sum (1.47) may, for specific cases of p and q, be written as a recurrence relation. Another related sum is given by Strehl [84], whom in an informative paper shows that, for all natural numbers n

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \tag{1.51}$$

$$= {}_4F_3\left[\begin{matrix} n+1,n+1,-n,-n \\ 1,1,1 \end{matrix}\middle| 1\right].$$

Strehl offers six different proofs of (1.51) based on:

- Bailey's bilinear generating function for the Jacobi polynomials in the special case when the Jacobi polynomials reduce to Legendre polynomials,
- A combinatorial approach to the Bailey identity,
- Legendre inverse pairs,
- the Pfaff-Saalschütz identity,
- Zeilberger's algorithm, and
- known recurrences for the Franel and Apèry numbers.

From (1.51), after various manipulations Strehl obtains

$$\sum_{k=0}^{n}\binom{n}{k}^2\left(\frac{(\lambda+1)^2}{\lambda}\right)^k=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\lambda+1}{\lambda}\right)^k\sum_{j=0}^{k}\lambda^j\binom{k}{j}^2 \tag{1.52}$$

$$= {}_2F_1\left[\begin{array}{c|c} -n,-n & \\ & 2+\lambda+\frac{1}{\lambda} \\ 1 & \end{array}\right]. \tag{1.53}$$

Given that $2\lambda_{1,2} = -3 \pm \sqrt{5}$ are the zeros of the quadratic $\lambda^2 + 3\lambda + 1$, then from (1.52)

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\lambda_{1,2}+1}{\lambda_{1,2}}\right)^k \sum_{j=0}^{k} \lambda_{1,2}^j \binom{k}{j}^2. \tag{1.54}$$

Identifying (1.53) with (1.50) for $q = 1$ we may also give the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k = {}_2F_1\left[\begin{array}{c|c} -n,-n & \\ & -1 \\ 1 & \end{array}\right] = \frac{2^{n+1}}{B\left(\frac{2+n}{2},\frac{1-n}{2}\right)}$$

and from (1.54) we can write, the new result

$$S_n(2,1) = \sum_{k=0}^{n} \binom{n}{k} \left(1+\frac{1}{\lambda_{1,2}}\right)^k \sum_{j=0}^{k} \lambda_{1,2}^j \binom{k}{j}^2$$

where a second order recurrence of (1.54) is $(n+2)\,S_{n+2}(2,1) + 4(n+1)\,S_n(2,1) = 0$, with $S_0(2,1) = 1$ and $S_1(2,1) = 0$; for n odd $S_n(2,1) = 0$, hence $(n+1)\,S_{2n+2}(2,1)+2(2n+1)\,S_{2n}(2,1)$ 0 and by iteration $S_{2n}(2,1) = (-2)^n \prod_{j=0}^{n-1} \frac{2j+1}{j+1}$.

1.2.9 Method of Sister Celine.

Let

$$f_n = \sum_{k=0}^{n} F(n,k) \tag{1.55}$$

where

$$F(n,k) = \binom{n}{k} \left(\frac{-1}{2}\right)^k \binom{2k}{k}. \tag{1.56}$$

Since the ratio of two subsequent terms of (1.22) is a rational function in both n and k then (1.56) is a proper hypergeometric function . Following Sister Celine [74] we require non-trivial solutions of the recurrence

$$\sum_{i=0}^{2}\sum_{j=0}^{2}\alpha_{i,j}(n)\,F(n-j,k-i)=0. \tag{1.57}$$

Utilizing the computer package added to "Mathematica", we can generate the recurrence

$$\begin{aligned}
&2\alpha(n-1)^2F(n-2,k-2)-(n-1)(\alpha n-2\beta n-\alpha)F(n-2,k-1)\\
&-\beta n(n-1)F(n-2,k)-\alpha(n-1)(2n-1)F(n-1,k-2)\\
&-(2n-1)(\beta n-\alpha n-\alpha)F(n-1,k-1)+\beta n(2n-1)F(n-1,k)\\
&-\alpha n(n-1)F(n,k-1)-\beta n^2F(n,k)=0.
\end{aligned}$$

Setting $\alpha=0,\beta=1$ and summing over k, we obtain a recursion equation for f_n, namely

$$nf_n=(n-1)\,f_{n-2}\;,f_0=1,f_1=0$$

and the Reed Dawson identity follows.

1.2.10 Method of creative telescoping.

The method of creative telescoping is described in the book of Petkovšek, Wilf and Zeilberger [74]. It utilizes the *Zb* algorithm in "Mathematica" so that the input

$$Zb[\text{Binomial}[n,k]\left(-\frac{1}{2}\right)^k\text{Binomial}[2k,k],k,n,2]$$

responds with a recurrence relation

$$(1+n)\,Sum[n]-(n+2)\,Sum[n+2]=0$$

and with initial conditions leads to the Reed Dawson identity.

1.2.11 WZ pairs method.

This method certifies a given identity as well as having some spin-offs. Given the identity (1.22) we may write

$$\sum_{k\geq 0}^{n} F(n,k) = 1 \tag{1.58}$$

where

$$F(n,k) = \frac{\begin{pmatrix} 2n \\ k \end{pmatrix} \left(-\frac{1}{2}\right)^k \begin{pmatrix} 2k \\ k \end{pmatrix} 4^n}{\begin{pmatrix} 2n \\ n \end{pmatrix}} = \frac{\left(-\frac{1}{2}\right)^k (2k)!n!^2 4^n}{(2n-k)!k!^3}. \tag{1.59}$$

Calling up the WZ package in "Mathematica" we obtain the certificate function

$$R(n,k) = \frac{k^2}{(2n-k+1)(k-2-2n)}. \tag{1.60}$$

Now, we define

$$G(n,k) = R(n,k)F(n,k) = \frac{-\left(-\frac{1}{2}\right)^k (2k)!n!^2}{k!(k-1)^2(2n-k+2)!} \tag{1.61}$$

such that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ is true. Sum that equation over all integers k, such that the right hand side telescopes to zero and therefore

$$\sum_{k\geq 0} F(n+1,k) = \sum_{k\geq 0} F(n,k). \tag{1.62}$$

The two discrete functions $F(n,k)$ and $G(n,k)$ are termed the WZ pairs. From (1.62) and with initial conditions we obtain the Reed Dawson identity. Petkovšek et al. [74] claim that the WZ pairs method provides extra information because of the existence of a dual WZ pair. To obtain the dual WZ pair make the substitution $(an+bk+c)!$ by $\frac{(-1)^{an+bk}}{(-an-bk-c-1)!}$ for $a+b\neq 0$ in (1.59) and (1.61) to obtain $\overline{F}$ and $\overline{G}$. Next change the variables (n,k) by $F^*(n,k) = \overline{G}(-k-1,-n); G^*(n,k) = \overline{F}(-k,-n-1)$, (this transformation maps WZ pairs to WZ pairs), such that we obtain

$$F^*(n,k) = \frac{(-1)^{n+1} 2^n (n-1)!n!^2 (2k-1-n)!}{4^{k+1}(2n-1)!k!^2} \tag{1.63}$$

and

$$G^*(n,k) = \frac{(-1)^{n+1} 2^{n+1} n!^3 (2k-2-n)!}{4^k (2n+1)! (k-1)!^2}. \tag{1.64}$$

As previously, we obtain $f_n^* = \sum_{k\geq 0} F^*(n,k)$ and because of the $(2k-1-n)$ term in (1.63) we shall define

$$f_n^* = \sum_{k\geq\left[\frac{n+1}{2}\right]} F^*(n,k), \tag{1.65}$$

where $[x]$ represents the integer part of x. Now, we need to sum over k, the recurrence

$$F^*(n+1,k) - F^*(n,k) = G^*(n,k+1) - G^*(n,k); \tag{1.66}$$

since the right hand side of (1.66) does not disappear, we sum for $k \geq 1 + \left[\frac{n}{2}\right]$, this however gives us an extra term, and distinguishing for n odd and n even, we obtain

$$F^*(n+2,k) - F^*(n,k) = G^*(n+1,k+1) - G^*(n+1,k) + G^*(n,k+1) - G^*(n,k).$$

For n even, let $n = 2m$, and summing for $k \geq 2+m$, we obtain

$$f^*(2+2m) - f^*(2m) + F^*(2m,m+1) = -G^*(2m+1,m+2) - G^*(2m,m+2),$$

and from (1.63) and (1.64) substituting for F^* and G^* we obtain

$$f^*(2+2m) = f^*(2m) + \frac{(3m+2)(2m+1)!(2m)!^2}{m!(4m+3)!(m+1)!}. \tag{1.67}$$

Iterating the recurrence (1.67) we have

$$f^*(2+2m) = f^*(2) + \sum_{j=1}^{m} \frac{(3j+2)(2j+1)!(2j)!^2}{j!(4j+3)!(j+1)!} \tag{1.68}$$

and from (1.63) and (1.65) we have

$$f^*(2) = -\frac{2}{3}\sum_{k\geq 2} \frac{(2k-3)!}{4^k k!^2}. \tag{1.69}$$

We can put (1.69) in "Mathematica, Algebra, SymbolicSum" and obtain

$$f^*(2) = \frac{1}{3} - \ln\sqrt{2}. \tag{1.70}$$

(We may also obtain (1.70) by starting with identity 2.5.16 in the book by Wilf [93]). Now from (1.70), (1.68), and (1.65) we obtain

$$\frac{4^m(2m-1)!\,(2m)!^2}{(4m-1)!}\sum_{k=m+1}^{\infty}\frac{(2k-1-2m)!}{4^{k+1}k!^2} = \ln\sqrt{2} - \frac{1}{3} - \sum_{j=1}^{m-1}\frac{(3j+2)\,(2j+1)!\,(2j)!^2}{j!\,(4j+3)!\,(j+1)!}. \tag{1.71}$$

From (1.67) and (1.70) we also obtain $f^*(0) = -\ln\sqrt{2}$ and from (1.71) putting $k^* = k - m$ and renaming k^* we have the new result

$$\sum_{k=1}^{\infty}\frac{(2k-1)!}{2^{2k}(m+k)!^2} = \frac{(4m-1)!}{(2m-1)!\,(2m)!^2}\left\{\ln 4 - 4\sum_{j=0}^{m-1}\frac{(3j+2)\,(2j+1)!\,(2j)!^2}{j!\,(4j+3)!\,(j+1)!}\right\}.$$

In the next chapter we develop and apply our procedure of 'domination of zeros' for the summation of series in closed form.

Chapter 2

Summing series arising from integro-differential-difference equations

In this chapter a first order differential-difference equation is considered and by the use of Laplace transform theory an infinite series is generated, which may be represented in closed form. The series, it turns out, arises in a number of areas including teletraffic problems, neutron behaviour, renewal processes, risk theory, grazing systems and demographic problems.

Related works to this area of study are considered, including Euler's and Jensen's investigations, Ramanujan's question, Cohen's modification and extension and finally a solution to Conolly's problem is given.[1]

[1] This chapter, in condensed form, is to be published in the Bull. Austral. Math. Soc.

2.1 Introduction.

Differential-difference equations occur in a wide variety of applications including: ship stabilization and automatic steering [72], the theory of electrical networks containing lossless transmission lines [17], the theory of biological systems [16], and in the study of distribution of primes [90]. The equation

$$f'(t) + \alpha f'(t-a) + \beta f(t) + \gamma f(t-a) + \delta f(t+a) = 0$$

is termed a first order linear delay, or retarded, differential-difference equation for $\alpha = 0, \delta = 0$ and $a > 0$. For $\alpha = 0, \delta = 0$ and $a < 0$ it is termed an advanced equation. In the case $\delta = 0$ and $a > 0$ it is referred to as a neutral equation and when $\alpha = 0, \beta = 0$ and $a > 0$, an equation of mixed type. A great deal of the studies for the stability of differential-difference equations necessitate an investigation of its associated characteristic function. Some of the early work in this area has been carried out by Pontryagin [77], Wright [96] and more recently by Cooke and van den Driessche [36] and Hao and Brauer [55]. In this chapter we will show that, by using Laplace transform techniques together with a reliance on asymptotics, series representations for the solution of differential-difference equations may be expressed in closed form. The series, in its region of convergence, it is conjectured, applies for all values of the delay parameter without necessarily relying on its association with the differential-difference equation. Unlike some of the series that are listed as high precision fraud by Borwein and Borwein [15] the series in this chapter will be shown to be exact by the use of Bürmann's theorem. The analysis also relies on the exact location of the zeros of the associated transcendental characteristic function. The technique developed in this chapter is then applied to particular examples that arise in teletraffic problems, neutron behaviour, renewal problems, ruin problems and to a model of a grazing system. We also investigate, briefly, equations with forcing terms, and equations with multiple delays, mixed and neutral equations. The fundamental series obtained in this chapter has also been investigated, using different methods than the author, by Euler, Jensen and Ramanujan. We shall describe their techniques and give in detail, a description of Cohen's modification and extension, and a solution to Conolly's problem.

2.2 Method.

Consider the first order linear homogeneous differential-difference equation with real parameters a, b and c and real variable t:

$$\left.\begin{array}{c} f'(t) + bf(t) + cf(t-a) = 0, \quad t \geq a \\ f'(t) + bf(t) = 0, \quad f(0) = 1, \quad 0 \leq t < a. \end{array}\right\} \tag{2.1}$$

Taking the Laplace transform of (2.1) and using the initial condition, results in

$$\pounds\left(f(t)\right) = F(p) = \frac{1}{p + b + ce^{-ap}} = \sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-an(p+b)} e^{anb}}{(p+b)^{n+1}}. \tag{2.2}$$

The inverse Laplace transform of (2.2) is

$$\pounds^{-1}\left(F(p)\right) = f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t-an)^n}{n!} H(t-an) \tag{2.3}$$

where the Heaviside unit function

$$H(x) = \begin{cases} 1, \text{for } x \geq 0 \\ 0, \text{for } x < 0. \end{cases}$$

The solution to (2.1), by Laplace transform theory may be written as

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} F(p)\, dp$$

for an appropriate choice of γ such that all the zeros of the characteristic function

$$g(p) = p + b + ce^{-ap} \tag{2.4}$$

are contained to the left of the line in the Bromwich contour. Now, using the residue theorem

$$f(t) = \sum \text{residues of } \left(e^{pt} F(p)\right)$$

which suggests the solution of $f(t)$ may be written in the form

$$f(t) = \sum_r Q_r e^{p_r t}$$

where the sum is over all the characteristic zeros p_r of $g(p)$ and Q_r is the residue of $F(p)$ at $p = p_r$. The poles of the expression (2.2) depend on the zeros of the characteristic function (2.4), namely, the roots of $g(p) = 0$. The dominant zero p_0 of $g(p)$ has the greatest real part and therefore asymptotically $f(t) \sim Q_0 e^{p_0 t}$, and so from (2.3),

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t-an)^n}{n!} H(t-an) \sim Q_0 e^{p_0 t}. \tag{2.5}$$

After some experimentation it is conjectured from (2.5) that:

$$\sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t-an)^n}{n!} = Q_0 e^{p_0 t} \tag{2.6}$$

$\forall t \in \Re$ in the region where the series on the left of (2.6) converges. Bürmann's theorem will be used, a little later, to prove the identity (2.6). By the use of the ratio test it can be shown that the series on the left of (2.6) converges in the region

$$\left| ace^{1+ab} \right| < 1. \tag{2.7}$$

In a similar fashion, the Laplace transform from (2.2) may be expressed as

$$F(p) = \frac{1}{p}\left(1 + \frac{b + ce^{-ap}}{p}\right)^{-1} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \frac{(-1)^n b^{n-r} c^r e^{-arp}}{p^{n+1}},$$

and the inverse Laplace transform may be written as

$$f(t) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \frac{(-1)^n b^{n-r} c^r (t-ar)^n}{n!} H(t-ar) \sim Q_0 e^{p_0 t}.$$

As previous it is conjectured that

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^n b^{n-r} c^r (t-ar)^n}{n!} = Q_0 e^{p_0 t} \tag{2.8}$$

whenever the double series converges.

Lemma 1 *The poles of the expression (2.2) are all simple for the inequality (2.7).*

Proof: Assume on the contrary that there is a repeated root of

$$p + b + ce^{-ap} = 0 \tag{2.9}$$

then by differentiation it is required that $1 - ace^{-ap} = 0$, in which case $p = \ln(ac)/a$. Substituting in (2.9) results in $\ln(ac) + ab + 1 = 0$ and therefore $ace^{1+ab} = 1$ which violates the inequality (2.7). Hence all the zeros in (2.9) are simple.

Now, the residue Q_0 of the dominant simple zero $p_0 = \xi$ is

$$\frac{1}{1+ab+a\xi}, \text{ where } \xi + b + ce^{-a\xi} = 0,$$

and so the expressions (2.6) and (2.8) become

$$\sum_{n=0}^{\infty}\frac{(-1)^n c^n e^{-b(t-an)} (t-an)^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^n b^{n-r} c^r (t-ar)^n}{n!} = \frac{e^{\xi t}}{1+ab+a\xi} \tag{2.10}$$

whenever the single and double series converge in a mutual region. Using the transformation

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} f(n,r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} f(n+r,r) \tag{2.11}$$

we obtain, from (2.10)

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\binom{n+r}{r}\frac{(-1)^{n+r} b^n c^r (t-ar)^{n+r}}{(n+r)!} = \frac{e^{\xi t}}{1+ab+a\xi}.$$

Lemma 2

1. *The single and double sum in (2.10) are solutions to (2.1) in their region of convergence.*

2. *The closed form expression in (2.10) is a solution to (2.1) for $t \geq a$.*

3. *The single and double sums in (2.10) are equal in their mutual region of convergence, which is no larger than that given by (2.7).*

Proof: 1. and 2. can be shown to be solutions of (2.1) by substitution and statement 3 of lemma 2 requires that we show

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^n b^{n-r} c^r (t-ar)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n c^n e^{-b(t-an)}(t-an)^n}{n!},$$

so that expanding the left hand side and summing each column from the left hand side results in

$$\begin{aligned}
&+\frac{c^0 (t-0)^0}{0!}\left[\frac{1}{0!}+\frac{b(t-0)^1}{1!}+\frac{b^2(t-0)^2}{2!}+\frac{b^3(t-0)^3}{3!}+\ldots\right]\\
&-\frac{c^1 (t-a)^1}{1!}\left[\frac{1}{0!}+\frac{b(t-a)^1}{1!}+\frac{b^2(t-a)^2}{2!}+\frac{b^3(t-a)^3}{3!}+\ldots\right]\\
&+\frac{c^2 (t-2a)^2}{2!}\left[\frac{1}{0!}+\frac{b(t-2a)^1}{1!}+\frac{b^2(t-2a)^2}{2!}+\frac{b^3(t-2a)^3}{3!}+\ldots\right]\\
&-\frac{c^3 (t-3a)^3}{3!}\left[\frac{1}{0!}+\frac{b(t-3a)^1}{1!}+\frac{b^2(t-3a)^2}{2!}+\frac{b^3(t-3a)^3}{3!}+\ldots\right]\\
&+\ldots
\end{aligned}$$

$$= \frac{c^0(t-0)^0 e^{-b(t-0)}}{0!} - \frac{c(t-a)e^{-b(t-a)}}{1!} + \frac{c^2(t-2a)^2 e^{-b(t-2a)}}{2!} - \frac{c^3(t-3a)^3 e^{-b(t-3a)}}{3!} + \ldots$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n c^n e^{-b(t-an)}(t-an)^n}{n!}.$$

Returning briefly to (2.10), put $b+c=0$, which implies that $\xi = 0$, also let $t = -a\tau$, so that

$$\sum_{n=0}^{\infty}(ab)^n\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r(\tau+r)^n}{n!} = \frac{1}{1+ab}.$$

The inner sum, which we shall generalise in chapter 4, is $(-1)^n$, and hence we obtain the common series

$$\sum_{n=0}^{\infty} (-ab)^n = \frac{1}{1+ab}. \tag{2.12}$$

Bürmann's theorem [92] will now be used to prove the explicit form of relationship (2.6).

2.3 Bürmann's theorem and application.

Theorem 3 *Let ϕ be a simple function in a domain D, zero at a point β of D, and let*

$$\theta(z) = \frac{z-\beta}{\phi(z)}, \theta(\beta) = \frac{1}{\phi'(\beta)}.$$

If $f(z)$ is analytic in D then $\forall z \in D$

$$f(z) = f(\beta) + \sum_{r=1}^{n} \frac{\{\phi(z)\}^r}{r!} \frac{d^{r-1}}{dt^{r-1}} \left[f'(t)\{\theta(t)\}^r\right]_{t=\beta} + R_{n+1}$$

$$\text{where } R_{n+1} = \frac{1}{2\pi i}\int_\Gamma d\nu \int_C \left[\frac{\phi(\nu)}{\phi(t)}\right]^n \frac{f'(t)\phi'(\nu)}{\phi(t)-\phi(\nu)} dt.$$

The ν integral is taken along a contour Γ in D from β to z, and the t integral along a closed contour C in D encircling Γ once positively.

We shall prove Bürmann's theorem in chapter three. However next, we shall apply Bürmann's theorem to equation (2.10).

The characteristic function (2.4) may be shown to have a simple dominant zero at $p = 0$ for $b + c = 0$ and $1 + ab > 0$. Thus from (2.6)

$$\sum_{n=0}^{\infty} \frac{b^n e^{-b(t-an)} (t-an)^n}{n!} = \frac{1}{1+ab}. \tag{2.13}$$

Let $t = -a\tau$, $ab = -\rho$, and hence from above

$$\sum_{n=0}^{\infty} \frac{(\rho e^{-\rho})^n (\tau+n)^n}{n!} = \frac{e^{\rho\tau}}{1-\rho}. \tag{2.14}$$

Identity (2.14) is now shown to be true by applying Bürmann's theorem. Let

$$f(z) = \frac{e^{xz}}{1-z}, \theta(z) = \frac{z}{\phi(z)} = e^{z}, \phi(z) = ze^{-z}, f(\beta)_{\beta=0} = 1,$$

and we will show, in chapter three, that $R_{n+1} \to 0$ as $n \to \infty$. From $f(t) = \frac{e^{xt}}{1-t}$,

$$f'(t) = e^{xt}\left(\frac{x}{1-t} + \frac{1}{(1-t)^2}\right) = e^{xt}\left(\sum_{j=0}^{\infty}(x+1+j)\,t^j\right),$$

and so $f'(t)\{\theta(t)\}^r = e^{t(r+x)}\Psi(t)$, where $\Psi(t) = \sum\limits_{j=0}^{\infty}(x+1+j)\,t^j$. The coefficients in this expression are the same as those in a Taylor series expansion $\Psi^{(j)}(0) = (x+1+j)\,j!$. Now let

$$\begin{aligned}
B_r(t) &= \frac{d^{r-1}}{dt^{r-1}}\left[f'(t)\{\theta(t)\}^r\right] \\
&= \frac{d^{r-1}}{dt^{r-1}}\left[e^{t(r+x)}\Psi(t)\right] \\
&= e^{t(r+x)}\left[\begin{array}{c}
(r+x)^{r-1}\binom{r-1}{0}\Psi^{(0)}(t) + (r+x)^{r-2}\binom{r-1}{1}\Psi'(t) + \\
(r+x)^{r-3}\binom{r-1}{2}\Psi''(t) + \ldots + (r+x)^{1}\binom{r-1}{r-2}\Psi^{(j-2)}(t) \\
+(r+x)^{0}\binom{r-1}{r-1}\Psi^{(j-1)}(t).
\end{array}\right]
\end{aligned}$$

Hence

$$B_r(0) = \left(\begin{array}{c}
(r+x)^{r-1}\binom{r-1}{0}(x+1) + (r+x)^{r-2}\binom{r-1}{1}(x+2) + \\
(r+x)^{r-3}\binom{r-1}{2}(x+3) + \ldots + (r+x)^{1}\binom{r-1}{r-2}(x+r-1)(r-2)! \\
+\binom{r-1}{r-1}(x+r)(r-1)!.
\end{array}\right)$$

If we now put $y = x + r$ we obtain

$$\begin{aligned} B_r(0) &= y^{r-1}(y-r+1) + y^{r-2}(r-1)(y-r+2) + y^{r-3}(r-1)(r-2)(y-r+3) \\ &\quad +\ldots + (r-1)!y(y-1) + (r-1)!y \\ &= y^r - (r-1)y^{r-1} + (r-1)y^{r-1} - (r-1)(r-2)y^{r-2} + (r-1)(r-2)y^{r-2} \\ &\quad -\ldots - (r-1)!y + (r-1)!y \\ &= y^r = (x+r)^r. \end{aligned}$$

Hence it follows that

$$\frac{e^{xz}}{1-z} = 1 + \sum_{r=1}^{\infty} \frac{\{ze^{-z}\}^r}{r!}(x+r)^r.$$

A modification of this sum also appears as a problem in the work of Pólya and Szegö [75].

By the ratio test the infinite sum (2.14) converges in the region $|\rho e^{1-\rho}| < 1$,(or $|abe^{1+ab}| < 1$ for (2.13)), and so considering ρ as a complex variable $\rho = x+iy$, then $\left(e^{2(1-x)}(x^2+y^2)\right)^{\frac{1}{2}} < 1$. The region is shown in figure 2.1.

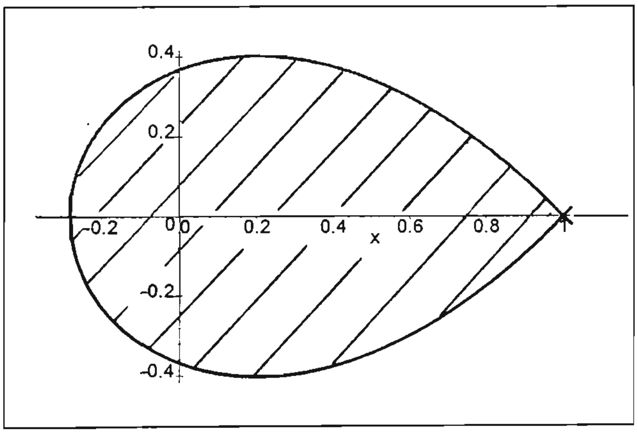

Figure 2.1: Convergence region, $|\rho e^{1-\rho}| < 1$.

On the boundary $\rho = 1$, from (2.14), the series

$$\sum_{n=0}^{\infty} \frac{e^{-(\tau+n)}(\tau+n)^n}{n!} \text{ may be shown to diverge.}$$

Consider the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, then by the limit comparison test $\lim_{n\to\infty} \left(\frac{e^{-(\tau+n)}(\tau+n)^n}{n!} n\right) > 0$ on utilizing Stirling's approximation $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ as $n \to \infty$. The divergence of the above series can also be ascertained from the closed form representation of the right hand side in (2.14).

The characteristic function (2.4) may be shown to have a dominant double zero at $p = 0$ for $b + c = 0$ and $1 + ab > 0$. From the general theory of linear functional differential equations [52] it follows that there exists constants α and β such that

$$\lim_{t\to\infty} (f(t) - \alpha t) = \beta.$$

From residue theory, the constants α and β can be shown to be $\frac{2}{a}$ and $\frac{2}{3}$ respectively, in which case

$$\lim_{t\to\infty} \left(f(t) - \frac{2t}{a}\right) = \frac{2}{3}.$$

From (2.10) and (2.2) it can be seen that

$$\lim_{a\to 0} \left(\sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t-an)^n}{n!}\right) = e^{-(b+c)t} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r} \frac{(-1)^n b^{n-r} c^r t^n}{n!}.$$

This result can be ascertained directly from the differential-difference equation (2.1).

2.4 Differentiation and Integration.

Rewriting (2.10) we have that

$$\sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{abn} (t-an)^n}{n!} = \frac{e^{t(b+\xi)}}{1+ab+a\xi}. \tag{2.15}$$

Differentiating both sides of (2.15) with respect to t, j times we have

$$\sum_{n=j}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t-an)^{n-j}}{(n-j)!} = \frac{(b+\xi)^j e^{t\xi}}{1+ab+a\xi}. \tag{2.16}$$

On the left hand side (2.16) put $n - j = n^*$ and rename $n^* = n$, also put $t + aj = x$, then the left hand side becomes

$$(-c)^j \sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(x-an)} (x - an)^n}{n!}$$

and from (2.15), is equal to

$$= (-c)^j \frac{e^{x\xi}}{1 + ab + a\xi}, \tag{2.17}$$

but from the characteristic function (2.4), since ξ is a zero

$$-c = (b + \xi) e^{a\xi} \tag{2.18}$$

and therefore (2.17) is equal to

$$\frac{(b + \xi)^j e^{x\xi}}{1 + ab + a\xi}$$

which is equivalent to the right hand side of (2.16) after renaming x as t.

We may also integrate (2.15) j times such that

$$\sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{abn} (t - an)^{n+j}}{(n + j)!} = \underbrace{\int \cdots\cdots \int}_{j-times} \frac{e^{t(b+\xi)}}{1 + ab + a\xi} dt. \tag{2.19}$$

For $j = 1$ we have, from (2.19)

$$\sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t - an)^{n+1}}{(n + 1)!} = \frac{e^{t\xi}}{(b + \xi)(1 + ab + a\xi)} + Ke^{-bt}, \tag{2.20}$$

where K is a constant of integration. Now putting $t = x - a$, $n^* = n + 1$ and renaming the counter n^*, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n c^n e^{-b(x-an)} (x - an)^n}{n!} = -c \left[\frac{e^{\xi(x-a)}}{(b + \xi)(1 + ab + a\xi)} + Ke^{-b(x-a)} \right],$$

to evaluate the constant K, adjust the counter on the left hand side and use the result (2.10);

on the right substitute (2.18) for $-c$, such that $K = -\frac{e^{-a(b+\xi)}}{b+\xi}$. Thus from (2.20)

$$\sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t-an)^{n+1}}{(n+1)!} = \frac{1}{b+\xi}\left[\frac{e^{t\xi}}{1+ab+a\xi} - e^{-b(t+a)-a\xi}\right]. \tag{2.21}$$

If $b+c=0$ and $1+ab \neq 0$ then $\xi = 0$, in which case from (2.21)

$$\sum_{n=0}^{\infty} \frac{b^n e^{-b(t-an)} (t-an)^{n+1}}{(n+1)!} = \frac{1}{b}\left[\frac{1}{1+ab} - e^{-b(t+a)}\right]. \tag{2.22}$$

2.5 Forcing terms.

The result (2.19) may be arrived at by considering a difference-delay equation with a forcing term. Let

$$f'(t) + bf(t) - bf(t-a) = \frac{t^{m-1}e^{-bt}}{\Gamma(m)},\ f(0) = 0, \tag{2.23}$$

for m a positive integer. Following the procedure of the previous sections, we have

$$F(p) = \frac{1}{(p+b)^m (p+b-be^{-ap})} \tag{2.24}$$

where $F(p)$ has a simple dominant pole at $\xi = 0$, and a pole of order m at $p = -b$. From these considerations we arrive at the result

$$\sum_{n=0}^{\infty} \frac{b^n e^{-b(t-an)} (t-an)^{n+m}}{(n+m)!} = \frac{1}{b^m(1+ab)} + \sum_{\nu=0}^{m-1} P_{m,\nu}(-b) \frac{t^{m-\nu-1}e^{-bt}}{(m-\nu-1)!} \tag{2.25}$$

where

$$\nu! P_{m,\nu}(-b) = \lim_{p\to -b}\left\{\frac{d^\nu}{dp^\nu}\left[\frac{1}{p+b-be^{-ap}}\right]\right\},\ \nu = 0,1,2,...,m-1.$$

For $m=1$, (2.25) gives the result (2.22), and for $m=3$ we have the result

$$\sum_{n=0}^{\infty} \frac{b^n e^{-b(t-an)} (t-an)^{n+3}}{(n+3)!} = \frac{1}{b^3(1+ab)} - \frac{e^{-bt}}{be^{ab}}\left[\frac{t^2}{2} + \frac{t\left(1+abe^{ab}\right)}{be^{ab}} - \frac{a^2}{2} + \frac{2\left(1+abe^{ab}\right)^2}{b^2e^{2ab}}\right].$$

Now let us consider the case where m may be a rational number. As an example if $m = \alpha/\beta$ then (2.24) has a simple dominant pole at $\xi = 0$ and a branch point at $p = -b$. Also, from

(2.24), $f(t) = \mathcal{L}^{-1}(F(p))$ where

$$\oint_C = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt} dp}{(p+b)^{\alpha/\beta} (p + b - be^{-ap})} \tag{2.26}$$

and the contour C in (2.26) may be

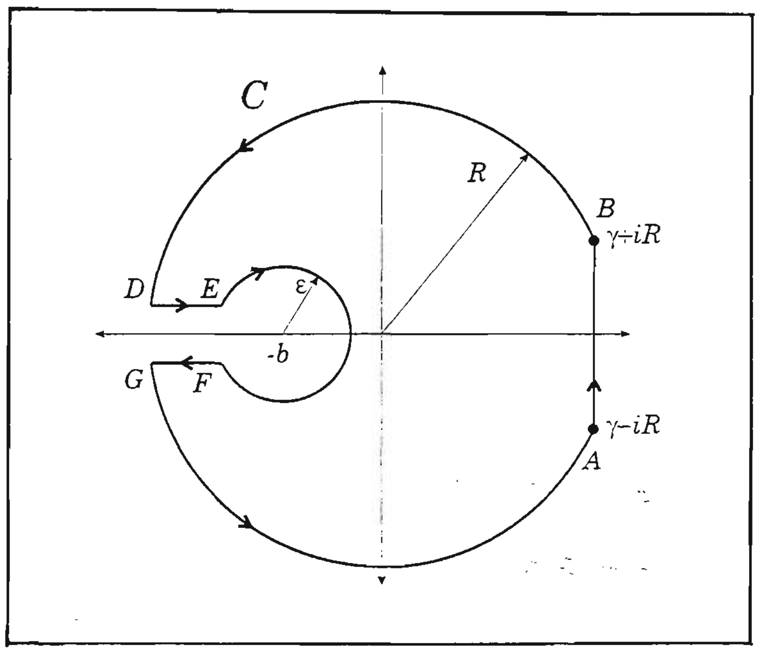

Figure 2.2: The contour C, in (2.26).

Now

$$\oint_C = \int_{AB} + \int_{BD} + \int_{DE} + \int_{EF} + \int_{FG} + \int_{GA} = 2\pi i \text{Res}(p = 0), \tag{2.27}$$

$\text{Res}(p = 0) = \frac{1}{b^{\alpha/\beta}(1+ab)}$ and along BD and EF,

$$\int_{BD} = 0 = \int_{EF}. \tag{2.28}$$

Along DE and FG we have

$$\int_{DE} = \int_{FG} = i \lim_{\varepsilon \to 0, R \to \infty} \int_{\varepsilon}^{R-b} \frac{e^{-(x+b)t} dx}{x^{\alpha/\beta} \left(x + be^{a(x+b)}\right)}. \tag{2.29}$$

From (2.27), (2.28) and (2.29)

$$\frac{1}{2\pi i}\int\limits_{AB} = \frac{1}{b^{\alpha/\beta}(1+ab)} - \frac{1}{\pi}\int\limits_{x=0}^{\infty} \frac{e^{-(x+b)t}dx}{x^{\alpha/\beta}\left(x+be^{a(x+b)}\right)}, \tag{2.30}$$

and if, as previous, our conjecture is to follow then

$$\sum_{n=0}^{\infty} \frac{b^n e^{-b(t-an)}(t-an)^{n+\frac{\alpha}{\beta}}}{\Gamma\left(n+\frac{\alpha}{\beta}+1\right)} =? \frac{1}{b^{\alpha/\beta}(1+ab)} - \frac{1}{\pi}\int\limits_{x=0}^{\infty} \frac{e^{-(x+b)t}dx}{x^{\alpha/\beta}\left(x+be^{a(x+b)}\right)} \tag{2.31}$$

however, since the integral in (2.30) is improper and divergent then (2.31) is not an identity. A similar improper divergent integral (2.30) may be obtained for any real number m.

2.6 Multiple delays, mixed and neutral equations.

Consider an equation with two delays

$$f'(t) + bf(t-\alpha) - bf(t-\beta) = 0, \; f(0) = 1, \alpha, \beta > 0. \tag{2.32}$$

Taking the Laplace transform of (2.32) we obtain

$$F(p) = \frac{1}{p + be^{-\alpha p} - be^{-\beta p}}$$

which has a simple dominant pole at $\xi = 0$. We may write $F(p)$ in series form such that

$$F(p) = \sum_{n=0}^{\infty} \frac{(-b)^n}{p^{n+1}} \sum_{r=0}^{n} (-1)^r \binom{n}{r} e^{-p(\alpha n+\beta r-\alpha r)}$$

and so using the techniques of the previous sections we have

$$\sum_{n=0}^{\infty} (-b)^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{(t-(\alpha n+\beta r-\alpha r))^n}{n!} = \frac{1}{1-b(\alpha-\beta)}. \tag{2.33}$$

If $\alpha = 0$ and $\beta = a$ (2.32) reduces to (2.1) and (2.33) is equivalent to (2.10). If $\beta = -\alpha$ (2.32) becomes a mixed equation, and (2.33) reduces to the identity

$$\sum_{n=0}^{\infty}(-b)^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(t-\alpha(n-2r))^n}{n!}=\frac{1}{1-2b\alpha}.$$

For the homogeneous neutral equation (forcing terms may also be added).

$$f'(t)+bf'(t-a)+cf(t-a)=0,\ f(0)=1$$

we obtain

$$F(p)=\frac{1}{p+pbe^{-ap}+ce^{-ap}},$$

and from the methods of the previous sections

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-b)^n\binom{n}{r}\left(\frac{c}{b}\right)^r\frac{(t-an)^r}{r!}=\frac{(c+b\xi)\,e^{t\xi}}{c-a\xi\,(c-b\xi)},\tag{2.34}$$

where ξ is the dominant zero of the characteristic function

$$g(p)=p+(c+bp)\,e^{-ap}.$$

Using the transformation (2.11), (2.34) reduces to the identity

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{n+r}\,b^nc^r\binom{n+r}{r}\frac{(t-a(n+r))^r}{r!}=\frac{(c+b\xi)\,e^{t\xi}}{c-a\xi\,(c-b\xi)},$$

and for the degenerate case of $a=0$ we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{ct}{b}\right)^r\frac{(-b)^n}{r!}=\frac{e^{-ct/(1+b)}}{1+b}.$$

A number of examples will next be investigated in which the methods of the previous sections are applicable and in which the identity (2.13) and its variations can be extracted.

2.7 Bruwier series.

Bellman and Cooke [7] refer to

$$f(t) = \sum_{n=0}^{\infty} \frac{\nu^n (t + n\omega)^n}{n!}$$

as the Bruwier series, see [20] and [21], which is a solution to the advanced equation

$$f'(t) - \nu f(t + \omega) = 0, \; f(0) = 1. \tag{2.35}$$

Comparing (2.35) with (2.1) it can be seen that $b = 0$, $c = -\nu$, $a = -\omega$ and from the series at (2.11)

$$\sum_{n=0}^{\infty} \frac{\nu^n (t + n\omega)^n}{n!} = \frac{e^{\xi t}}{1 - \omega\xi},$$

where ξ is the dominant real root of $\xi - \nu e^{\omega\xi} = 0$ and $|\nu\omega e| < 1$, is the region of convergence of the series.

2.8 Teletraffic example.

Erlang [40], see also Brockmeyer and Halstrom [18], considers the delay in answering of telephone calls. The problem is to determine the function $f(t)$, representing the probability of the waiting time not exceeding time t. Hence for an $M/M/1$ regimen, Erlang shows

$$f(t) = \int_{y=0}^{\infty} f(t + y - a)\, e^{-y} dy.$$

The probability that, at the moment a call arrives, the time having elapsed since the preceding call confined between y and dy, is $e^{-y}dy$. The probability that the waiting time of the preceding call has been less than $t + y - a$ is $f(t + y - a)$, where a is the connection time of a call. Differentiating the integral equation with respect to t and partially integrating the result gives the differential-difference equation (2.1) with $b = -1$ and $c = 1$, from which (2.10) follows. It may be shown that the characteristic function (2.9), with $b = -1$ and $c = 1$, has the following real root distribution:

- One root at $p = 0$ for $a \le 0$,
- one negative root, plus $p = 0$ for $0 < a < 1$,
- a double(repeated) root at $p = 0$ for $a = 1$ and,
- one positive root, plus $p = 0$ for $a > 1$.

The following results apply for all real values of t , in the region of convergence$\left|ae^{1-a}\right| < 1$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n e^{t-an} (t-an)^n}{n!} = \begin{cases} \frac{e^{\xi t}}{1-a+a\xi} & \text{for } a > 1 \\ \frac{1}{1-a} & \text{for } a < 1 \end{cases}$$

which on putting $t = -a\tau$, we may write

$$\sum_{n=0}^{\infty} \frac{(ae^{-a})^n (\tau+n)^n}{n!} = \begin{cases} \frac{e^{(1-\xi)a\tau}}{1-a+a\xi} & \text{for } a > 1 \\ \frac{e^{a\tau}}{1-a} & \text{for } a < 1 \end{cases}$$

where ξ is the positive dominant root of $\xi - 1 + e^{-a\xi} = 0$. Erlang considered only the case of $0 < a < 1$. In the case when $a = 1$ there is a double pole which results in, from a previous statement in section 2.3

$$\lim_{t\to\infty} (f(t) - 2t) = \frac{2}{3}. \tag{2.36}$$

This fact has also been noted, in a different context, by Feller [44]. Bloom [12] proposes the problem of evaluating $\lim_{t\to\infty} (f(t) - 2t)$ given that, for t a positive integer $f(t) = \sum_{0\le n\le t} \frac{(-1)^n e^{t-n}(t-n)^n}{n!}$. The W.M.C. problems group [94] and Holzsager [60] both solve this problem, and in particular Holzsager considers $f(t), \forall t > 0$. Now, $f(t)$ satisfies the differential-difference equation $f'(t) = f(t) - f(t-1)$, $t \ge 1$ and using the theory of linear functional differential equations, Holzsager shows the result (2.36). Holzsager's work relates only to the asymptotic of the finite sum whereas in this chapter it is shown that the infinite sum is equal to the asymptotic expression for all t. We may also prove (2.36) in the following way.

Theorem 4

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n-1} \frac{e^{-(k-n)} (k-n)^k}{k!} - 2n \right) = \frac{2}{3}. \tag{2.37}$$

Proof: Let $a_n = \sum_{k=0}^{n} \frac{e^{-(k-n)}(k-n)^k}{k!}, n = 1, 2, 3, ...$ and consider the generating function

$$
\begin{aligned}
F(z) &= 1 + \sum_{n=1}^{\infty} a_n z^n \\
&= 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{e^{-(k-n)}(k-n)^k}{k!} z^n \\
&= 1 + \sum_{n=1}^{\infty} (ez)^n + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{e^{-(k-n)}(k-n)^k}{k!} z^n, \text{ let } r = n-k, \\
&= 1 + \sum_{n=1}^{\infty} (ez)^n + \sum_{r=0}^{\infty} (ez)^r \sum_{k=1}^{\infty} \frac{(-rz)^k}{k!}, \text{ rename the counter } r, \\
&= \sum_{n=0}^{\infty} (ez)^n + \sum_{n=0}^{\infty} (ez)^n \left(e^{-zn} - 1\right) \\
&= \sum_{n=0}^{\infty} \left(ze^{1-z}\right)^n \\
&= \frac{1}{1 - ze^{1-z}}.
\end{aligned}
$$

We therefore have a pole at $z = 1$, hence $F(z)$ is analytic on $C : |z| \leq 1$. Other poles of $F(z)$ are outside the unit circle. A Laurent expansion of $F(z)$ about $z = 1$,after putting $t = 1 - z$ is

$$
\begin{aligned}
F(t) &= \frac{1}{1 - (1-t)e^t} \\
&= \frac{2}{t^2}\left(1 - \frac{2t}{3} + ...\right)
\end{aligned}
$$

and at $t = 0, (z = 1)$ there is a pole of order 2. The principal part

$$
\begin{aligned}
G(z) &= \frac{2}{(1-z)^2} - \frac{4}{3(1-z)} \\
&= \sum_{n=0}^{\infty} 2(n+1) z^n - \frac{4}{3} \sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} \left(2n + \frac{2}{3}\right) z^n.
\end{aligned}
$$

Hence

$$\begin{aligned}(F-G)(z) &= 1+\sum_{n=1}^{\infty} a_n z^n - \sum_{n=0}^{\infty}\left(2n+\frac{2}{3}\right) z^n \\ &= \frac{1}{3}+\sum_{n=1}^{\infty}\left(a_n - 2n - \frac{2}{3}\right) z^n,\end{aligned}$$

is analytic for $|z| \leq 1$, and so converges for $z = 1$. Thus (2.37) follows and the proof is complete.

An elementary approach leading to (2.37) is suggested by Haigh [51]. Using (probability) renewal theory arguments Haigh demonstrates, that, given that X is a random positive variable, then $E(X) = \sum_{n=1}^{\infty} \text{Prob.}(X \geq n)$ and if $N(t)$ is the number of random numbers we need to sum until we exceed some target t, then $E(N(t)) = \sum_{k=0}^{[t]} \frac{e^{t-k}(k-t)^k}{k!}$. Cox [37] and Feller [44] then show the result (2.37).

2.9 Neutron behaviour example.

In the slowing down of neutrons Teichmann [86] introduces Laplace transform techniques to analyze the renewal equation. This example involves the Placzek function

$$F(p) = \frac{1-e^{-(1+p)u_0}}{(1+p)(1-\alpha)-1+e^{-(1+p)u_0}} \tag{2.38}$$

before inversion, where α is a constant depending on the mass of the moderating nuclei and $u_0 = -\ln\alpha$ is the maximum lethargy change in a single collision. Keane [67] obtains

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n e^{\alpha t/(1-\alpha)}}{(1-\alpha)\, n!}\left\{\frac{(t-nu_0)^n}{(1-\alpha)^n} + \frac{n(t-nu_0)^{n-1}}{(1-\alpha)^{n-1}}\right\} e^{-nu_0/(1-\alpha)} H(t-nu_0)$$

where t is lethargy and $H(t-nu_0)$ is the Heaviside function. From (2.38), it may be shown that $F(p)$ has a simple dominant pole at $p = 0$ and for $1-\alpha+\alpha\ln\alpha \neq 0$ its residue contribution is $A = 1 + \frac{\alpha}{1-\alpha}\ln\alpha$. Using the techniques developed in section 2.2 it will now be shown that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left\{\left(\frac{t-nu_0}{1-\alpha}\right)^n + n\left(\frac{t-nu_0}{1-\alpha}\right)^{n-1}\right\} e^{-nu_0/(1-\alpha)} = \frac{(1-\alpha)\, e^{-\alpha t/(1-\alpha)}}{A}. \tag{2.39}$$

From (2.10), for $b = 0$ and $c = 1$ we have

$$h(y) = \sum_{n=0}^{\infty} \frac{(-1)^n (y - an)^n}{n!} = \frac{e^{y\eta}}{1 + a\eta}, \text{ where } \eta + e^{-a\eta} = 0. \tag{2.40}$$

Rewriting the left hand side of (2.39) gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{ \frac{te^{-u_0/(1-\alpha)}}{1-\alpha} - \frac{nu_0e^{-u_0/(1-\alpha)}}{1-\alpha} \right\}^n -$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{ \frac{te^{-u_0/(1-\alpha)}}{1-\alpha} - \frac{(n+1)\, u_0e^{-u_0/(1-\alpha)}}{1-\alpha} \right\}^n e^{-\frac{u_0}{1-\alpha}}$$

$$= h(t) - h(t - u_0)\, e^{-\frac{u_0}{1-\alpha}}. \tag{2.41}$$

Now it is required to show that (2.41) is identically equal to the right hand side of (2.39). Let

$$y = \frac{te^{-u_0/(1-\alpha)}}{1-\alpha}, a = \frac{u_0e^{-u_0/(1-\alpha)}}{1-\alpha} \text{ and } B = \frac{u_0}{1-\alpha} \text{ then } a = Be^{-B},$$

so that from (2.40)

$$h(t) = \frac{e^{\frac{\eta te^{-B}}{1-\alpha}}}{1 + a\eta}.$$

From $\eta + e^{-\eta Be^{-B}} = 0$ put $a\eta = -E$ then $E = ae^E$ and hence $Ee^{-E} = Be^{-B}$, which is satisfied by the relationship $E = \alpha B$. From (2.41)

$$\frac{e^{-\alpha t/(1-\alpha)}}{1 + \frac{\alpha}{1-\alpha}\ln\alpha} - \frac{e^{-\alpha(t-u_0)/(1-\alpha)}e^{-u_0/(1-\alpha)}}{1 + \frac{\alpha}{1-\alpha}\ln\alpha}$$

$$= \frac{(1-\alpha)\, e^{-\alpha t/(1-\alpha)}}{1 + \frac{\alpha}{1-\alpha}\ln\alpha}, \text{ as required.}$$

Identities (2.39) and (2.41) hold in the region of convergence $\left|\frac{u_0}{1-\alpha}e^{\frac{1-\alpha-u_0}{1-\alpha}}\right| < 1$. From (2.38) a double pole occurs at $p = 0$ when $1 - \alpha + \alpha\ln\alpha = 0$, therefore

$$\lim_{t\to\infty} \left(f(t) - \frac{2\alpha t}{1-\alpha} \right) = \frac{2\alpha\,(2\alpha + 1)}{3\,(1-\alpha)}.$$

2.10 A renewal example.

In determining the availability of a renewed component Pagēs and Gondran [73] consider the case of a constant failure rate. Given that $A(t)$ is the availability of a Markovian component, λ is the constant failure rate, and $g(t)$ is a density function, then the integro-differential equation satisfied by $A(t)$ is

$$\frac{d}{dt}A(t) = -\lambda A(t) + (1 - A_0)\, g(t) + \lambda \int_0^t g(u)\, A(t-u)\, du, \quad A(0) = A_0.$$

Taking the Laplace transform results in

$$\pounds(A(t)) = \frac{A_0 + (1 - A_0)\, g(p)}{p + \lambda - \lambda g(p)}.$$

Considering the case of constant repair time, that is Mean Time To Repair, M.T.T.R., is a, then $g(t) = \delta(t-a)$, where $\delta(t)$ is the Impulse function, resulting in

$$\begin{aligned} \pounds(A(t)) &= \frac{A_0 + (1 - A_0)\, e^{-ap}}{p + \lambda - \lambda e^{-ap}} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{(p+\lambda)^{n+1}} \left\{ A_0 e^{-apn} + (1 - A_0)\, e^{-ap(n+1)} \right\} \end{aligned} \tag{2.42}$$

and by inversion

$$A(t) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left[\begin{array}{c} A_0 e^{-\lambda(t-an)} (t-an)^n\, H(t-an) + \\ (1 - A_0)\, e^{-\lambda(t-a(n+1))} (t - a(n+1))^n\, H(t - a(n+1)) \end{array} \right]$$

where $H(x)$ is the Heaviside function. From (2.42) the residue at the dominant zero $p = 0$, of the characteristic equation $p + \lambda - \lambda e^{-ap} = 0$ for $a > 0$ and $1 + a\lambda \neq 0$,is $\frac{1}{1+a\lambda}$, hence, by utilizing the results of the previous sections, the result becomes

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left[A_0 e^{-\lambda(t-an)} (t-an)^n + (1 - A_0)\, e^{-\lambda(t-a(n+1))} (t - a(n+1))^n \right] = \frac{1}{1 + a\lambda}$$

in its region of convergence $\left|a\lambda e^{1+a\lambda}\right| < 1$ and $\forall t \in \Re$. The value of the availability limit sum is independent of the initial value A_0 and the closed form solution is independent of the value of t. It may also be seen that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda(t-an)} (t-an)^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda(t-a(n+1))} (t-a(n+1))^n = \frac{1}{1+a\lambda}$$

by putting $t-a=T$ in the second sum. Utilizing (2.10) and putting $t=-a\tau$ results in

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(a\lambda e^{a\lambda}\right)^n}{n!} (\tau+n)^n = \frac{e^{-a\lambda\tau}}{1+a\lambda} = e^{-a\lambda\tau} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r}{n!} \binom{n}{r} (a\lambda)^n (\tau+r)^n$$

whenever the double sum converges. From (2.42), a double pole occurs at $p=0$ when $1+a\lambda=0$, resulting in

$$\lim_{t\to\infty} \left(A(t) + \frac{2t}{a}\right) = \frac{2(3A_0-2)}{3}.$$

2.11 Ruin problems in compound Poisson processes.

The integro-differential equation

$$R'(t) = \frac{\alpha}{c_1} \left\{ R(t) - \int_0^t R(t-x)\, dF(x) \right\} \tag{2.43}$$

is derived by Tijms [87] and Feller [44] and has applications in collective risk theory, storage problems and scheduling of patients. Here, α is the Poisson parameter and c_1 a positive rate. Taking the Laplace transform of (2.43), it follows that

$$R^*(p) = \pounds(R(t)) = \frac{R(0)}{1-\frac{\alpha}{pc_1}(1-F^*(p))} \cdot \frac{1}{p}.$$

Given that F is a distribution concentrated at a point a, F^* is its Laplace transform, μ is the expectation of F and $R(0) = 1-k\mu$, where $k=\frac{\alpha}{c_1}$ results in

$$R^*(p) = \frac{1-k\mu}{p-k+ke^{-ap}}. \tag{2.44}$$

Comparing (2.44) with (2.2), $b = k, c = -k$ results in

$$R(t) = (1 - k\mu) \sum_{n=0}^{\infty} \frac{(-k)^n}{n!} e^{k(t-an)} (t - an)^n H(t - an).$$

The characteristic equation $p - k + ke^{-ap} = 0$ has a simple dominant zero at $p = 0$, for $1 - ak \neq 0$ and therefore

$$\sum_{n=0}^{\infty} \frac{(-k)^n}{n!} e^{k(t-an)} (t - an)^n = \frac{1}{1 - ak}, \forall t \in \Re$$

and in the region of convergence $\left|ake^{1-ak}\right| < 1$.

From (2.44) a double pole occurs at $p = 0$ when $1 - ak = 0$, therefore

$$\lim_{t \to \infty} \left(R(t) + \frac{2t(1 - k\mu)}{a} \right) = \frac{2(1 - k\mu)}{3}.$$

2.12 A grazing system.

Woodward and Wake [95] consider the differential-delay model

$$w'(t) + (r_1 - g) w(t) + ge^{-\tau r_1} w(t - \tau) = 0 \tag{2.45}$$

describing a linear continuous grazing system. $w'(t)$ represents the change of pasture mass over time t, r_1 is a constant grazing pressure, g and τ are positive constants representing growing conditions. From the work of section 2.2, the following results are inferred

$$\sum_{n=0}^{\infty} \frac{(-ge^{-\tau r_1})^n}{n!} e^{-(r_1 - g)(t-\tau)} (t - \tau)^n = \frac{e^{t\xi}}{1 + \tau(\xi - g + r_1)} \tag{2.46}$$

where ξ is the dominant zero of

$$h(p) = p - g + r_1 + ge^{-\tau(p + r_1)} \tag{2.47}$$

and in fact for $t > \tau$, the right hand side of (2.46) is a solution of (2.45). Woodward and Wake describe a neutral stability condition for

$$g - r_1 = ge^{-\tau r_1} \tag{2.48}$$

and applying (2.48) to (2.47) gives us the dominant zero, $\xi = 0$, in which case (2.46) reduces to

$$\sum_{n=0}^{\infty} \frac{(r_1 - g)^n}{n!} e^{-(r_1-g)(t-\tau)} (t-\tau)^n = \frac{1}{1+\tau(r_1 - g)},$$

representing a constant steady state solution for $r_1 > g$. The characteristic function (2.47) has a double pole for $p = -r_1$ and $g\tau = 1$, in which case, from residue theory

$$\lim_{t \to \infty} \left(w(t) - \frac{2t}{\tau} \right) = 2.$$

Other examples occur in stochastic processes, see Hall [53] and in the demographic problem of a counter model with fixed dead time, see Biswas [11].

2.13 Zeros of the transcendental equation.

Equation (2.4) is the transcendental function associated with the differential-difference equation (2.1). The zeros of this function are well documented and since many research papers have been interested in the stability of the solution of the differential-difference equation, conditions are given for the existence of complex conjugate roots with negative real part. Firstly, ξ satisfies (2.4) if and only if $\ln(\xi + b) = \ln\left(-ce^{-a\xi}\right)$; ξ complex, hence $a\xi + \ln(\xi + b) = (2n+1)\pi i + \ln c$, in which case $a\,\mathrm{Re}(\xi) = -\mathrm{Re}(\ln(b+\xi)) + \ln c$. If $a > 0 \Rightarrow$as $|\xi| \to \infty$ then $\mathrm{Re}(\xi) \to -\infty$ and if $a < 0 \Rightarrow$as $|\xi| \to \infty$ then $\mathrm{Re}(\xi) \to \infty$. Since (2.4) is an analytic function of ξ , there are therefore only a finite number of zeros to the right of any line $\mathrm{Re}(\xi) = \gamma$ for $a > 0$. Similarly, if $a < 0$, there are a finite number of zeros to the left of any line $\mathrm{Re}(\xi) = \gamma$. We may also note that as $a \to 0^+, \mathrm{Re}(\xi) \to -\infty$ unless $|\xi + b| = 1$, and as $a \to 0^-, \mathrm{Re}(\xi) \to \infty$ unless $|\xi + b| = 1$. A proof of the following theorem may be seen in Bellman and Cooke [7].

Theorem 5 *A necessary and sufficient condition for the characteristic function (2.4) to have zeros with negative real part is:*

1. $ab > 1$,

2. $-ab < ac < \sqrt{\eta^2 + (ab)^2}$, *where* η *is the zero of* $\eta + ab\tan\eta = 0; 0 < \eta < \pi$ *or* $\eta = \frac{\pi}{2}$ *if* $ab = 0$.

Lemma 6 *The characteristic function (2.4) has at most 2 real zeros.*

Proof: From (2.4), let $z = ap$, $\alpha = ab$, $\beta = ac$, $a > 0$, $g(ap) = G(z)$ and so

$$G(z) = z + \alpha + \beta e^{-z}. \qquad (2.49)$$

Putting $Y(z) = \frac{(z+\alpha)e^z}{\beta} = -1$ then at the turning point $z^* = -(1+\alpha)$, we have $Y(z^*) = \frac{1}{\beta e^{1+\alpha}}$, hence since, $|\beta e^{1+\alpha}| < 1$ then if $Y(z^*) < -1$ there are at most 2 real zeros as can be seen from figure 2.3.

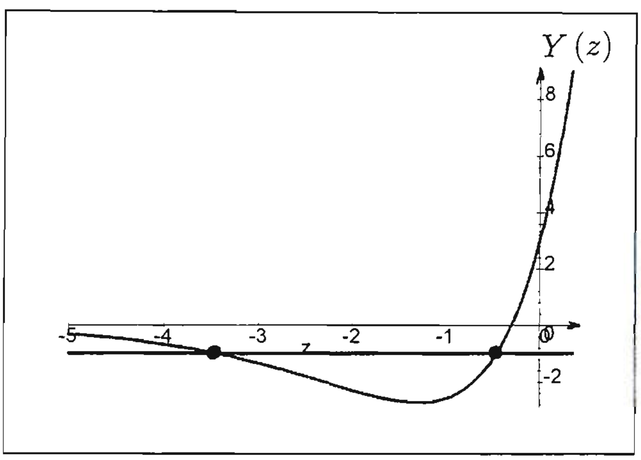

Figure 2.3: The real zeros of $G(z)$ defined in (2.49).

Lemma 7 *The transcendental function (2.49) has a finite number of complex zeros with positive real part.*

Proof: Let $z = x + iy$, then from (2.49)

$$\left.\begin{aligned} x + \alpha + \beta e^{-x}\cos y = 0 \\ y - \beta e^{-x}\sin y = 0 \end{aligned}\right\}. \qquad (2.50)$$

The zeros of $G(z)$ depend continuously on β, and for $\beta > 0$ all zeros will be in the half plane $\mathrm{Re}(z) \leq \beta$. If $G(z) = G'(z) = 0$ there will be a double zero at $z+1+\alpha = 0$ and therefore zeros cannot bifurcate or merge, as β varies, in the half plane $x > -1$. Utilizing similar arguments to that of Cooke and Grossman [35] it can be seen that if $z = z(\beta)$ is an isolated simple zero with $\mathrm{Re}(z) \geq 0$, then it moves to the right of the half plane for increasing β, since

$$\frac{dz}{d\beta} = -\frac{dG/d\beta}{dG/dz} = \frac{z+\alpha}{\beta(z+1+\alpha)} \text{ and}$$

$$\mathrm{Re}\left(\frac{dz}{d\beta}\right) = \frac{(x+\alpha)(x+\alpha+1)+y^2}{(x+\alpha+1)^2+y^2} > 0.$$

Suppose a pure imaginary zero exists, then $z = iy$ and a manipulation of (2.50) gives

$$y^2 = \beta^2 - \alpha^2, \text{ and } \alpha + \beta\cos\sqrt{\beta^2 - \alpha^2} = 0.$$

For β increasing from α to ∞ there exists an increasing sequence $0 < \beta_1(\alpha) < \beta_2(\alpha) < ...$,

$$\text{so that } \lim_{k\to\infty} \beta_k(\alpha) = \infty \text{ with } \sin\sqrt{\beta^2 - \alpha^2} > 0.$$

Here $\beta \in (\beta_k(\alpha), \beta_{k+1}(\alpha))$ and equation (2.49) has precisely k complex zeros with positive real part. Also, whenever $\beta = \beta_k(\alpha)$ there exists a pair of complex conjugate imaginary zeros $\pm iy_k$ such that

$$(4k+1)\frac{\pi}{2} < y_k < (2k+1)\pi; k = 0, 1, 2, 3, ... \, .$$

It appears, from (2.50), that a zero must remain in the region where $\sin y > 0$ and $\cos y < 0$ and in the specific case where $\alpha = 0$ then $\beta = y_k = (4k+1)\frac{\pi}{2}; k = 0, 1, 2, 3, ...$.

2.14 Numerical examples.

The zeros of the characteristic function (2.4) can be located using Mathematica. Let $p = x+iy$ then

$$\left.\begin{aligned} \mathrm{Re}(x,y) = 0 &= x + b + ce^{-ax}\cos ay \\ \mathrm{Im}(x,y) = 0 &= y - ce^{-ax}\sin ay \end{aligned}\right\} \quad (2.51)$$

and if for any x, y is a solution then so is $-y$. Therefore non-real zeros occur in complex conjugate pairs. Putting $t = -a\tau$ in (2.10), we may write

$$\sum_{n=0}^{\infty} \frac{(ace^{ab})^n (\tau+n)^n}{n!} = \frac{e^{-a\tau(b+\xi)}}{1+ab+a\xi} = e^{-ab\tau} \sum_{n=0}^{\infty} \frac{(ab)^n}{n!} \sum_{r=0}^{n} \binom{n}{r} \left(\frac{c}{b}\right)^r (\tau+r)^n \tag{2.52}$$

where ξ is the dominant zero of (2.51). As an example for $(a,b,c) = (8,-1,6)$, $\xi = .997954$ and for $\tau = 2$, from (2.52), the sum to six significant figures is 1.050472.

In the next section we look at the related works of Euler, Jensen and Ramanujan. We shall describe their techniques and give in detail, a description of Cohen's modification and extension, and a solution to Conolly's problem.

2.15 Euler's work.

Euler's work is related to our equation (2.10); his work was published in Latin in 1779 and we give an English translation of the main points that are pertinent to our work. Euler [41], see also [42], considers a series given by Lambert and investigates several of its notable properties. Euler rewrites the Lambert series in the form

$$S = 1 + nv + n \sum_{r=1}^{\infty} \frac{v^{r+1}}{(r+1)!} \prod_{j=0}^{r-1} (n + (j+1)\,\alpha + (r-j)\,\beta) \tag{2.53}$$

and given that for constants α and β we may put

$$x^\alpha - x^\beta = (\alpha - \beta)\, v x^{\alpha+\beta} \tag{2.54}$$

then $S = x^n$. Euler makes several observations about (2.53) for particular cases.

1. Take the constant and the factor n to the left hand side of (2.53), then investigate the limit as $n \to 0$, such that

$$\ln x = v + \sum_{r=1}^{\infty} \frac{v^{r+1}}{(r+1)!} \prod_{j=0}^{r-1} ((j+1)\,\alpha + (r-j)\,\beta) \tag{2.55}$$

where from a rearrangement of (2.54)

$$v = \frac{x^{-\beta} - x^{-\alpha}}{\alpha - \beta}. \tag{2.56}$$

2. For $\beta = 0$, from(2.53) and utilizing $\ln(1 - \alpha v) = -\alpha \ln x$, Euler obtains

$$\ln(1 - \alpha v) = -\sum_{j=1}^{\infty} \frac{(\alpha v)^j}{j!}.$$

3. For $\beta = \alpha$, Euler writes

$$\frac{x^n - 1}{n} = v + \sum_{r=1}^{\infty} \frac{v^{r+1}}{(r+1)!} \prod_{j=0}^{r-1} (n + \alpha(r+1)) \tag{2.57}$$

and again taking the limit as $n \to 0$, Euler obtains, from (2.57)

$$\ln x = \sum_{r=0}^{\infty} \frac{v(\alpha v(r+1))^r}{(r+1)!},$$

moreover, substituting $\ln x = vx^{\alpha}$ we obtain

$$x^{\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha v(r+1))^r}{(r+1)!}. \tag{2.58}$$

Now substitute $\alpha v = u$ and $x^{\alpha} = y$, and from (2.58)

$$y = \sum_{r=0}^{\infty} \frac{(u(r+1))^r}{(r+1)!}, \text{ where } uy = \ln y. \tag{2.59}$$

Multiply both sides of (2.59) by u, differentiate with respect to u, multiply by u, and then again replacing $uy = \ln y$, Euler obtains

$$\frac{\ln y}{1 - \ln y} = \sum_{r=1}^{\infty} \frac{(ur)^r}{r!}, \tag{2.60}$$

in its region of convergence $|u| < e^{-1}$. Euler gives some numerical values of (2.60) and in particular for $\ln y = 1/2$, from (2.60) he obtains

$$1 = \sum_{r=0}^{\infty} \frac{u^{r+1}(r+1)^r}{r!} \quad \text{where } u = \frac{1}{2}e^{-\frac{1}{2}}.$$

From this work, Euler states his theorem:

$$\frac{x^s}{1-\ln x} = \sum_{n=0}^{\infty} \frac{(n+s)^n}{n!} \left(\frac{\ln x}{x}\right)^n, \tag{2.61}$$

and (2.61) is identical to our equation (2.13) upon putting $sa = -t$ and $x = e^{-ab}$.

2.16 Jensen's work.

Jensen's [63] work is related to our equation (2.10); his work was published in French in 1902 and we give an English translation of the main points that are pertinent to our work. Jensen, who appears to have been unaware of Euler's work, obtains the equivalent of identity (2.13) by an application of the Lagrange inversion formula. Lagrange's inversion formula, circa 1770, arose as a tool in the solution of implicit equations or the reversion of series.

Theorem 8 *:*

$$\phi(z) = \phi(a) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{d^{n-1}}{dz^{n-1}} \left[\phi'(z) f^{(n)}(z)\right]_{z=a}$$

where $z = a + wf(z)$.

An alternative form of theorem 8, given by Jensen, who says he learned it from Hermite, circa 1881, but apparently known to Jacobi, at least as early as 1826 is;

Theorem 9 *:*

$$\frac{g(z)}{1-wf'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{d^n}{dz^n} \left[g(z) f^{(n)}(z)\right]_{z=a}$$

where $z = a + wf(z)$.

If we take the partial derivative with respect to w in theorem 9 and put $g = \phi' f$ we obtain theorem 8. Jensen puts $\phi(z) = e^{\alpha z}, f(z) = e^{\beta z}$ and obtains, from theorems 8 and 9,

$$e^{\alpha z} = \sum_{j=0}^{\infty} \frac{\alpha (\alpha + j\beta)^{j-1} u^j}{j!} \quad \text{and} \tag{2.62}$$

$$\frac{e^{\alpha z}}{1 - \beta z} = \sum_{j=0}^{\infty} \frac{(\alpha + j\beta)^j u^j}{j!} \tag{2.63}$$

where $u = ze^{-\beta z}$; moreover if we put $\beta z = \rho$ and $\alpha = \beta \tau$ in (2.63) we obtain (2.13). Now, Jensen puts $\alpha = a$ and $\alpha = b$ in (2.62) and multiplies the two series together to obtain

$$e^{(a+b)z} = \sum_{j=0}^{\infty} \frac{(a+b)(a+b+j\beta)^{j-1} u^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{a(a+j\beta)^{j-1} u^j}{j!} \sum_{j=0}^{\infty} \frac{b(b+j\beta)^{j-1} u^j}{j!}$$

$$= \sum_{j=0}^{\infty} \left(\sum_{\nu=0}^{n} \binom{n}{\nu} \frac{a(a+\nu\beta)^{\nu-1} b(b+(n-\nu)\beta)^{n-\nu-1} u^j}{j!} \right). \tag{2.64}$$

Equating the coefficients of u^j in (2.64), Jensen obtains

$$(a+b)(a+b+n\beta)^{n-1} = \sum_{\nu=0}^{n} \binom{n}{\nu} a(a+\nu\beta)^{\nu-1} b(b+(n-\nu)\beta)^{n-\nu-1}$$

and similarly, from (2.63)

$$(a+b+n\beta)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} a(a+\nu\beta)^{\nu-1} b(b+(n-\nu)\beta)^{n-\nu} \tag{2.65}$$

which is one form of the celebrated generalization of the Binomial theorem given by Abel in 1839. Putting $x = b + n\beta, a = \alpha$ and $\beta = -\beta$ in (2.65) we obtain

$$(x+\alpha)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} \alpha(a-\nu\beta)^{\nu-1} (x+\nu\beta)^{n-\nu} \tag{2.66}$$

and in a note dedicated to the memory of Abel, Lie noted that (2.66) is in fact a special case of a formula noted by Cauchy:

$$(x+\alpha+n)^n-(x+\alpha)^n=\sum_{\nu=0}^{n-1}\binom{n}{\nu}\alpha(\alpha+n-\nu)^{n-\nu-1}(x+\nu)^\nu.$$

2.17 Ramanujan's question.

In the collected papers of Ramanujan, edited by Hardy, Seshu Aiyar and Wilson [56] on page 332, Ramanujan states in question 738, if

$$\phi(x)=\sum_{r=0}^{\infty}\frac{x^r(r+1)^{r-1}e^{-x(r+1)}}{r!} \tag{2.67}$$

show that $\phi(x)=1$ for $0<x<1$ and that $\phi(x)\neq 1$ for $x>1$ and also find

$$\lim_{\varepsilon\to 0}\left(\frac{\phi(1+\varepsilon)-\phi(1)}{\varepsilon}\right).$$

We can see that (2.67) is identical to (2.62) when $\alpha=1$ and $\beta=1$. B.C. Berndt [9] gives an excellent short historical account of (2.62) and (2.63).

2.18 Cohen's modification and extension.

Putting $\alpha=-1$ and $\beta=-a$ in (2.62) and (2.63) Cohen [30], in his notation, extends the results of the two sums

$$\sum_{n=0}^{\infty}\frac{w^n(1+an)^{n-1}}{n!}=e^{-z}\text{ and} \tag{2.68}$$

$$\sum_{n=0}^{\infty}\frac{w^n(1+an)^{n}}{n!}=\frac{e^{-z}}{1+az} \tag{2.69}$$

where $w=-ze^{az}$. Cohen considers

$$(Dx)\{(1-x^a)^n\}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+ak)x^{ak}\text{ and}$$

$$(Dx)^n\{(1-x^a)^n\}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+ak)^n x^{ak}$$

where $D=\frac{d}{dx}$. From

$$\int_{x=0}^{1} x^{a\beta-1}(Dx)^n\{(1-x^a)^n\}\,dx=\int_{x=0}^{1} x^{a\beta-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+ak)^n x^{ak}dx$$

Cohen obtains

$$\sum_{k=0}^{n}(-1)^k\beta\binom{n}{k}\frac{(1+ak)^n}{k+\beta}=\frac{(1-\beta a)^n n!}{(\beta+1)_n}$$

which corrects a minor misprint error in his equation (1.9). Cohen then considers

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}(-1)^k\beta\binom{n}{k}\frac{(1+ak)^n}{k+\beta}=\sum_{n=0}^{\infty}z^n\left\{\frac{(1-\beta a)^n}{(\beta+1)_n}\right\}$$

and applying the transformation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/s]}f(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}f(n+sk,k)$$

he obtains the new result

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-ze^{az})^k\beta(1+ak)^k}{k!\,(k+\beta)} &= e^{-z}\sum_{n=0}^{\infty}z^n\left\{\frac{(1-\beta a)^n}{(\beta+1)_n}\right\}\\ &= e^{-z}\,{}_1F_1\left[\begin{array}{c|c}1 & \\ \beta+1 & z(1-\beta a)\end{array}\right],\end{aligned}$$

which upon putting $\beta a=1$ we obtain the result (2.68) and letting $\beta\to\infty$ we obtain the identity (2.69). Cohen goes on, in this article and in [31], to obtain other results, especially related to Laguerre, Hermite and other special functions of mathematical physics. It appears that other results may also be obtained by gainfully employing the ideas of Cohen. The following is one such result. Let

$$(xD) = xD\left\{x^{\alpha}\left(1-x^{a}\right)^{n}\right\} = \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(\alpha+ak)\,x^{\alpha+ak} \text{ hence,}$$

$$(xD)^{n-m}\left\{x^{\alpha}\left(1-x^{a}\right)^{n}\right\}\Big]_{x=1} = \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(\alpha+ak)^{n-m} \text{ and similarly}$$

$$(xD)^{m-n}\left\{x^{\beta}\left(1-x^{b}\right)^{m}\right\}\Big]_{x=1} = \sum_{\rho=0}^{m}(-1)^{\rho}\binom{m}{\rho}(\beta+b\rho)^{m-n}.$$

We can now write, from above

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{m}y^{n}}{m!n!}(xD)^{n-m}\left\{x^{\alpha}\left(1-x^{a}\right)^{n}\right\}(xD)^{m-n}\left\{x^{\beta}\left(1-x^{b}\right)^{m}\right\}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{m}y^{n}}{m!n!}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(\alpha+ak)^{n-m}\sum_{\rho=0}^{m}(-1)^{\rho}\binom{m}{\rho}(\beta+b\rho)^{m-n}. \qquad (2.70)$$

On the right hand of (2.70) we can use the transformation suggested by Cohen [31], namely

$$\left.\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(n,k) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}f(n+k,k)\\ \sum_{m=0}^{\infty}\sum_{\rho=0}^{m}f(m,\rho) &= \sum_{\rho=0}^{\infty}\sum_{m=0}^{\infty}f(n+\rho,\rho)\end{aligned}\right\} \qquad (2.71)$$

which after some manipulation gives

$$\sum_{k=0}^{\infty}\sum_{\rho=0}^{\infty}\frac{(-1)^{k+\rho}y^{k}x^{\rho}}{k!\rho!}\left(\frac{\alpha+ak}{\beta+b\rho}\right)^{k-\rho}e^{y\left(\frac{\alpha+ak}{\beta+b\rho}\right)+x\left(\frac{\beta+b\rho}{\alpha+ak}\right)}. \qquad (2.72)$$

Also, from the left hand side of (2.70) we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}y^{n}}{n!\beta^{n}}\binom{n}{k}(\alpha+ak)^{n}+\sum_{m=1}^{\infty}(-b\alpha x)^{m} \qquad (2.73)$$

and applying the transformation (2.71) to (2.73), it reduces to

$$\frac{1-\frac{abxy}{\alpha\beta}}{\left(1+\frac{ay}{\beta}\right)\left(1+\frac{bx}{\alpha}\right)}. \tag{2.74}$$

From (2.72) and (2.74) we obtain the result, whenever the double series converges,

$$\sum_{k=0}^{\infty}\sum_{\rho=0}^{\infty}\frac{(-1)^{k+\rho}y^k x^\rho}{k!\rho!}\left(\frac{\alpha+ak}{\beta+b\rho}\right)^{k-\rho}e^{y\left(\frac{\alpha+ak}{\beta+b\rho}\right)+x\left(\frac{\beta+b\rho}{\alpha+ak}\right)}=\frac{1-\frac{abxy}{\alpha\beta}}{\left(1+\frac{ay}{\beta}\right)\left(1+\frac{bx}{\alpha}\right)}.$$

2.19 Conolly's Problem.

Brian Conolly [34] proposes, "for $\lambda \in [0,1]$ and $m \geq 0$, let

$$S_m(\lambda)=\sum_{n\geq 1}\frac{(\lambda n)^{n-m}}{n!}e^{-\lambda n}. \tag{2.75}$$

Show that

$$S_0(\lambda)=\lambda/(1-\lambda),\ S_1(\lambda)=1,\ S_2(\lambda)=1/\lambda-1/2, \text{ and } S_3(\lambda)=1/\lambda^2-3/(4\lambda)+1/6\text{''}.$$

The infinite sum (2.75), for $m=0$, is a very specialized case of the generalized sum (2.13). For $t=0,\ b=-1$, and $a=\lambda$ we have, from (2.13)

$$S_0(\lambda)=\sum_{n=1}^{\infty}\frac{(\lambda n)^n}{n!}e^{-\lambda n}=\frac{\lambda}{1-\lambda} \tag{2.76}$$

with convergence region $\left|\lambda e^{1-\lambda}\right|<1$, which is different than that given by Conolly and indeed, we have previously shown that (2.76) diverges on the boundary $\lambda=1$, and converges on the boundary $\lambda=-.278464...$.

From (2.75), writing the exponential in series form, we have

$$S_m(\lambda)=\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r(\lambda n)^{n+r-m}}{n!r!}. \tag{2.77}$$

Expanding the double infinite sum in (2.77) term by term, and then summing diagonally from the top left hand corner, thus collecting coefficients of λ^{n-m}, we have

$$S_m(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^{n-m}}{n!} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^{n-m}. \tag{2.78}$$

For the special case of $m = 0$, from (2.78)

$$S_0(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^n, \tag{2.79}$$

and since it is well known that the inner sum is equal to $n!$, we have from (2.79)

$$S_0(\lambda) = \sum_{n=1}^{\infty} \lambda^n = \frac{\lambda}{1-\lambda}$$

which is identical to (2.76). From (2.78), for $m \geq 1$

$$\begin{aligned} S_m(\lambda) &= \sum_{n=1}^{m} \frac{\lambda^{n-m}}{n!} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^{n-m} + \\ & \sum_{n=m+1}^{\infty} \frac{\lambda^{n-m}}{n!} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^{n-m}, \end{aligned} \tag{2.80}$$

in the second term, the inner sum

$$\sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^{n-m} = 0 \text{ for } n \geq m+1$$

and therefore from (2.80)

$$S_m(\lambda) = \sum_{n=1}^{m} \frac{\lambda^{n-m}}{n!} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^{n-m}, \text{ for } m \geq 1. \tag{2.81}$$

The convergence region of (2.75) for $m \geq 1$ is as previous however, by an application of the limit comparison test and applying Stirling's approximation for $n!$, (2.75) converges on the boundary $\lambda = 1$, and $\lambda = -.278464...$.

Putting $m = 1, 2$ and 3 into (2.81) we obtain Conolly's results. Obviously other $S_m(\lambda)$ results are available from (2.81), three other such values are;

$$S_4(\lambda) = \frac{1}{\lambda^3} - \frac{7}{8\lambda^2} + \frac{11}{36\lambda} - \frac{1}{4!},$$

$$S_5(\lambda) = \frac{1}{\lambda^4} - \frac{15}{16\lambda^3} + \frac{85}{216\lambda^2} - \frac{25}{288\lambda} + \frac{1}{5!} \text{ and}$$

$$S_6(\lambda) = \frac{1}{\lambda^5} - \frac{31}{32\lambda^4} + \frac{575}{1296\lambda^3} - \frac{415}{3456\lambda^2} + \frac{137}{7200\lambda} - \frac{1}{6!}.$$

An alternative procedure, via a recurrence relation, for determining closed form representations of (2.75) is the following. Differentiate (2.75) with respect λ, algebraically manipulate the terms, and finally we obtain the recurrence relation

$$\frac{d}{d\lambda}(S_m(\lambda)) + \frac{m}{\lambda}S_m(\lambda) = \frac{1-\lambda}{\lambda^2}S_{m-1}(\lambda). \tag{2.82}$$

We can obtain an integrating factor of (2.82) such that

$$\frac{d}{d\lambda}(\lambda^m S_m(\lambda)) = \lambda^m \left(\frac{1-\lambda}{\lambda^2}\right) S_{m-1}(\lambda) \text{ and}$$

$$\lambda^m S_m(\lambda) = \int \lambda^{m-2}(1-\lambda) S_{m-1}(\lambda)\, d\lambda, \tag{2.83}$$

given that for $\lambda = 0$ the constant of integration is zero and for $m = 0$, we have the value $S_0(\lambda) = \lambda/(1-\lambda)$. Putting $m = 1, 2$ and 3 into (2.83) we obtain Conolly's results. For $m = 0$, and using (2.82) we have

$$\frac{d}{d\lambda}(S_0(\lambda)) = \frac{1-\lambda}{\lambda^2}S_{-1}(\lambda),$$

and since $\frac{d}{d\lambda}(S_0(\lambda)) = \frac{1}{(1-\lambda)^2}$ we get the additional result

$$S_{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{(\lambda n)^{n+1}}{n!} e^{-\lambda n} = \frac{\lambda^2}{(1-\lambda)^3}, \tag{2.84}$$

which we shall generalize further in chapter 5. I consider this procedure less transparent than the first procedure leading to (2.81). The sum (2.13) will be generalized further in chapter 5.

Chapter 3

Bürmann's theorem

A detailed proof of Bürmann's theorem is given in this chapter. For the principle series (2.14), obtained in chapter two, it will be shown that the remainder term of the series vanishes for large n, and discuss in detail its radius of convergence.

3.1 Introduction.

Bürmann's theorem [22] essentially allows, under various conditions, for the expansion of one function in terms of positive powers of another function. A sketch proof of Bürmann's theorem is given in Whittaker and Watson [92] and at a first look it appears that the proof contains a line which seems misprinted. The third last line on page 129 of [92] is

$$\ldots\int_a^z f'(\xi)\,d\xi = \frac{1}{2\pi i}\int_a^z\int_\gamma \frac{f'(t)\,\phi'(\xi)\,dt.d\xi}{\phi(t)-\phi(\xi)}$$

and it may seem that $\phi'(\xi)$ in the numerator ought to be $\phi'(t)$ arising out of applying Cauchy's integral to the $f'(\xi)$ function in the left hand side. This point will be clarified in the author's proof. Whittaker and Watson, list Bürmann's theorem and Lagrange's theorem as two separate results, however according to Henrici [58], except for matters of notation they are identical. A formal proof of the Lagrange-Bürmann theorem, that is different than the author's, is given by Henrici [59] and a part of his proof is based on elementary notions in the theory of matrices. We also show that the remainder term of Bürmann's theorem when applied to the sum (2.14), obtained in chapter two, goes to zero.

3.2 Bürmann's theorem and proof.

The following treatment is based on that given by Whittaker and Watson, however there are some deficiencies and obscurities which we shall attempt to clarify. In our proof we shall suppose that $\phi(a) = 0$, which involves no loss of generality, but does shorten the formulae. If $\phi(a) \neq 0$ we simply replace $\phi(z)$ by $\phi(z) - \phi(a)$ throughout the theorem as set out below.

Theorem 10 *Let $\phi(z)$ be a simple function in a domain D, zero at a point a of D, and let*

$$\theta(z) = \frac{z-a}{\phi(z)}\ ,\theta(a) = \frac{1}{\phi'(a)}.$$

If $f(z)$ is analytic in D then $\forall z \in D$

$$f(z) = f(a) + \sum_{r=1}^{n} \frac{\{\phi(z)\}^r}{r!} \frac{d^{r-1}}{dt^{r-1}} \left[f'(t) \{\theta(t)\}^r \right]_{t=a} + R_{n+1} \tag{3.1}$$

$$\textit{where } R_{n+1} = \frac{1}{2\pi i} \int_{\Gamma} d\xi \int_{C} \left[\frac{\phi(\xi)}{\phi(t)} \right]^n \frac{f'(t)\,\phi'(\xi)}{\phi(t) - \phi(\xi)} dt. \tag{3.2}$$

The ξ integral is taken along a contour Γ in D from a to z, and the t integral along a closed contour C in D encircling Γ once positively.

The following **properties** of simple functions are required for the proof of Bürmann's theorem and may be found in the book by Titchmarsh [88]:

1. A function $\phi(z)$ of a complex variable z is called simple in a domain D if it is analytic in D and takes no value twice in D.

2. If $\phi(z)$ is simple in a domain D then $\phi'(z) \neq 0$ in D.

3. The inverse function $\phi^{-1}(w)$ exists and is simple in D_w, where D_w is the map of D in the w-plane by $w = \phi(z)$.

4. A domain D means an open connected set of points in the plane; that is , every point of D has a neighbourhood, $\aleph$, in D and every two points of D can be joined by a continuous curve in D.

Proof of Bürmann's theorem.

The proof will be given in five parts.

(i). Let C_w be the map of C by $w = \phi(z)$, and D_w the map of D . Since ϕ is simple, C_w is a closed contour and D_w is a domain containing it. Also since C encircles every ξ of Γ, C_w encircles $\phi(\xi)$ for all such ξ. Now $g(w) = f'\left(\phi^{-1}(w)\right)$ is analytic in D_w, since ϕ^{-1} is analytic in D_w by property 3, and f' is analytic in D the map of D_w by $z = \phi^{-1}(w)$. So Cauchy's integral gives, for every ξ of Γ

$$g(\phi(\xi)) = \frac{1}{2\pi i} \int\limits_{C_w} \frac{g(s)\,ds}{s - \phi(\xi)} = \frac{1}{2\pi i} \int\limits_{C} \frac{g(\phi(t))\,\phi'(t)}{\phi(t) - \phi(\xi)} dt$$

upon changing the variable by $s = \phi(t)$, so that t goes along C. But $g(\phi(t)) = f'(\phi^{-1}(\phi(t))) = f'(t)$ and similarly for $g(\phi(\xi))$; so that, for every ξ on Γ

$$f'(\xi) = \frac{1}{2\pi i}\int\limits_C \frac{f'(t)\,\phi'(t)}{\phi(t)-\phi(\xi)}dt. \tag{3.3}$$

(ii). Each ξ of D has a neighbourhood $\aleph(\xi)$ in D. Now, by Taylor's theorem, for $t \in \aleph(\xi)$ we have

$$\phi(t) = \phi(\xi) + (t-\xi)\,\phi'(\xi) + \frac{(t-\xi)^2}{2!}\phi''(\xi) + \frac{(t-\xi)^3}{3!}\phi'''(\xi) + \ldots$$

and

$$\phi'(t) = \phi'(\xi) + (t-\xi)\,\phi''(\xi) + \frac{(t-\xi)^2}{2!}\phi'''(\xi) + \frac{(t-\xi)^3}{3!}\phi''''(\xi) + \ldots.$$

Since $\phi(z)$ is simple in D and hence in the subdomain $\aleph(\xi)$, $\phi(t) - \phi(\xi) \neq 0\ \forall t \in \aleph(\xi) - \{\xi\}$ so that

$$\frac{\phi'(t)-\phi'(\xi)}{\phi(t)-\phi(\xi)} = \frac{\phi''(\xi) + \frac{(t-\xi)}{2!}\phi'''(\xi)}{\phi'(\xi) + \frac{(t-\xi)}{2!}\phi''(\xi)}. \tag{3.4}$$

Now $\phi'(\xi) \neq 0$ in D by property 2. Hence, by Knopp [68] page 180, the quotient in (3.4) of power series is expressible as a power series for $|t-\xi|$ sufficiently small, so that it is an analytic function of t in some $\aleph$ of ξ. Further the quotient on the left hand side of (3.4) is a quotient of a function of t analytic in D, with denominator non-zero in $D - \{\xi\}$ since ϕ is simple in D. So the quotient is analytic in $D - \{\xi\}$, as well as analytic in a neighbourhood of ξ. Hence (3.4) is analytic in the whole of D.

(iii). It therefore follows that

$$\frac{\phi'(t)-\phi'(\xi)}{\phi(t)-\phi(\xi)}f'(t)$$

is analytic in D, being a product of functions analytic in D. So by Cauchy's theorem

$$\int\limits_C \frac{\phi'(t)-\phi'(\xi)}{\phi(t)-\phi(\xi)}f'(t)\,dt = 0,$$

and this with (3.3) , gives for each ξ on Γ

$$f'(\xi) = \frac{1}{2\pi i}\int\limits_C \frac{f'(t)\,\phi'(\xi)}{\phi(t)-\phi(\xi)}dt \tag{3.5}$$

(iv). Since a is on Γ and not on C, $\phi(t) \neq \phi(a)$ for t on C, since ϕ is simple. But $\phi(a) = 0$; so $\phi(t) \neq 0$ for t on C. For such t

$$\frac{1}{\phi(t) - \phi(\xi)}\left[1 - \left(\frac{\phi(\xi)}{\phi(t)}\right)^n\right] = \frac{1}{\phi(t)}\sum_{r=0}^{n-1}\left(\frac{\phi(\xi)}{\phi(t)}\right)^r$$

by summing the geometric series on the right;

$$\frac{1}{\phi(t) - \phi(\xi)} = \frac{\phi(\xi)^n}{\phi(t)^n[\phi(t) - \phi(\xi)]} + \frac{1}{\phi(t)}\sum_{r=0}^{n-1}\left(\frac{\phi(\xi)}{\phi(t)}\right)^r.$$

For each z in D we have, using (3.5)

$$f(z) - f(a) = \int_\Gamma f'(\xi)\,d\xi = \frac{1}{2\pi i}\int_\Gamma d\xi \int_C \frac{\phi'(t) - \phi'(\xi)}{\phi(t) - \phi(\xi)}dt$$

$$= \frac{1}{2\pi i}\int_\Gamma \phi'(\xi)\,d\xi \int_C \frac{f'(t)}{\phi(t)}\sum_{r=0}^{n-1}\left(\frac{\phi(\xi)}{\phi(t)}\right)^r dt + R \tag{3.6}$$

where

$$R = \frac{1}{2\pi i}\int_\Gamma \phi'(\xi)\,d\xi \int_C \frac{f'(t)\,\phi(\xi)^n}{\phi(t)^n[\phi(t) - \phi(\xi)]}dt$$

and this agrees with the expression for R_{n+1} stated in the theorem at (3.2).

(v). The function

$$\theta(z) = \frac{z-a}{\phi(z)} = \frac{z-a}{\phi(z) - \phi(a)}, \theta(a) = \frac{1}{\phi'(a)}$$

is defined and analytic throughout D being a quotient of analytic functions with non-zero denominator since $\phi(z) - \phi(a) \neq 0$ for z in $D - \{a\}$, and being continuous at a. Taking the summation in (3.6) outside the integration, the typical term is

$$\frac{1}{2\pi i}\int_\Gamma d\xi \int_C \frac{f'(t)\,\phi(\xi)^r\,\phi'(\xi)}{\phi(t)^{r+1}}dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f'(t)}{\phi(t)^{r+1}}dt \int_\Gamma \phi(\xi)^r\,\phi'(\xi)\,d\xi$$

$$= \frac{1}{2\pi i}\int_C \frac{f'(t)\,\theta(t)^{r+1}}{(t-a)^{r+1}}dt\frac{\phi(z)^{r+1}-\phi(a)^{r+1}}{r+1}$$
$$= \frac{1}{r!}\frac{d^r}{dt^r}\left[f'(t)\{\theta(t)\}^{r+1}\right]_{t=a}\frac{\phi(z)^{r+1}}{r+1}$$
$$= \frac{\phi(z)^{r+1}}{(r+1)!}\frac{d^r}{dt^r}\left[f'(t)\{\theta(t)\}^{r+1}\right]_{t=a}$$

by Cauchy's integral for the r^{th} derivative. This gives the expression (3.1) stated in the theorem by means of (3.6), after replacing $r-1$ instead of r and so completing the proof.

3.2.1 Applying Bürmann's theorem.

From Chapter two, we have evaluated

$$\frac{e^{xz}}{1-z} = 1+\sum_{r=1}^{\infty}\frac{(ze^{-z})^r}{r!}(x+r)^r, \tag{3.7}$$

and the hypothesis of Bürmann's theorem requires that ϕ be simple in a domain D containing the origin, and that f be analytic in D. Now

$$f(z)=\frac{e^{xz}}{1-z},\ \phi(z)=ze^{-z},\ \theta(z)=\frac{z}{\phi(z)}=e^z,\ \text{and } \theta(a)=\frac{1}{\phi'(a)}=1 \text{ for } a=0.$$

For $f(z)$ to be analytic in D the disc $|z|<1$ will be adequate. The definition of a simple function is given by property 1, and the following lemma proves that ϕ is simple.

Lemma 11 *The function $\phi(z)=ze^{-z}$ is simple in the disc $D=\left\{z:|z|<\frac{1}{2}\right\}$.*

Proof: Assume on the contrary that $\phi(z)$ is not simple in D, so that there are unequal z and z' such that

$$ze^{-z}=z'e^{-z'}$$

for $|z|<\frac{1}{2}$ and $|z'|<\frac{1}{2}$. Now

$$\frac{z}{z'}=e^{z-z'},$$

$$\frac{z-z'}{z'}=e^{z-z'}-1=\sum_{r=1}^{\infty}\frac{\left(z-z'\right)^r}{r!},$$

$$\frac{1}{z'} = \sum_{r=1}^{\infty} \frac{\left(z - z'\right)^{r-1}}{r!} \text{ since } z - z' \neq 0,$$

$$\begin{aligned} \text{and so } \frac{1}{|z'|} &\leq \sum_{r=1}^{\infty} \frac{\left|z - z'\right|^{r-1}}{r!} \\ &= 1 + \sum_{r=2}^{\infty} \frac{\left|z - z'\right|^{r-1}}{r!}. \end{aligned}$$

Using the fact that $\sum\limits_{r=2}^{\infty} \frac{1}{r^2} \leq \sum\limits_{r=2}^{\infty} \frac{1}{2^{r-1}}$ we have

$$\begin{aligned} \frac{1}{|z'|} &\leq 1 + \sum_{r=2}^{\infty} \frac{\left|z - z'\right|^{r-1}}{2^{r-1}} \\ &= 1 + \frac{\frac{1}{2}\left|z - z'\right|}{1 - \frac{1}{2}\left|z - z'\right|}; \end{aligned}$$

since the geometric series is convergent, and by the ratio test we require $\frac{1}{2}\left|z - z'\right| < 1$ so that we may write

$$\frac{1}{|z'|} \leq \frac{1}{1 - \frac{1}{2}\left|z - z'\right|}$$

and we want

$$\begin{aligned} \left|z'\right| &> 1 - \frac{1}{2}\left|z - z'\right| \\ &> 1 - \frac{1}{2} \\ &> \frac{1}{2}. \end{aligned}$$

This inequality contradicts the assumption $\left|z'\right| < \frac{1}{2}$, hence $\phi(z)$ is both simple and analytic in D. All conditions of Bürmann's theorem are now met. Now we need to show that the remainder goes to zero for large n.

3.2.2 The remainder.

To obtain an infinite series for $f(z_0)$ from Bürmann's theorem we need to choose Γ and C so that $R_{n+1} \to 0$ as $n \to \infty$. Choose a fixed z_0 such that $|z_0| \le c_1$. Then Γ can be the line from 0 to z_0; and C can be the circle with radius $c_2 > c_1$ centred at the origin, described once positively. From the remainder R_{n+1}, defined in (3.2), consider, since $\phi(z) = ze^{-z}$,

$$\begin{aligned}\left|\frac{\phi(\xi)}{\phi(t)}\right| &= \left|\frac{\xi}{t}e^{t-\xi}\right| \\ &\le \frac{|z_0|}{|t|}e^{\text{real}(t-\xi)}\end{aligned}$$

where, real $(t-\xi) \le |t-\xi| \le |t| + |\xi| < c_2 + c_1 = c_3$,

$$\left|\frac{\phi(\xi)}{\phi(t)}\right| \le \frac{c_1}{c_2}e^{c_3}$$

and for appropriate positive constants c_1 and c_2, $\left|\frac{\phi(\xi)}{\phi(t)}\right| < 1$. Also

$$|\phi(t)| = |t|\, e^{-\,\text{real}\, t} \ge c_2 e^{-c_2} > 0$$

so that

$$\begin{aligned}|\phi(t) - \phi(\xi)| &\ge |\phi(t)| - |\phi(\xi)| \\ &= |\phi(t)|\left(1 - \frac{|\phi(\xi)|}{|\phi(t)|}\right) \\ &\ge c_2 e^{-c_2}\left(1 - \frac{c_1}{c_2}e^{c_3}\right) \\ &= m > 0, \quad \text{say.}\end{aligned}$$

The functions $\phi(z) = ze^{-z}$ and $f(z) = \frac{e^{xz}}{1-z}$ are analytic in $D = \left\{z : |z| < \frac{1}{2}\right\}$ and so are $\phi'(z)$ and $f'(z)$. Consequently $\phi'(z)$ is continuous on the compact set Γ, and so bounded on Γ. Similarly $f'(t)$ is continuous on the compact set C, and so bounded on C. So there is an M independent of ξ and t such that $|\phi'(\xi) f'(t)| \le M$ for ξ on Γ and t on C. The inner integral in (3.2) has modulus

$$\left|\int_C \left[\frac{\phi(\xi)}{\phi(t)}\right]^n \frac{f'(t)\,\phi'(\xi)}{\phi(t)-\phi(\xi)}dt\right| \leq 2\pi\,(c_2)\left(\frac{c_1}{c_2}e^{c_3}\right)^n \frac{M}{m}$$

which is independent of ξ. So that

$$\begin{aligned} |R_{n+1}| &\leq \frac{1}{2\pi}(c_1)\,2\pi\,(c_2)\left(\frac{c_1}{c_2}e^{c_3}\right)^n \frac{M}{m} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

since both M and m are independent of n.

In chapter two we apply (3.1) and arrive at (3.7). If we put $x = -t/z$, for $z \neq 0$ we obtain

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(nz-t)^n}{n!} e^{t-nz}$$

and for the trivial case $x = 0$, hence $t = 0$ we have

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} \frac{(nze^{-z})^n}{n!} \\ &= 1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + \dots . \end{aligned}$$

3.3 Convergence region.

For convergence of (3.7) we apply the ratio test , such that

$$\begin{aligned} &\lim_{r\to\infty} \left| \frac{r+1+x}{r+1} \left(\frac{r+1+x}{r+x}\right)^r \left(ze^{-z}\right) \right| \\ = &\left|ze^{-z}\right| \lim_{r\to\infty} \left|\left(1+\frac{1}{r+x}\right)^{r+x}\right| \left|\left(1+\frac{1}{r+x}\right)^{-x}\right| \\ = &\left|ze^{1-z}\right| . \end{aligned}$$

So the series (3.7) converges $\forall z$ such that

$$\left|ze^{1-z}\right| < 1 \tag{3.8}$$

and diverges outside this region. If $z = x+iy$, in chapter two we stated that the series converges in the region $\left(\left(x^2+y^2\right)e^{2(1-x)}\right)^{\frac{1}{2}} < 1$ and so the curve $ze^{1-z} = 1$, $\left(\left(x^2+y^2\right)e^{2(1-x)}\right)^{\frac{1}{2}} = 1$ will separate the regions of convergence and divergence. Similarly in polar coordinates, $z = re^{i\theta}$, the series converges inside a certain oval whose polar equation is $re^{1-r\cos\theta} = 1$. See the sketch in chapter 2, figure 2.1.

3.3.1 Extension of the series.

The series (3.7) cannot hold outside the region described by (3.8). To investigate if the series holds everywhere inside this region, which is where both sides of (3.7) have meaning, we need corollary (ii), the principle of analytic continuation, as given on page 89 of Titchmarsh [88].

Corollary 12 *If two functions are analytic in a domain D, and are equal at the points of a set S which has a limit point in D, then they are equal throughout D.*

From my knowledge of analysis apparantly this corollary has no counterpart for real functions. To apply it to the series (3.7), let D be the inside of the oval curve, and let the two sides of (3.7) be the two functions. They are equal in $|z| \leq c_1$, a set of points S which has 0 as a limit point; and 0 is in D. So if both sides of (3.7) are analytic in D they must be equal throughout D by the above corollary. The left hand side of (3.7) is analytic in D, because it is analytic everywhere except at $z = 1$ which is not a point of D. So now we need to show that the right hand side of (3.7) is also analytic in D. Consider

$$F(w) = \sum_{n=0}^{\infty} \frac{(n+x)^n}{n!} w^n \tag{3.9}$$

where $w = ze^{-z}$, and $F(w)$ is therefore the sum of a power series which, by the use of the ratio test, converges in the disc $|w| < e^{-1}$. The following lemma is required and may be found on page 66 of Titchmarsh [88].

Lemma 13 *A power series represents an analytic function inside its circle of convergence.*

By this lemma we have that $F(w)$ is an analytic function of w in $|w| < e^{-1}$. Now an analytic function of an analytic function of z is an analytic function of z, if the ranges are correctly matched. In our case ze^{-z} is an analytic function of z in the whole z-plane, and its values $w = ze^{-z}$ satisfy $|w| < e^{-1}$ if z lies in the oval region D. Thus $F(ze^{-z})$ is an analytic function of z in D. Hence (3.7) holds for all z in the disc D. It may be difficult to determine whether (3.7) holds for all z values on the curve. In chapter two we demonstrated that (3.7) is divergent at the point $z = 1$. At the other intersection $w = -e^{-1}$, $z \sim -.2784...$, the series (3.9) has real terms alternating in sign which decrease in modulus by the ratio test and tend to zero as $n \rightarrow \infty$ and so the series is convergent at $z \sim -.2784...$.

Chapter 4

Binomial type sums.

A procedure which will allow a specific finite Binomial type sum to be expressed in closed polynomial form will be developed in this chapter. The Binomial type sum will be useful in the next chapter, where the results of chapter two will be generalized.[1]

[1] This chapter, in extended form, has been published in the Australian Mathematical Society Gazette, Vol.24, pp.66-73, 1997.

4.1 Introduction.

In the next chapter we shall extend, in various directions the results of chapter two. In the course of these investigations we shall come across the finite sum

$$P_m(n) = \frac{(-1)^{n+m}}{(n+m)!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} r^{n+m}. \tag{4.1}$$

Therefore the major aim of this chapter is to develop a procedure that will allow us to express (4.1), and its generalisation, in closed polynomial form. Initially an infinite double sum will be obtained by the consideration of a Volterra integral equation and the double sum then expanded to obtain coefficients in terms of a recurrence relation, which upon further expansion will lead to sums of the form (4.1). We shall then prove that the finite sum (4.1) can be written as a polynomial in n of degree m, and a procedure for the evaluation of the polynomial will be given.

4.2 Problem statement.

Volterra integral equations of the form

$$\psi(t) = F(t) + \int_0^t \psi(t-x)\,\phi(x)\,dx \tag{4.2}$$

occur in a wide variety of applications. If $F(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function, and we take the Laplace transform of (4.2) then

$$\Psi(p) = \frac{1}{1-\Phi(p)}, \tag{4.3}$$

where $\Psi(p)$ and $\Phi(p)$ are the Laplace transforms of $\psi(t)$ and $\phi(t)$ respectively. Now, consider the rectangular wave $\phi(t) = H(a-t) = 1 - H(t-a)$, where $H(t)$ is the Heaviside function, and taking the Laplace transform of $\phi(t)$ we have

$$\Phi(p) = \frac{1-e^{-ap}}{p} \tag{4.4}$$

and the n^{th} moment of $\phi(t)$ is given by

$$M_n = \lim_{p \to 0} \left[(-1)^n \frac{d^n}{dp^n} \{\Phi(p)\} \right] = \frac{a^{n+1}}{n+1}. \tag{4.5}$$

Substituting (4.4) into (4.3) results in

$$\Psi(p) = \frac{1}{p - 1 + e^{-ap}} \tag{4.6}$$

and an expansion of (4.6) gives

$$\Psi(p) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \sum_{k=0}^{\infty} (-1)^k \frac{(ar)^k p^{k-n}}{k!}. \tag{4.7}$$

It may be noticed from (4.7) that $\Psi(p)$ can be written in the form

$$\Psi(p) = \sum_{m=0}^{\infty} \beta_m(a) p^m$$

where

$$\beta_m(a) = \sum_{n=0}^{\infty} a^{n+m} P_m(n) \tag{4.8}$$

and $P_m(n)$ is as given in (4.1).

4.3 A recurrence relation.

Lemma 14 : *A recurrence relation for $\beta_m(a)$ is*

$$\left. \begin{array}{c} (1-a)\beta_m(a) = \sum_{k=0}^{m-1} (-1)^{m-k} \frac{a^{m-k+1}}{(m-k+1)!} \beta_k(a), m = 1,2,3... \\ \\ \text{with } \beta_0(a) = \frac{1}{1-a}, \; a \neq 1 \end{array} \right\}$$

Proof: From (4.3) and (4.4) we may write

$$\beta_m(a) = \lim_{p \to 0} \left[\frac{1}{m!} \frac{d^m}{dp^m} \left\{ \frac{1}{1 - \Phi(p)} \right\} \right], \quad m = 1,2,3,... \tag{4.9}$$

and its easy to see from (4.9) and (4.4) that $\beta_0(a) = \frac{1}{1-\Phi(0)} = \frac{1}{1-a}$, $a \neq 1$. From (4.9),

$$
\begin{aligned}
m!\beta_m(a) &= \lim_{p \to 0}\left[\frac{d^m}{dp^m}\left\{1 + \frac{\Phi(p)}{1-\Phi(p)}\right\}\right] \\
&= \lim_{p \to 0}\left[\frac{d^m}{dp^m}\left\{\frac{\Phi(p)}{1-\Phi(p)}\right\}\right] \\
&= \lim_{p \to 0}\left[\sum_{k=0}^{m}\binom{m}{k}\frac{d^{m-k}}{dp^{m-k}}\{\Phi(p)\}\frac{d^k}{dp^k}\left\{\frac{1}{1-\Phi(p)}\right\}\right].
\end{aligned}
\tag{4.10}
$$

Hence, using (4.5) and (4.9), we may write (4.10) as

$$
m!\beta_m(a) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\frac{a^{m-k+1}k!}{m-k+1}\beta_k(a) \tag{4.11}
$$

and from (4.11) the $\beta_m(a)$ are given by the recurrence relation

$$
\left.
\begin{aligned}
(1-a)\beta_m(a) = \sum_{k=0}^{m-1}(-1)^{m-k}\frac{a^{m-k+1}}{(m-k+1)!}\beta_k(a), m = 1,2,3.. \\
\text{with } \beta_0(a) = \tfrac{1}{1-a},\ a \neq 1,
\end{aligned}
\right\}
\tag{4.12}
$$

hence the lemma is proved.

From (4.8), $\beta_m(a)$ may be expanded in a Maclaurin series

$$
\beta_m(a) = \sum_{q=0}^{\infty}\beta_m^{(q)}(0)\frac{a^q}{q!}, \tag{4.13}
$$

and the coefficients $\beta_m^{(q)}(0)$ can be calculated from the recurrence relation in (4.12) as follows. From the left hand side of (4.12)

$$
\frac{d^q}{da^q}\{(1-a)\beta_m(a)\} = \sum_{r=0}^{q}\binom{q}{r}\frac{d^{q-r}}{da^{q-r}}\{1-a\}\beta_m^{(r)}(0),\ q = 0,1,2,...
$$

$q\backslash m$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2	2!	-1	0	0	0	0	0	0
3	3!	-6	1	0	0	0	0	0
4	4!	-36	14	-1	0	0	0	0
5	5!	-240	150	-30	1	0	0	0
6	6!	-1800	1560	-540	62	-1	0	0
7	7!	-15120	16800	-8400	1806	-126	1	0
8	8!	-141120	191520	-126000	40824	-4914	254	-1

Table 4.1: The beta coefficients of (4.16).

and this term is non-zero only for $r = q$ and $r = q - 1$, so that

$$\frac{d^q}{da^q}\left\{(1-a)\,\beta_m(a)\right\} = (1-a)\,\beta_m^{(q)}(0) - \begin{pmatrix} q \\ q-1 \end{pmatrix}\beta_m^{(q-1)}(0)\,. \tag{4.14}$$

Further from the right hand side of (4.12)

$$\frac{d^q}{da^q}\left[\sum_{k=0}^{m-1}(-1)^{m-k}\frac{a^{m-k+1}}{(m-k+1)!}\beta_k(a)\right] = \sum_{k=0}^{m-1}(-1)^{m-k}\begin{pmatrix} q \\ q-m+k-1 \end{pmatrix}\beta_k^{(q-m+k-1)}(a)$$

$$+\sum_{k=0}^{m-1}(-1)^{m-k}\sum_{r=0}^{q}\begin{pmatrix} q \\ r \end{pmatrix}\frac{a^{m-k+1-q+r}}{(m-k+1-q+r)!}\beta_k^{(r)}(a)\,, \tag{4.15}$$

and setting $a = 0$ in (4.14) and (4.15) gives, after equating the right hand sides and rearranging, the recurrence relation for the coefficients of the Maclaurin series (4.13),

$$\beta_m^{(q)}(0) = \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} q \\ q-m+k-1 \end{pmatrix}\beta_k^{(q-m+k-1)}(0)\,,\ q = 0,1,2,\dots\,. \tag{4.16}$$

These coefficients are demonstrated in table 4.1.

Some observations that may be made from (4.16) and table 4.1 are $\beta_0^{(0)}(0) = 1$, $\beta_0^{(q)}(0) = q!$, $\beta_m^{(0)}(0) = 0$ for $m \geq 1$, $\beta_m^{(1)}(0) = \beta_m^{(0)}(0)$ and $\beta_m^{(m+1)}(0) = (-1)^m$. Further, $\beta_m^{(q)}(0) = 0$ for $0 < q \leq m$, and so the leading power of (4.13) is a^{m+1}.

Theorem 15 : *The finite sum, $P_m(n)$, (4.1) is a polynomial in n of degree m.*

Proof: From (4.8)

$$\beta_m(a) = \sum_{n=1}^{m} \frac{a^{n+m}\beta_m^{(n+m)}(0)}{(n+m)!} + \sum_{n=m+1}^{\infty} \frac{(-a)^{n+m}}{(n+m)!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} r^{n+m}. \qquad (4.17)$$

From (4.12)

$$(-1)^m (1-a)\beta_m(a) = \frac{a^{m+1}\beta_0(a)}{(m+1)!} - \frac{a^m\beta_1(a)}{m!} + \frac{a^{m-1}\beta_2(a)}{(m-1)!}$$

$$-\frac{a^{m-2}\beta_3(a)}{(m-2)!} + ... + \frac{a^3\beta_{m-2}(a)}{3!} + \frac{a^2\beta_{m-1}(a)}{2!}, m = 1, 2, 3, ...$$

$$\frac{(-1)^m (m+1)!(1-a)}{a^{m+1}}\beta_m(a) = \frac{1}{1-a} + \frac{(m+1)a^2}{2a(1-a)^2} + \frac{(m+1)ma^3(a+2)}{12a^2(1-a)^3}$$

$$+\frac{(m+1)m(m-1)a^4(1+2a)}{24a^3(1-a)^4} + \frac{(m+1)m(m-1)(m-2)a^5\left(6+32a+8a^2-a^3\right)}{3!5!a^4(1-a)^5}$$

$$+... + \frac{(m+1)!a^m X(a)}{2a^{m-1}(1-a)^m}$$

where the function $X(a)$ is a polynomial in a, to be determined from the particular $\beta_{m-1}(a)$, that is

$$\frac{(-1)^m (m+1)!(1-a)^{m+1}}{a^{m+1}}\beta_m(a) = (1-a)^{m-1} + \frac{(m+1)a}{2(1-a)^{2-m}} + \frac{(m+1)ma(a+2)}{12(1-a)^{3-m}} +$$

$$\frac{(m+1)m(m-1)a(1+2a)}{24(1-a)^{4-m}} + \frac{(m+1)m(m-1)(m-2)a\left(6+32a+8a^2-a^3\right)}{3!5!(1-a)^{5-m}}$$

$$+... + \frac{(m+1)!a^m X(a)}{2}$$

and so

$\frac{(-1)^m (m+1)!}{a^{m+1}}\beta_m(a) =$

$$(m+1)!\beta_m(a) = \sum_{n=0}^{\infty}\binom{n+m}{n}a^n\left\{\begin{array}{r}\sum_{k=0}^{m-1}\binom{m-1}{k}(-a)^k - \frac{m+1}{2}\sum_{k=0}^{m-2}\binom{m-2}{k}(-a)^{k+1}+\\ \frac{m(m+1)}{12}\sum_{k=0}^{m-3}\binom{m-3}{k}(-1)^k\left[2a^{k+1}+a^{k+2}\right]+\\ \frac{m(m+1)(m-1)}{24}\sum_{k=0}^{m-4}\binom{m-4}{k}(-1)^k\left[a^{k+1}+2a^{k+2}\right]+\\ \frac{m(m+1)(m-1)(m-2)}{3!5!}\sum_{k=0}^{m-5}\binom{m-5}{k}(-1)^k\left[6a^{k+1}+32a^{k+2}+8a^{k+3}-a^{k+4}\right]\\ +...+\frac{(m+1)!aX(a)}{2}\end{array}\right.$$

$$\sum_{n=0}^{\infty}\binom{n+m}{n}a^{n+m+1}\left\{\begin{array}{r}a^{m-1}\left[-1+\frac{m+1}{2}-\frac{(m+1)m}{12}+\frac{(m+1)m(m-1)(m-2)}{3!5!}+...\right]\\ +a^{m-2}\left[(m-1)-\frac{(m+1)(m-2)}{2}+\frac{(m+1)m(m-1)}{12}+...\right]+...+a^0[...]\end{array}\right\}.$$

Thus $\beta_m(a)$ may be expressed in the form

$$\beta_m(a) = \sum_{n=0}^{\infty}\binom{n+m}{n}\left\{a^{n+2m}F_1(m)+a^{n+2m-1}F_2(m)+...+a^{n+m+1}F_m(m)\right\}$$

where the $F_j(m), j = 1,2,3,...,m$, are functions dependent on the fixed parameter m only. The summation indices are now adjusted to obtain coefficients of common powers of a in the following manner,

$$\beta_m(a) = \sum_{n=m-1}^{\infty}\binom{n+1}{n-m+1}a^{n+m+1}F_1(m)+\sum_{n=m-2}^{\infty}\binom{n+2}{n-m+2}a^{n+m+1}F_2(m)+$$
$$...+\sum_{n=1}^{\infty}\binom{n+m-1}{n-1}a^{n+m+1}F_{m-1}(m)+\sum_{n=0}^{\infty}\binom{n+m}{n}a^{n+m+1}F_m(m),$$

and so,

$$\beta_m(a) = \binom{m}{0}a^{2m}F_1(m)+\sum_{n=m}^{\infty}\binom{n+1}{n-m+1}a^{n+m+1}F_1(m)+$$

$$\left[\binom{m+1}{1}a^{2m}+\binom{m}{0}a^{2m-1}\right]F_2(m)+\sum_{n=m}^{\infty}\binom{n+2}{n-m+2}a^{n+m+1}F_2(m)+...+$$

$$\left[\binom{2m-2}{m-2}a^{2m}+...+\binom{m}{0}a^{m+2}\right]F_{m-1}(m)+\sum_{n=m}^{\infty}\binom{n+m-1}{n-1}a^{n+m+1}F_{m-1}(m)$$

$$+\left[\binom{2m-1}{m-1}a^{2m}+...+\binom{m}{0}a^{m+1}\right]F_{m}(m)+\sum_{n=m}^{\infty}\binom{n+m}{n}a^{n+m+1}F_{m}(m). \quad (4.18)$$

Grouping of terms gives

$$\begin{aligned}\beta_m(a) &= \sum_{n=m}^{\infty}a^{n+m+1}\left[\begin{array}{c}\binom{n+1}{n-m+1}F_1(m)+\binom{n+2}{n-m+2}F_2(m)+...+\\ \binom{n+m-1}{n-1}F_{m-1}(m)+\binom{n+m}{n}F_m(m)\end{array}\right]\\ &+a^{2m}G_1(m)+a^{2m-1}G_2(m)+...+a^{m+2}G_{m-1}(m)+a^{m+1}G_m(m),\end{aligned}$$

and so

$$\begin{aligned}\beta_m(a) &= \sum_{n=m+1}^{\infty}a^{n+m}\left[\begin{array}{c}\binom{n}{n-m}F_1(m)+\binom{n+1}{n-m+1}F_2(m)+...+\\ \binom{n+m-2}{n-2}F_{m-1}(m)+\binom{n+m-1}{n-1}F_m(m)\end{array}\right]\\ &+\sum_{n=1}^{m}a^{n+m}G_{m-n+1}(m)\end{aligned} \quad (4.19)$$

where the functions $G_j(m)$, like $F_j(m)$, are dependent only on the fixed parameter m. From the right hand side of (4.17) and (4.19) it may be seen that

$$\sum_{n=1}^{m}\frac{a^{n+m}}{(n+m)!}\beta_m^{(n+m)}(0)=\sum_{k=1}^{m}a^{m+k}G_{m-k+1}(m) \quad (4.20)$$

and equating the powers of a^{m+j}, where $j=m+1,m+2,m+3,...$

$$P_m(n)=\frac{(-1)^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{n+m}=\sum_{k=0}^{m-1}\binom{n+k}{n-m+k}F_{k+1}(m). \quad (4.21)$$

Since

$$\left(\begin{array}{c} n+k \\ n-m+k \end{array}\right)=\frac{(n+k)(n+k-1)\ldots(n+k-m+1)}{m!}, \; k=0,1,2,\ldots,(m-1)$$

is a polynomial in n of degree m and the $F_{k+1}(m)$ functions depend on the fixed parameter m, then the right hand side of (4.21) is a polynomial in n of degree m. Hence the theorem is proved.

4.4 Relations between $G_k(m)$ and $F_{k+1}(m)$.

From (4.18) and (4.19) it can be seen that, on equating coefficients of a^{m+j}, where $j=1,2,\ldots,m$ gives,

$$G_m(m)=\left(\begin{array}{c} m \\ 0 \end{array}\right) F_m(m)$$

$$G_{m-1}(m)=\left(\begin{array}{c} m \\ 0 \end{array}\right) F_{m-1}(m)+\left(\begin{array}{c} m+1 \\ 1 \end{array}\right) F_m(m)$$

....

....

....

$$G_2(m)=\left(\begin{array}{c} m \\ 0 \end{array}\right) F_2(m)+\left(\begin{array}{c} m+1 \\ 1 \end{array}\right) F_3(m)+\ldots+\left(\begin{array}{c} 2m-3 \\ m-3 \end{array}\right) F_{m-1}(m)+\left(\begin{array}{c} 2m-2 \\ m-2 \end{array}\right) F_m(m)$$

$$G_1(m)=\left(\begin{array}{c} m \\ 0 \end{array}\right) F_1(m)+\left(\begin{array}{c} m+1 \\ 1 \end{array}\right) F_2(m)+\ldots+\left(\begin{array}{c} 2m-2 \\ m-2 \end{array}\right) F_{m-1}(m)+\left(\begin{array}{c} 2m-1 \\ m-1 \end{array}\right) F_m(m),$$

and therefore

$$G_k(m)=\sum_{j=0}^{m-k}\left(\begin{array}{c} m+j \\ j \end{array}\right) F_{j+k}(m), k=1,2,3,\ldots,m.$$

The functions $F_j(m)$, $j = 1, 2, 3, ..., m$ in (4.21) can be recursively obtained from

$$\mathbf{MF} = \mathbf{G} \tag{4.22}$$

where $\mathbf{M}$ is an $(m \times m)$ upper triangular matrix, $\mathbf{F}$ and $\mathbf{G}$ are $(m \times 1)$ column vectors. Similarly, from (4.20)

$$G_{m-k+1}(m) = \frac{\beta_m^{(m+k)}(0)}{(m+k)!}, \; k = 0, 1, 2, 3, ..., m \; ,$$

putting $q = m + k$

$$G_{2m-q+1}(m) = \frac{\beta_m^{(q)}(0)}{q!}, \; q = m+1, m+2, ..., 2m \; ,$$

and for the counter $j = 2m - q + 1$

$$G_j(m) = \frac{\beta_m^{(2m-j+1)}(0)}{(2m-j+1)!}, \; j = m, m-1, ..., 2, 1 \; ,$$

where the $\beta_m^{(q)}(0)$ values can be obtained from (4.16). Therefore (4.22), may be written as $\mathbf{MF} = \mathbf{B}$, $\mathbf{B}$ is a $(m \times 1)$ vector and in explicit form,

$$\begin{bmatrix} \binom{m}{0} & \binom{m+1}{1} & \cdots & & \binom{2m-1}{m-1} \\ & \binom{m}{0} & \cdots & & \binom{2m-2}{m-2} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ & & & \binom{m+1}{1} & \binom{m+2}{2} \\ & & & \binom{m}{0} & \binom{m+1}{1} \\ & & & & \binom{m}{0} \end{bmatrix} \begin{bmatrix} F_1(m) \\ F_2(m) \\ \cdot \\ \cdot \\ \cdot \\ F_{m-2}(m) \\ F_{m-1}(m) \\ F_m(m) \end{bmatrix} = \begin{bmatrix} \frac{\beta_m^{(2m)}(0)}{(2m)!} \\ \frac{\beta_m^{(2m-1)}(0)}{(2m-1)!} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\beta_m^{(m+3)}(0)}{(m+3)!} \\ \frac{\beta_m^{(m+2)}(0)}{(m+2)!} \\ \frac{\beta_m^{(m+1)}(0)}{(m+1)!} \end{bmatrix}$$

m	$P_m(n)$
0	1
1	$-n/2$
2	$n(3n+1)/4!$
3	$-n^2(n+1)/(2\times 4!)$
4	$n\left(15n^3+30n^2+5n-2\right)/(8\times 6!)$
5	$-n^2(n+1)\left(3n^2+7n-2\right)/(16\times 6!)$
6	$n\left(63n^5+315n^4+315n^3-91n^2-42n+16\right)/\left(2^3\times 9!\right)$
7	$-n^2(n+1)\left(9n^4+54n^3+51n^2-58n+16\right)/\left(2^4\times 9!\right)$
8	$n\left(135n^7+1260n^6+3150n^5+840n^4-2345n^3+540n^2-404n-144\right)/\left(3\times 2^7\times 10!\right)$
9	$-n^2(n+1)\left(15n^6+165n^5+465n^4-17n^3-648n^2+548n-144\right)/\left(3\times 2^8\times 10!\right)$

Table 4.2: The polynomials of (4.21).

This matrix setup therefore allows a recursive evaluation of the functions $F_j(m), j = 1, 2, 3, ..., m$, in terms of the coefficients $\beta_m^{(q)}(0)$ in the Maclaurin series (4.13). In particular $F_1(m)$ takes the form

$$(m+1)!F_1(m) = -1 + \frac{m+1}{2} - \frac{(m+1)m}{12} + 0 + \frac{(m+1)m(m-1)(m-2)}{6!} + ... \tag{4.23}$$

and for a particular value of m, that same number of terms are used on the right hand side of (4.23). Some values of $F_1(m)$ are $F_1(1) = -\frac{1}{2}$, $F_1(2) = \frac{1}{12}$, $F_1(3) = 0$, $F_1(4) = -\frac{1}{6!}$, $F_1(5) = 0, ...$. Various closed form polynomial representations of (4.1) are given in table 4.2.

Cerone, Sofo and Watson [24], have shown a connection of the finite sum (4.1) with Stirling numbers of the second kind and and association of the finite sum (4.1) with an application of a problem using the idea of a multinomial distribution. Moreover, the author has generalized (4.1): namely, given that

$$V_m(n,x) = \frac{(-1)^{n+m}}{(n+m)!}\sum_{r=0}^{n}(-1)^r \binom{n}{r} (x+r)^{n+m} \tag{4.24}$$

for x a real number, then $V_m(n,x)$ can be expressed as a polynomial in x and n of degree m for both x and n.

Theorem 16 : *$V_m(n,x)$ is a polynomial in x and n of degree m for both x and n.*

Proof: The following result is needed and is quoted by Feller [44] on page 65.

$$R_i(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^i = \begin{cases} 0, & i = 0, 1, 2, ..., n-1 \\ (-1)^n n!, & i = n. \end{cases} \tag{4.25}$$

From (4.24),

$$V_0(n,x) = \frac{(-1)^n}{n!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (x+r)^n = \sum_{r=0}^{n} \sum_{k=0}^{n} \frac{(-1)^{r+n}}{n!} \binom{n}{r} \binom{n}{k} x^{n-k} r^k. \tag{4.26}$$

Changing the order of summation on the right hand side of (4.26), gives upon using the above result (4.25)

$$V_0(n,x) = \frac{(-1)^n}{n!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (x+r)^n = 1. \tag{4.27}$$

The result (4.27) can be integrated m times with respect to x to evaluate $V_m(n,x)$ as defined in (4.24). For example, integrating (4.27) and using the initial condition $V_1(n,0) = P_1(n) = -n/2$, from table 4.2 for $m = 1$, results in

$$V_1(n,x) = \frac{(-1)^{n+1}}{(n+1)!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (x+r)^{n+1} = -\frac{1}{2}(n+2x).$$

Using this procedure we see that (4.24) is a polynomial in x and n of degree m for both x and n. Alternatively, the recurrence relation

$$V_m(n,x) = P_m(n) - \int_0^x V_{m-1}(n,t)\,dt, \; V_0(n,x) = 1,$$

may be used to evaluate $V_m(n,x)$. The table 4.3 lists some of the $V_m(n,x)$ in closed form.

In the next chapter we shall extend the results of chapter 2, and also utilize the polynomials obtained in this chapter.

m	$V_m(n,x)$
0	1
1	$-(n+2x)/2$
2	$\left(3n^2+n(1+12x)+12x^2\right)/4!$
3	$-\left(n^3+n^2(1+6x)+n\left(12x^2+2x\right)+8x^3\right)/(2\times 4!)$
4	$\left(15n^4+30n^3(1+4x)+5n^2(1+24x(3x+1))+2n\left(60x^2(4x+1)-1\right)+240x^4\right)/(8\times 6!)$

Table 4.3: The polynomials of (4.24).

Chapter 5

Generalization of Euler's identity.

In this chapter an investigation of a generalization of the identity (2.13) of chapter two is undertaken. The investigation will also make use of the finite binomial type sums obtained in chapter four. A connection with renewal processes will be made. It will be proved that generated infinite sums may be represented in closed form that depend on k-dominant zeros of an associated transcendental characteristic function.[1]

[1] This chapter, in modified form, has been published in the Journal of Mathematical Analysis and Applications. Vol.214, pp.191-206, 1997.

5.1 Introduction.

We shall consider a forced differential-difference equation, of arbitrary order, and by the procedure developed in chapter 2, generate sums which, by the use of residue theory, may be represented in closed. As in chapter 2 the generalized identity, as it will be shown, depends on a dominant zero of an associated transcendental characteristic function. We shall also develop recurrence relations for use in determining specific closed form expressions of the infinite sum. We shall employ an induction argument to prove the closed form representation of the infinite sum and then give a functional relationship of the sum. An extension to our main results will be indicated and in the process utilize the finite binomial type sums obtained in chapter 4; a connection with renewal processes can also be made.

5.2 1-dominant zero.

5.2.1 The system.

Consider, for a well behaved function $f(t)$, the forced dynamical system with constant real coefficients b and c, real delay parameter a, and all initial conditions at rest,

$$\left.\begin{array}{r} \sum\limits_{n=0}^{R}\binom{R}{R-n}c^{R-n}\sum\limits_{r=0}^{n}\binom{n}{r}b^{n-r}f^{(r)}\left(t-(R-n)\,a\right)=w\left(t\right);\; t>Ra \\ \sum\limits_{r=0}^{R}\binom{R}{r}b^{R-r}f^{(r)}\left(t\right)=w\left(t\right);\; 0<t\leq Ra. \end{array}\right\} \tag{5.1}$$

In the system (5.1) $w(t)$ is a forcing term, t a real variable, and R is a positive integer, being the order of the differential-delay equation. Taking the Laplace transform of (5.1) and utilizing the property

$$\pounds\left(f^{(n)}\left(t-k\right)\right)=e^{-kp}\left(p^{n}F\left(p\right)-\sum_{j=1}^{n}p^{n-j}f^{(j-1)}\left(0\right)\right),$$

we obtain

$$\left(\sum_{j=0}^{R}\binom{R}{r}\left(p+b\right)^{j}\left(ce^{-ap}\right)^{R-j}\right)F\left(p\right)=W\left(p\right). \tag{5.2}$$

From (5.2)

$$F(p) = \frac{W(p)}{(p + b + ce^{-ap})^R} \tag{5.3}$$

where $F(p)$and $W(p)$ are the Laplace transforms of $f(t)$ and $w(t)$ respectively. Equation (5.3) may be expanded in series so that

$$F(p) = \frac{W(p)}{(p+b)^R \left(1 + \frac{ce^{-ap}}{p+b}\right)^R}$$

$$= \sum_{n=0}^{\infty} \binom{n+R-1}{n} \frac{W(p)(-ce^{-ap})^n}{(p+b)^{n+R}}. \tag{5.4}$$

To bring out the essential features of our results we may choose the forcing term $w(t) = \delta(t)$, the Dirac delta function, such that $W(p) = 1$. Substituting for $W(p)$ into (5.4) and taking the inverse Laplace transform, we have

$$f(t) = \sum_{n=0}^{\infty} \binom{n+R-1}{n} \frac{(-c)^n e^{-b(t-an)} (t - an)^{n+R-1}}{(n+R-1)!} H(t - an) \tag{5.5}$$

where, $H(x)$ is the unit Heaviside step function. The inverse of (5.3), a solution of the system (5.1) by Laplace transform theory may also be written as

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} F(p)\, dp,$$

for an appropriate choice of γ such that all the zeros of the characteristic function

$$g_1(p) = p + b + ce^{-ap} \tag{5.6}$$

are contained to the left of the line in the Bromwich contour, and $F(p)$ is defined by (5.3). Now by the residue theorem

$$f(t) = \sum \text{residues of } \left(e^{pt} F(p)\right)$$

which suggests the solution of $f(t)$ may be written in the form

$$f(t) = \sum_r Q_r e^{p_r t}$$

where the sum is over all the characteristic zeros p_r of $g_1(p)$ and Q_r is the contribution of the residues in $F(p)$ at $p = p_r$. The zeros of the characteristic function (5.6) with restriction

$$\left|ace^{1+ab}\right| < 1 \tag{5.7}$$

are all distinct. The poles of the expression (5.3) depend on the zeros of the characteristic function (5.6), namely, the zeros of $g_1(p)$. The dominant distinct root p_0, of $g_1(p_0) = 0$ is defined as one with the greatest real part and therefore we have that asymptotically

$$f(t) \sim \sum_{k=0}^{R-1} Q_{R,k}(p_0) \frac{t^{R-k-1}e^{p_0 t}}{(R-k-1)!}. \tag{5.8}$$

From (5.5) and (5.8)

$$f(t) = \sum_{n=0}^{\left[\frac{t}{a}\right]} \binom{n+R-1}{n} \frac{(-c)^n e^{-b(t-an)} (t-an)^{n+R-1}}{(n+R-1)!} \tag{5.9}$$

$$\sim \sum_{k=0}^{R-1} Q_{R,k}(p_0) \frac{t^{R-k-1}e^{p_0 t}}{(R-k-1)!}$$

where $[x]$ represents the integer part of x and the residue contribution $Q_{R,k}(p_0)$, is given by

$$k!Q_{R,k}(p_0) = \lim_{p \to p_0} \left[\frac{d^k}{dp^k}\left((p-p_0)^R F(p)\right)\right]; k = 0, 1, 2, ..., (R-1), \tag{5.10}$$

since (5.3) has a pole of order R at the distinct dominant zero, $p = p_0$ for $1 - ac \neq 0$. From now on we may take, without any loss of generality, $b + c = 0$ and $1 + ab \neq 0$. These conditions simply allow the distinct dominant zero, p_0, of the characteristic function (5.6), with restriction (5.7), to occur at $p_0 = 0$, and therefore from (5.6) and (5.10) respectively

$$g(p) = p + b - be^{-ap} \tag{5.11}$$

and

$$k!Q_{R,k}(0) = \lim_{p\to 0}\left[\frac{d^k}{dp^k}\left(\left(\frac{p}{g(p)}\right)^R\right)\right]; k = 0,1,2,...,(R-1). \tag{5.12}$$

Theorem 17 . *Let*

$$T_n(b,R,a,t) = \binom{n+R-1}{n}\frac{b^n e^{-b(t-an)}(t-an)^{n+R-1}}{(n+R-1)!} \tag{5.13}$$

and

$$S_R(b,a,t) = \sum_{n=0}^{\infty} T_n(b,R,a,t) \tag{5.14}$$

which is convergent for all values of b, R, a and t in the region of convergence (5.7). Then by the suggestive behaviour of (5.9)

$$S_R(b,a,t) = \sum_{k=0}^{R-1} Q_{R,k}(0)\frac{t^{R-k-1}}{(R-k-1)!}. \tag{5.15}$$

The series (5.14) is known as an Abel type series, because of the $(t-an)^{n+R-1}$ term and, the convergence region (5.7) may be obtained by applying the ratio test to the term $T_n(b,R,a,t)$ in (5.13). A proof of the main theorem 17 will follow shortly. Firstly we shall develop two useful recurrence relations for the evaluation of the terms $Q_{R,k}(0)$ in (5.12) and an identity for the $Q_{R,k}(0)$ terms. Secondly, using the terms $Q_{R,k}(0)$ we shall give some closed form representations of the infinite sum (5.15). Thirdly, a recurrence relation for the series (5.14) will be developed, and finally an induction argument on the integer R will be applied to prove the main theorem 17.

5.2.2 $Q_{R,k}(0)$ recurrences and closed forms.

Lemma 18 . *A recurrence relation for the evaluation of the terms $Q_{R,k+1}(0)$ in (5.12) is*

$$(k+1)Q_{R,k+1}(0) = R\sum_{\mu=0}^{k}\frac{(-1)^\mu ba^{\mu+2}(\mu+1)}{(\mu+2)!}Q_{R+1,k-\mu}(0) \tag{5.16}$$

with

$$Q_{R,0}(0) = \frac{1}{(1+ab)^R}.$$

Proof: From (5.12)

$$Q_{R,0}(0) = \lim_{p\to 0}\left[\left(\frac{p}{g(p)}\right)^R\right] = \frac{1}{(1+ab)^R}.$$

Also from (5.12)

$$(k+1)!Q_{R,k+1}(0) = \lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{\frac{d}{dp}\left(\frac{p}{g(p)}\right)^R\right\}\right]; k = 0,1,2,...,(R-1) \quad (5.17)$$

$$= R\lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{\frac{p^{R-1}(g(p)-pg'(p))}{(g(p))^{R+1}}\right\}\right]$$

where $g(p)$ is defined in (5.11) and its first derivative

$$g'(p) = 1 + abe^{-ap}.$$

Letting $h(p) = g(p) - pg'(p)$ we find that $h(0) = 0$ and $h'(0) = 0$ and therefore expanding $h(p)$ as a Taylor series about $p = 0$ we may write, from (5.17)

$$(k+1)!Q_{R,k+1}(0) = R\lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{\left(\frac{p}{g(p)}\right)^{R+1}\frac{h(p)}{p^2}\right\}\right] \quad (5.18)$$

where

$$\frac{h(p)}{p^2} = \sum_{j=2}^{\infty}\frac{(-a)^j bp^{j-2}(j-1)}{j!} = B(p).$$

Hence from (5.18)

$$(k+1)!Q_{R,k+1}(0) = R\lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{\left(\frac{p}{g(p)}\right)^{R+1}B(p)\right\}\right]$$

$$= R\lim_{p\to 0}\left[\sum_{\mu=0}^{k}\binom{k}{\mu}\left\{\left(\frac{p}{g(p)}\right)^{R+1}\right\}^{(k-\mu)}B^{(\mu)}(p)\right] \quad (5.19)$$

by the Leibniz rule of differentiation, where

$$B^{(\mu)}(p) = \frac{d^{\mu}}{dp^{\mu}} B(p).$$

Now since

$$\lim_{p \to 0} \left[\frac{d^{\mu}}{dp^{\mu}} B(p) \right] = \frac{(-1)^{\mu} b a^{\mu+2}}{(\mu+2)}$$

and substituting in (5.19) we find that

$$(k+1) Q_{R,k+1}(0) = R \sum_{\mu=0}^{k} \frac{(-1)^{\mu} b a^{\mu+2} (\mu+1)}{(\mu+2)!} Q_{R+1,k-\mu}(0)$$

which completes the proof of lemma 18. The following lemma regarding moments of the generator function $\phi(x)$ will be proved and required in the evaluation of another comprehensive recurrence relation for the contribution $Q_{R,k}(0)$ to the residues.

Lemma 19 . *The n^{th} moment of the R^{th} convolution of $\phi(x) = -bH(a-x)$ is $(-ab)^R (-a)^n n! C_n^R$.*

Proof: Consider the rectangular wave $\phi(x) = -bH(a-x) = b(-1 + H(x-a))$, which has a Laplace transform of $\Phi(p) = \frac{b(-1+e^{-ap})}{p}$. The R^{th} convolution of $\Phi(p)$ can be expressed as

$$\Phi^R(p) = b^R \left(\frac{-1 + e^{-ap}}{p} \right)^R$$

$$= b^R \left(\frac{-1 + \sum_{r=0}^{\infty} \frac{(-ap)^r}{r!}}{p} \right)^R \quad ; R = 1, 2, 3...$$

$$= b^R \left(-a \sum_{r=0}^{\infty} \frac{(-ap)^r}{(r+1)!} \right)^R$$

$$= (-ab)^R \sum_{r=0}^{\infty} C_r^R (-ap)^r. \tag{5.20}$$

The convolution constants, C_r^R in (5.20) can be evaluated recursively as follows

$$\left.\begin{array}{c} C_r^1 = \beta_r = \frac{1}{(r+1)!};\ R = 1 \\ C_r^R = \sum\limits_{j=0}^{r} \beta_{r-j} C_j^{R-1};\ R = 2,3,4... \end{array}\right\},$$

and they are polynomials in R of degree r; in fact they are related to Stirling polynomials of the second kind so that $C_r^R = (-1)^r P_r(R)$ where $P_r(R)$ are the polynomials fully described in chapter 4. The n^{th} moment of the R^{th} convolution can be obtained by differentiating (5.20) n times with respect to p, so that

$$\frac{d^n}{dp^n}\left[\Phi^R(p)\right] = (-ab)^R \sum_{r=n}^{\infty} C_r^R (-a)^r r(r-1)...(r-n+1)p^{r-n},$$

and therefore

$$\lim_{p\to 0} \frac{d^n}{dp^n}\left[\Phi^R(p)\right] = (-ab)^R (-a)^n n! C_n^R,$$

hence the proof of the lemma is complete. This lemma may now be used to determine a recurrence for $Q_{R,k}(0)$ which, it is argued to be more computationally efficient than directly using (5.12).

Lemma 20 . *A recurrence relation for the evaluation of the terms $Q_{R,k}(0)$ in (5.12) is:*

$$Q_{R,k}(0) = \frac{1}{\left(1-(-ab)^R\right)}\left[\sum_{j=0}^{R}\binom{R}{j}\sum_{r=0}^{k}(-a)^{k-r}(-ab)^j C_{k-r}^j Q_{j,r}(0) - (-ab)^R Q_{R,k}(0)\right] \tag{5.21}$$

with initial values $C_0^0 = 1$ and $Q_{0,0}(0) = 1$.

Proof: From (5.12)

$$k!Q_{R,k}(0) = \lim_{p\to 0}\frac{d^k}{dp^k}\left[\left(\frac{1}{1-\Phi(p)}\right)^R\right];k = 0,1,2,...,(R-1)$$

$$= \lim_{p\to 0}\frac{d^k}{dp^k}\left[\left(1+\frac{\Phi(p)}{1-\Phi(p)}\right)^R\right]$$

k	$k!Q_{R,k}(0)$
0	$\frac{1}{(1+ab)^R}$
1	$\frac{1}{2(1+ab)^{R+1}}\left\{Rba^2\right\}$
2	$\frac{Rba^3}{12(1+ab)^{R+2}}\left\{ab(3R-1)-4\right\}.$
3	$\frac{Rba^4}{8(1+ab)^{R+3}}\left\{2-4abR+a^2b^2R(R-1)\right\}.$
4	$\frac{Rba^5}{240(1+ab)^{R+4}}\left\{\begin{array}{c} a^3b^3\left(15R^3-30R^2+5R+2\right)-a^2b^2\left(120R^2+40R-16\right)+ \\ ab(200R+56)-48 \end{array}\right\}.$
5	$\frac{Rba^6}{96(1+ab)^{R+5}}\left\{\begin{array}{c} a^4b^4\left(3R^4-10R^3+5R^2+2R\right)-a^3b^3\left(40R^3-40R^2-16R\right)+ \\ a^2b^2\left(140R^2+36R-8\right)-ab(128R+64)+16 \end{array}\right\}.$

Table 5.1: Some values of the recurrence (5.12).

$$=\lim_{p\to 0}\sum_{j=0}^{R}\frac{d^k}{dp^k}\left[\binom{R}{j}\Phi^j(p)\frac{1}{(1-\Phi(p))^j}\right]$$

$$=\lim_{p\to 0}\sum_{j=0}^{R}\binom{R}{j}\sum_{r=0}^{k}\binom{k}{r}\frac{d^{k-r}}{dp^{k-r}}\left[\Phi^j(p)\right]\frac{d^r}{dp^r}\left\{\frac{1}{(1-\Phi(p))^j}\right\}.$$

Utilizing lemma 19, for the $(k-r)th$ moment of $\Phi^j(p)$ implies that

$$k!Q_{R,k}(0)=\sum_{j=0}^{R}\binom{R}{j}\sum_{r=0}^{k}\binom{k}{r}(-a)^{k-r}(-ab)^j(k-r)!C_{k-r}^{j}r!Q_{j,r}(0)$$

$$Q_{R,k}(0)=\sum_{j=0}^{R}\binom{R}{j}\sum_{r=0}^{k}\binom{k}{r}(-a)^{k-r}(-ab)^jC_{k-r}^{j}Q_{j,r}(0),$$

using the fact that $C_0^R=1$ and taking the term at $j=R, r=k$ to the left hand side we obtain the recurrence (5.21). Now using recurrence (5.16), or (5.21), we can list some values of $Q_{R,k}(0)$ as given in table 5.1.

The following lemma on a functional relationship of $Q_{R,k}(0)$ will be useful in the proof of the main theorem 17.

Lemma 21 .

$$\begin{aligned} R(1+ab)Q_{R+1,k+1}(0)+ab\frac{d}{db}Q_{R,k}(0) &= (R-(k+1))Q_{R,k+1}(0), \\ k &= 0,1,2,3,...,(R-1). \end{aligned} \tag{5.22}$$

Proof: From (5.12)

$$\frac{d}{db}\left\{k!Q_{R,k}(0)\right\} = \frac{d}{db}\left\{\lim_{p\to 0}\left[\frac{d^k}{dp^k}\left(\frac{p}{g(p)}\right)^R\right]\right\}.$$

Interchanging the order of differentiation in the second term, and after some simplification, we obtain

$$\frac{d}{db}\left\{k!Q_{R,k}(0)\right\} = \lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{\frac{-Rp^R\frac{d}{db}g(p)}{g^{R+1}(p)}\right\}\right]$$

and since

$$\frac{d}{db}g(p) = \frac{g(p)-p}{b}$$

we have

$$k!\frac{d}{db}Q_{R,k}(0) = -\frac{R}{b}\lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{\frac{p^R}{g^R(p)} - \frac{p^{R+1}}{g^{R+1}(p)}\right\}\right],$$

and using (5.12)

$$b\frac{d}{db}Q_{R,k}(0) = R\left(Q_{R+1,k}(0) - Q_{R,k}(0)\right). \tag{5.23}$$

Now, the $Q(0)$ terms may be associated by constants $c_1, c_2,$ and c_3 such that

$$Q_{R+1,k+1}(0) + c_1(R-(k+1))\,Q_{R,k+1}(0) +$$

$$c_2Q_{R,k}(0) + c_3Q_{R+1,k}(0) = 0; \tag{5.24}$$

the $(R-(k+1))$ factor in $Q_{R,k+1}(0)$ is required since it does not contribute for $R = k+1$. From table 5.1 we choose three k values and substitute the respective $Q(0)$ values in (5.24), then solving for $c_1, c_2,$ and c_3 we obtain

$$c_1 = -\frac{1}{R(1+ab)},\; c_2 = -\frac{a}{1+ab},\; \text{and } c_3 = \frac{a}{1+ab}.$$

Hence from (5.24)

$$Q_{R+1,k+1}(0) - \frac{(R-(k+1))}{R(1+ab)}Q_{R,k+1}(0) - \frac{a}{1+ab}Q_{R,k}(0)$$

$$+\frac{a}{1+ab}Q_{R+1,k}(0) = 0. \tag{5.25}$$

R	The closed form of (5.15).
1	$\frac{1}{1+ab}$.
2	$\frac{t}{(1+ab)^2}+\frac{ba^2}{(1+ab)^3}$.
3	$\frac{t^2}{2(1+ab)^3}+\frac{3ba^2t}{2(1+ab)^4}-\frac{ba^3(1-2ab)}{2(1+ab)^5}$.
4	$\frac{t^3}{6(1+ab)^4}+\frac{ba^2t^2}{(1+ab)^5}-\frac{ba^3(4-11ab)t}{6(1+ab)^6}+\frac{ba^4\left(1-8ab+6a^2b^2\right)}{6(1+ab)^7}$.
5	$\frac{t^4}{24(1+ab)^5}+\frac{5ba^2t^3}{12(1+ab)^6}-\frac{5ba^3(2-7ab)t^2}{(1+ab)^7}+\frac{5ba^4\left(1-10ab+10a^2b^2\right)t}{4(1+ab)^8}-\frac{ba^5\left(3-66ab+199a^2b^2-72a^3b^3\right)}{3(1+ab)^9}$.

Table 5.2: Some closed form expressions of (5.15).

Substituting (5.23) in (5.25), and after some minor manipulation, we obtain the result (5.22), and the proof of lemma 21 is complete. Using the $Q_{R,k}(0)$ in table 5.1, some closed form representation of the infinite series (5.15) are given in table 5.2.

For the specialized case of $R = 2, b = -1$ and $a = \lambda$ from (5.14) and (5.15) we obtain the result, (2.84) in chapter 2, that Conolly missed. In the next section we give a proof of the main theorem 17.

5.2.3 Lemma and proof of theorem 17.

The following lemma will be useful for the proof of theorem 17.

Lemma 22 . *A recurrence relation for the infinite series (5.14) is*

$$R\,(1+ab)\,S_{R+1} + ab\frac{d}{db}S_R - tS_R = 0. \tag{5.26}$$

Proof: From (5.13) and (5.14)

$$S_{R+1} = \sum_{n=0}^{\infty} \binom{n+R}{n} \frac{b^n e^{-b(t-an)}\,(t-an)^{n+R}}{(n+R)!}$$

$$= \frac{1}{R}\left(tS_R - a\sum_{n=0}^{\infty} nT_n\right). \tag{5.27}$$

Also, from (5.13) and (5.14)

$$\frac{d}{db}S_R = \frac{1+ab}{b}\sum_{n=0}^{\infty} nT_n - tS_R. \tag{5.28}$$

Now multiplying (5.27) and (5.28) by $R(1+ab)$ and ab respectively and substituting into the left hand side of (5.26), gives

$$
\begin{aligned}
& (1+ab)\,tS_R - a(1+ab)\sum_{n=0}^{\infty} nT_n + \\
& a(1+ab)\sum_{n=0}^{\infty} nT_n - (1+ab)\,tS_R \\
= \; & 0
\end{aligned}
$$

which is identical to the right hand side of (5.26) and the proof is complete.

Proof of theorem 17.

The proof of theorem 17 will involve an induction argument on the parameter R. Lemma 22 proves the left hand side of (5.15). For the basis, $R=1$, a proof of (5.15) has been given in chapter 2. For $R=2$, a proof of (5.15), by Bürmann's theorem has been given by Sofo and Cerone [83]. The induction argument for the right hand side of (5.15) will involve the recurrence relation (5.26). From (5.26)

$$
\begin{aligned}
S_{R+1} &= \frac{1}{R(1+ab)}\left[tS_R - ab\frac{d}{db}S_R\right] \\
&= \frac{1}{R(1+ab)}\left[\begin{array}{c} t\sum\limits_{k=0}^{R-1}\frac{t^{R-k-1}}{(R-k-1)!}Q_{R,k}(0) \\ -ab\sum\limits_{k=0}^{R-1}\frac{t^{R-k-1}}{(R-k-1)!}\frac{d}{db}Q_{R,k}(0)\end{array}\right]
\end{aligned}
$$

$$
= \frac{1}{R(1+ab)}\left[\begin{array}{c} \frac{Rt^R}{R!}Q_{R,0}(0) + \sum\limits_{k=1}^{R}\frac{(R-k)t^{R-k}}{(R-k)!}Q_{R,k}(0) \\ -ab\sum\limits_{k=1}^{R}\frac{t^{R-k}}{(R-k)!}\frac{d}{db}Q_{R,k-1}(0)\end{array}\right] \tag{5.29}
$$

where the counter in the third term has been adjusted. Now collecting terms in (5.29) we have that

$$
\begin{aligned}
S_{R+1} = \; & \frac{1}{R(1+ab)}\sum_{k=1}^{R}\left[(R-k)\,Q_{R,k}(0) - ab\frac{d}{db}Q_{R,k-1}(0)\right]\frac{t^{R-k}}{(R-k)!} + \\
& \frac{t^R}{(1+ab)\,R!}Q_{R,0}(0)\,.
\end{aligned} \tag{5.30}
$$

From lemma 21, after adjusting the counter k

$$R(1+ab)\,Q_{R+1,k}(0)=(R-k)\,Q_{R,k}(0)-ab\frac{d}{db}Q_{R,k-1}(0) \tag{5.31}$$

so that by substituting (5.31) into the square bracket of (5.30) we have that

$$S_{R+1}=\frac{t^R}{(1+ab)\,R!}Q_{R,0}(0)+\sum_{k=1}^{R}\frac{t^{R-k}}{(R-k)!}Q_{R+1,k}(0)$$

$$=\frac{t^R}{R!}Q_{1,0}(0)\,Q_{R,0}(0)+\sum_{k=1}^{R}\frac{t^{R-k}}{(R-k)!}Q_{R+1,k}(0) \tag{5.32}$$

where $Q_{1,0}(0)$ is identified in (5.16) or table 5.1. By the convolution nature of the $Q(0)$ terms we may write equation (5.32) as

$$\begin{aligned} S_{R+1} &= \frac{t^R}{R!}Q_{R+1,0}(0)+\sum_{k=1}^{R}\frac{t^{R-k}}{(R-k)!}Q_{R+1,k}(0) \\ &= \sum_{k=0}^{R}\frac{t^{R-k}}{(R-k)!}Q_{R+1,k}(0) \end{aligned}$$

which completes the proof of theorem 17.

It is now worthwhile to briefly indicate a functional relationship for the infinite sum (5.14). From the left hand side of (5.15), let

$$t=a\tau,\rho=R-1 \text{ and } \gamma=abe^{ab} \text{ then}$$

$$\sigma_\rho(\tau)=\sum_{n=0}^{\infty}\frac{(-\gamma)^n(\tau+n)^{n+\rho}}{n!},\rho=0,1,2,3...$$

and

$$\sigma_\rho(\tau)+\gamma\sigma_\rho(\tau+1)=\tau\sigma_{\rho-1}(\tau). \tag{5.33}$$

Pyke and Weinstock [78] gave a functional relationship of (5.33) for the case of $R=1$ only. Sofo and Cerone [83] have given a proof of the functional form (5.33) for the general case of integer R.

5.2.4 Extension of results.

The dynamical system (5.1) may take other functional values of the forcing terms $w(t)$, other than $\delta(t)$, such that consequent results of (5.15) may be extended. We shall consider two other cases.

Case1: Let

$$w(t)=\frac{e^{-bt}t^{m-1}}{(m-1)!}$$

in the system (5.1), where m is a positive integer and following the procedure of section 5.2, we obtain

$$\sum_{n=0}^{\infty}\binom{n+R-1}{n}\frac{b^n e^{-b(t-an)}(t-an)^{n+m+R-1}}{(n+m+R-1)!}$$
$$=\sum_{\mu=0}^{R-1}\frac{t^{R-\mu-1}Q_{R,\mu}(0)}{(R-\mu-1)!}+\sum_{\nu=0}^{m-1}\frac{t^{m-\nu-1}}{(m-\nu-1)!}P_{m,\nu}(-b). \tag{5.34}$$

In identity (5.34) we have that

$$\nu!P_{m,\nu}(-b)=\lim_{p\to-b}\left[\frac{d^\nu}{dp^\nu}\left\{(p+b)^m F(p)\right\}\right];\nu=0,1,2,3,...,(m-1),$$

$$\mu!Q_{R,\mu}(0)=\lim_{p\to0}\left[\frac{d^\mu}{dp^\mu}\left\{p^R F(p)\right\}\right];\mu=0,1,2,3,...,(R-1)$$

and

$$F(p)=\frac{1}{(p+b)^m(p+b-be^{-ap})^R}.$$

For $R=1$ and $m=2$ we have

$$\sum_{n=0}^{\infty}\frac{b^n e^{-b(t-an)}(t-an)^{n+2}}{(n+2)!}=-\frac{e^{-bt}}{be^{ab}}\left[t+\frac{1+abe^{ab}}{be^{ab}}\right]+\frac{1}{b^2(1+ab)}$$

and for $R=2$ and $m=3$

$$\sum_{n=0}^{\infty}\binom{n+1}{n}\frac{b^n e^{-b(t-an)}(t-an)^{n+4}}{(n+4)!}$$

$$= e^{-bt}\left[\frac{t^2}{2\left(be^{ab}\right)^2}+\frac{2t\left(1+abe^{ab}\right)}{\left(be^{ab}\right)^3}+\frac{2\left(abe^{ab}\right)^2+6abe^{ab}+3}{\left(be^{ab}\right)^4}\right]$$
$$+\frac{t}{b^3\left(1+ab\right)^2}+\frac{a^2}{2b^2\left(1+ab\right)^3}-\frac{3}{b^4\left(1+ab\right)^2}.$$

In the degenerate case, for $a=0$, from (5.34) we obtain the impressive identity

$$\sum_{n=0}^{\infty}\left(\begin{array}{c} n+R-1 \\ n \end{array}\right)\frac{b^n e^{-bt}t^{n+m+R-1}}{(n+m+R-1)!}$$
$$= \frac{(-1)^R}{e^{bt}}\sum_{\nu=0}^{m-1}\frac{(R)_\nu \ t^{m-\nu-1}}{b^{R+\nu}\ \nu!\,(m-\nu-1)!}+\sum_{\mu=0}^{R-1}\frac{(-1)^\mu (m)_\mu \ t^{R-\mu-1}}{b^{m+\mu}\ \mu!\,(R-\mu-1)!},$$

where $(x)_p$ is known as Pochhammer's symbol. The identities (5.34) and (5.15) may be differentiated and integrated with respect to t to produce more identities.

Case2: Let

$$w(t)=\frac{t^{m-1}}{(m-1)!}$$

in the system (5.1), where m is a positive integer and following the procedure of section 5.2, we obtain

$$\sum_{n=0}^{\infty}(-b)^n\left(\begin{array}{c} n+R-1 \\ n \end{array}\right)\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{(-1)^r\,(t-ar)^{n+m+R-1}}{(n+m+R-1)!}=\sum_{k=0}^{m+R-1}\frac{t^{m+R-k-1}Q_{R,k}\left(0\right)}{(m+R-k-1)!}. \tag{5.35}$$

$$k!Q_{R,k}\left(0\right) = \lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{p^{m+R}F\left(p\right)\right\}\right]=\lim_{p\to 0}\left[\frac{d^k}{dp^k}\left\{\left(\frac{p}{g\left(p\right)}\right)^R\right\}\right], \tag{5.36}$$
$$k = 0,1,2,...,m+R-1; m\geq 1; R\geq 1$$

where

$$F\left(p\right)=\frac{1}{p^m\left(p+b-be^{-ap}\right)^R}$$

which has a pole of order $m+R$ at the singularity $p=0$, and $g\left(p\right)$ is defined by (5.11). Let

$t = -ax$ and from (5.35) we obtain

$$\sum_{n=0}^{\infty}(-ab)^n \binom{n+R-1}{n} \frac{(-1)^{n+m+R-1}}{(n+m+R-1)!}\sum_{r=0}^{n}\binom{n}{r}(-1)^r (x+r)^{n+m+R-1}$$
$$= \sum_{k=0}^{m+R-1} \frac{(-x)^{m+R-k-1} Q_{R,k}(0)}{a^k (m+R-k-1)!} \tag{5.37}$$

where the inner sum on the left hand side is the general polynomial investigated in chapter 4, namely

$$V_{m+R-1}(n,x) = \frac{(-1)^{n+m+R-1}}{(n+m+R-1)!}\sum_{r=0}^{n}\binom{n}{r}(-1)^r (x+r)^{n+m+R-1}$$

and hence from (5.37)

$$\sum_{n=0}^{\infty}(-ab)^n \binom{n+R-1}{n} V_{m+R-1}(n,x) = \sum_{k=0}^{m+R-1} \frac{(-x)^{m+R-k-1} Q_{R,k}(0)}{a^k (m+R-k-1)!}. \tag{5.38}$$

By an application of the ratio test the infinite sum in (5.38) converges in the region $|ab| < 1$, since $\lim_{n\to\infty}\left|\frac{V_{m+R-1}(n+1,x)}{V_{m+R-1}(n,x)}\right| = 1$. In the special case, that of $x = 0\,(a \neq 0)$, $V_{m+R-1}(n,0) = P_{m+R-1}(n)$, where $P_{m+R-1}(n)$ are the polynomials in chapter 4, described by (4.21), and therefore from (5.38)

$$\sum_{n=0}^{\infty}(-ab)^n \binom{n+R-1}{n} P_{m+R-1}(n) = \frac{Q_{R,m+R-1}(0)}{a^{m+R-1}}. \tag{5.39}$$

From (5.35), for the degenerate case $a = 0$, we have the identity

$$\sum_{n=0}^{\infty}(-b)^n \binom{n+R-1}{n} \frac{t^n}{(n+m+R-1)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \sum_{k=0}^{m+R-1} \frac{t^{-k} Q_{R,k}(0)}{(m+R-k-1)!},$$

since on the left, the inner sum is unity for $n = 0$ and zero otherwise and on the right, using (5.36) $Q_{R,k}(0) = 1$ for $k = 0$, and zero otherwise. Some examples are now illustrated. Putting

$\alpha = -ab$ in (5.38), we have for the case of $R = 1$ and $m = 1$ that

$$\sum_{n=0}^{\infty} \alpha^n (n + 2x) = \frac{2x}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2},$$

and when $x = 0$, $\sum\limits_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}$. However Jolley [64], entry 40 on page 8, gives the listing $\sum\limits_{n=0}^{\infty} n\alpha^n = \frac{\alpha(2-\alpha)}{(1-\alpha)^2}$, which is obviously incorrect. Jolley gives no other entries of this form, apart from chapter 2, (2.12). We have given a general method for closed form representations. For $R = 2$ and $m = 2$ we have that

$$\begin{aligned}\sum_{n=0}^{\infty} \alpha^n (n+1)\left(n^3 + n(1+6x)(n+2x) + 8x^3\right) &= \frac{8x^3}{(1-\alpha)^2} + \frac{24\alpha x^2}{(1-\alpha)^3} + \frac{4\alpha x(5\alpha+4)}{(1-\alpha)^4} \\ &\quad + \frac{4\alpha\left(\alpha^2+4\alpha+1\right)}{(1-\alpha)^5},\end{aligned}$$

which may also be checked on, say, 'Mathematica'.

5.2.5 Renewal processes.

In the theory of renewal processes, let $M(t) = E(N(t))$, be the expected number of renewals in the time interval $[0, t]$ such that $M(t) = \sum\limits_{n=1}^{\infty} \int_0^t f^{(n)}(x)\,dx$ and the Laplace transform is

$$\bar{M}(p) = \frac{1}{p}\left[\frac{\bar{f}(p)}{1-\bar{f}(p)}\right]. \tag{5.40}$$

Also, the expected instantaneous renewal rate $m(t) = \frac{d}{dt}M(t)$, whenever $\frac{d}{dt}M(t)$ exists such that

$$\bar{m}(p) = \frac{\bar{f}(p)}{1-\bar{f}(p)}. \tag{5.41}$$

$\bar{M}(p), \bar{m}(p)$ and $\bar{f}(p)$ are, respectively, the Laplace transforms of $M(t), m(t)$ and $f(t)$. Feller [44] obtains the average number of registrations of an event $M(t)$, for a type 1 counter as

$$M(t) = 1 - e^{-bt} + \int_0^t M(t-x)\, be^{-b(x-a)} H(x-a)\,dx. \tag{5.42}$$

From (5.40) or (5.42) we may obtain

$$\bar{M}(p) = \frac{1}{p(p+b-be^{-ap})}$$

and upon inversion the result is identical to that given by (5.35) with $R = 1$ and $m = 1$. Chaudhry [26] considers various other forms of the function $\bar{f}(p)$ from which expected instantaneous renewal rates are evaluated. For the shifted exponential function

$$f(t) = \left(1 - e^{-b(t-a)}\right) H(t-a)$$

and from (5.41) we have

$$\bar{m}(p) = \frac{be^{-ap}}{p^2 + bp - be^{-ap}}$$

where

$$g(p) = p^2 + bp - be^{-ap}. \tag{5.43}$$

From the work of the previous section we may write

$$m(t) \sim \sum_{n=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r} \frac{(-1)^{n+r} b^{n+1} (t - a(r+1))^{n+r+1}}{(n+r+1)!} = \sum_{j=0}^{1} \frac{\xi_j (\xi_j + b) e^{\xi_j t}}{a\xi_j^2 + (2+ab)\xi_j + b},$$

where ξ_j, $j = 0, 1$ are the two distinct dominant zeros of the characteristic function $g(p)$ in (5.43). In the next part we shall prove closed form representations of infinite sums which depend on k dominant zeros of an associated characteristic function.

5.3 The k-dominant zeros case.

5.3.1 The k-system.

Consider, for a well behaved function $f(t)$, the forced dynamical system with constant real coefficients b and c, real delay parameter a, and all initial conditions at rest,

$$\left.\begin{array}{l} cf(t-a)+\sum\limits_{r=0}^{k}\begin{pmatrix} k \\ r \end{pmatrix} b^{k-r} f^{(r)}(t) = w(t)\,;\; t>a \\ \sum\limits_{r=0}^{k}\begin{pmatrix} k \\ r \end{pmatrix} b^{k-r} f^{(r)}(t) = w(t)\,;\; 0<t\le a. \end{array}\right\}. \tag{5.44}$$

In the system (5.44) $w(t)$ is a forcing term, t a real variable, and k is a positive integer. If we let $w(t)=\delta(t)$, for illustration, and use the methods of the previous section we obtain

$$F(p)=\frac{1}{(p+b)^{k}+ce^{-ap}}. \tag{5.45}$$

Expanding (5.45), inverting and considering the residue, we get

$$\sum_{n=0}^{\infty}\frac{(-c)^{n}e^{-b(t-an)}(t-an)^{nk+k-1}}{(nk+k-1)!}=\sum_{\nu=0}^{k-1}Q(\xi_{\nu})e^{\xi_{\nu}t} \tag{5.46}$$

where

$$Q(\xi_{\nu})=\lim_{p\to\xi_{\nu}}\left[(p-\xi_{\nu})F(p)\right],\; \nu=0,1,2,3,...,k-1 \tag{5.47}$$

and ξ_{ν} are defined as the k dominant distinct zeros of the characteristic function $g(p)=(p+b)^{k}+ce^{-ap}$. To simplify the algebra let us take $c+b^{k}=0$, which allows one dominant zero of the characteristic function

$$g_{k}(p)=(p+b)^{k}-b^{k}e^{-ap}, \tag{5.48}$$

with $k+ab>0$, to occur at the origin. The condition $k+ab\neq 0$ will ensure the distinct nature of the zeros of (5.48). From these considerations and (5.46) we have the following theorem.

Theorem 23 . *Let*

$$T_n(k,b,a,t) = \frac{b^{nk} e^{-b(t-an)} (t-an)^{nk+k-1}}{(nk+k-1)!} \quad \textit{and} \tag{5.49}$$

$$S(k,b,a,t) = \sum_{n=0}^{\infty} T_n(k,b,a,t) \tag{5.50}$$

which is convergent for all values of k, b, a *and* t *in the region*

$$\left|(ab)^k e^{ab}\right| < \left(ke^{-1}\right)^k. \tag{5.51}$$

Then

$$S(k,b,a,t) = \sum_{\nu=0}^{k-1} Q(\xi_\nu) e^{\xi_\nu t}, \tag{5.52}$$

where $Q(\xi_\nu)$ *is defined in (5.47) and* ξ_ν *are the* k *dominant distinct zeros of the characteristic function (5.48).*

The following two lemmas, regarding the location of dominant zeros, will be useful in the proof of Theorem 23.

Lemma 24 . *The characteristic function (5.48) has* k *simple dominant zeros lying in the region* $\Gamma : |p| < \frac{k+ab}{a}; a, b > 0$ *and* k *is a positive integer.*

Proof: We have previously defined a dominant zero as the one with the greatest real part. It is known, see [7], that (5.48) has an infinite number of zeros lying in the left (or right) half plane. Using the same method as described in chapter 2, lemma 6, it can be shown that (5.48) has at most three (and at least one) real zeros with restriction (5.51) one of which is at the origin, $\xi_0 = 0$. Applying Rouche's theorem it is required to show that $|A(w)| > |B(w) - A(w)|$ for $w = p + b, A(w) = w^k, B(w) = w^k - b^k e^{ab-aw}$ in the region $\Gamma' : |w| < \frac{k+2ab}{a}$. Now $A(w)$ has k zeros lying in the region Γ' and since $\left|w^k\right| > \left|-b^k e^{ab-aw}\right|$ implies that $(k+2ab)^k > (ab)^k$; then $B(w)$ has k dominant zeros lying in the region Γ' and hence (5.48) has k dominant zeros lying in Γ.

Lemma 25 . *The characteristic function*

$$q_j(p) = p + b - be^{(2\pi ij - ap)/k} \tag{5.53}$$

for $j = 0, 1, 2, 3, ..., k-1$ *has one dominant zero for each* j *lying in the region* Γ *as defined in lemma 24.*

Proof: Now, $A_1(w) = w$ has one dominant zero lying in the region Γ' and $B_1(w) = w - be^{(2\pi ij + ab - aw)/k}$. Therefore $|w| > \left|-be^{(2\pi ij+ab-aw)/k}\right|$ implies that $(k + 2ab) > ab$, hence $B_1(w)$ has one zero lying inside the region Γ' and it follows that for $j = 0, 1, 2, 3, ..., k-1$, (5.53) has one dominant zero lying in the region Γ.

Proof of Theorem 23: Firstly, we evaluate $Q(\xi_\nu)$ from (5.47) and from (5.52) we may write

$$S(k, b, a, t) = \sum_{n=0}^{\infty} \frac{b^{nk} e^{-b(t-an)} (t - an)^{nk+k-1}}{(nk + k - 1)!} = \sum_{\nu=0}^{k-1} \frac{e^{\xi_\nu t}}{(b + \xi_\nu)^{k-1} (k + ab + a\xi_\nu)}. \tag{5.54}$$

The characteristic function (5.48) may be expressed as the product of factors such that,

$$g_k(p) = (p + b)^k - b^k e^{-ap} = \prod_{j=0}^{k-1} \left(p + b - be^{(2\pi ij - ap)/k}\right) = \prod_{j=0}^{k-1} q_j(p).$$

Lemmas 24 and 25 show that the dominant zeros, α_j, of $q_j(\alpha_j)$ for each $j = 0, 1, 2, 3, ..., k-1$ are the same as the k dominant zeros of $g_k(p)$. Using (5.47), the contribution $\Omega(\alpha_j)$ to each of the factors $q_j(\alpha_j)$ is

$$\begin{aligned} \Omega(\alpha_j) &= \lim_{p \to \alpha_j} \left[(p - \alpha_j) F_j(p)\right] \\ &= \lim_{p \to \alpha_j} \left[\frac{p - \alpha_j}{q_j(p)}\right] = \frac{k}{k + ab + a\alpha_j} \end{aligned}$$

and using this result, we have from (2.10) that

$$\sum_{n=0}^{\infty} \frac{\left(be^{2\pi ij/k}\right)^n e^{-b\left(t - \frac{an}{k}\right)} \left(t - \frac{an}{k}\right)^n}{n!} = \frac{ke^{\alpha_j t}}{k + ab + a\alpha_j} \tag{5.55}$$

for each $k = 1, 2, 3, \ldots$ and $j = 0, 1, 2, 3, \ldots, k-1$. Note that the sum (5.55) may in fact be a complex number. The summation of all the k dominant zeros , for each of the factors $q_j(\alpha_j)$ implies from (5.55) that

$$\begin{aligned}\sum_{j=0}^{k-1} \frac{k e^{\alpha_j t}}{k + ab + a\alpha_j} &= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} \frac{\left(b e^{2\pi i j/k}\right)^n e^{-b\left(t-\frac{an}{k}\right)} \left(t - \frac{an}{k}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{b^n e^{-b\left(t-\frac{an}{k}\right)} \left(t - \frac{an}{k}\right)^n}{n!} \sum_{j=0}^{k-1} e^{2\pi i j n/k}.\end{aligned}$$

Rescaling the infinite sum, by putting $n = (n^* + 1)k$, (and then renaming n^* as n) gives the result,

$$kb^k \sum_{n=-1}^{\infty} \frac{b^{nk} e^{-b(t-an-a)} (t - an - a)^{nk+k}}{(nk+k)!} = \sum_{j=0}^{k-1} \frac{k e^{\alpha_j t}}{k + ab + a\alpha_j}.$$

Now letting $y = t - a$ and, from $q_j(\alpha_j)$, using the fact that $e^{a\alpha_j} = \left(\frac{b}{b+\alpha_j}\right)^k$ then

$$\sum_{n=0}^{\infty} \frac{b^{nk} e^{-b(y-an)} (y - an)^{nk+k}}{(nk+k)!} = \sum_{j=0}^{k-1} \frac{e^{\alpha_j y} e^{-b(a+y)}}{(b+\alpha_j)^k (k + ab + a\alpha_j) b^k}. \tag{5.56}$$

Differentiating (5.56) with respect to y, which is permissible within the radius of convergence (5.51), gives after some algebraic manipulation

$$\sum_{n=0}^{\infty} \frac{b^{nk} e^{-b(y-an)} (y - an)^{nk+k-1}}{(nk+k-1)!} = \sum_{j=0}^{k-1} \frac{e^{\alpha_j y}}{(b+\alpha_j)^{k-1} (k + ab + a\alpha_j)}. \tag{5.57}$$

Renaming y as t shows that (5.57) is the same as (5.54) since by lemmas 24 and 25, $\alpha_j = \xi_\nu$ for $j = 0, 1, 2, 3, \ldots, k-1$; $\nu = 0, 1, 2, 3, \ldots, k-1$, and therefore theorem 23 is proved. Some examples are now given to illustrate the above theorem.

5.3.2 Examples.

(i). For k **even** there are 2 real dominant distinct zeros and $(k-2)$ complex conjugate zeros of the characteristic function (5.48) that need to be considered for determining the right hand side of (5.54). Consider, in particular, the case $k = 2$, then

$$\sum_{n=0}^{\infty} \frac{b^{2n} e^{-b(t-an)} (t-an)^{2n+1}}{(2n+1)!} = \sum_{\nu=0}^{1} \frac{e^{\xi_\nu t}}{(b+\xi_\nu)(2+ab+a\xi_\nu)}.$$

For $(a,b,t) = (.1, 2, 2)$ then $(\xi_0, \xi_1) = (0, -4.5053)$ and the sum takes the value, to four significant digits, .2272.

(ii). For k **odd** there are 3 real dominant distinct zeros and $(k-1)$ complex conjugate zeros of the characteristic function (5.48) that need to be considered for determining the right hand side of (5.54). Consider, in particular, $k = 3$, in this case there will be one real zero ξ_0 and two complex conjugate zeros $\xi_1 = (x+iy)$, $\bar{\xi}_1 = (x-iy)$ and ξ_ν satisfies $(\xi_\nu + b)^3 - b^3 e^{-a\xi_\nu} = 0, \nu = 0, 1, 2$. Hence we have

$$\sum_{n=0}^{\infty} \frac{b^{3n} e^{-b(t-an)} (t-an)^{3n+2}}{(3n+2)!} = \frac{e^{\xi_0 t}}{(b+\xi_0)^2 (3+ab+a\xi_0)} + \frac{2e^{xt}\left[(x_2x_4 - x_3x_5)\cos yt + (x_2x_5 + x_3x_4)\sin yt\right]}{(x_1^2+y^2)^2 (x_5^2+x_4^2)}$$

where $x_1 = (x+b)$, $x_2 = x_1^2 - y^2$, $x_3 = 2yx_1$, $x_4 = 3 + ax_1$ and $x_5 = ay$. For $(a,b,t) = (.8, 1, 2)$ then $(\xi_0, \xi_1, \bar{\xi}_1) = (0, -1.2193 + 1.3668i, -1.2193 - 1.3668i)$ and the sum takes the value, to four significant digits, .2769. Again the previous results (5.54) may be extended in various directions, we briefly mention one extension.

5.3.3 Extension.

Consider, for a well behaved function $f(t)$, the forced dynamical system with constant real coefficients b and c, real delay parameter a, and all initial conditions at rest,

$$\left.\begin{array}{l} \sum\limits_{j=0}^{R} \binom{R}{R-j} b^{k(R-j)} \sum\limits_{r=0}^{jk} \binom{jk}{r} b^{jk-r} f^{(r)}(t-(R-j)a) = w(t)\,;\ t > Ra \\ \sum\limits_{r=0}^{Rk} \binom{Rk}{r} b^{Rk-r} f^{(r)}(t) = w(t)\,;\ 0 < t \le Ra. \end{array}\right\} \tag{5.58}$$

In the system (5.58) $w(t)$ is a forcing term, t a real variable, and R and k are positive integers. Let $w(t) = \frac{e^{-bt}t^{m-1}}{(m-1)!}$, for $m = 1, 2, 3, ...$ and taking the Laplace transform of (5.58) we have

$$F(p) = \frac{1}{(p+b)^m \left((p+b)^k - b^k e^{-ap}\right)^R}. \tag{5.59}$$

By the methods of the previous section 5.3.1 we finally obtain

$$\sum_{n=0}^{\infty} \binom{n+R-1}{n} \frac{b^{nk} e^{-b(t-an)} (t-an)^{nk+Rk+m-1}}{(nk+Rk+m-1)!} = \sum_{r=0}^{m-1} \frac{e^{-bt} t^{m-r-1} P_{m,r}(-b)}{(m-r-1)!}$$

$$+ \sum_{\nu=0}^{k-1} \sum_{\mu=0}^{R-1} \frac{e^{-\xi_\nu t} t^{R-\mu-1} Q_{R,\mu}(\xi_\nu)}{(R-\mu-1)!} \tag{5.60}$$

where

$$r! P_{m,r}(-b) = \lim_{p \to -b} \left[\frac{d^r}{dp^r}\left((p+b)^m F(p)\right)\right]; r = 0, 1, 2, ..., (m-1),$$

$$\mu! Q_{R,\mu}(\xi_\nu) = \lim_{p \to \xi_\nu} \left[\frac{d^\mu}{dp^\mu}\left((p-\xi_\nu)^R F(p)\right)\right]; \mu = 0, 1, 2, ..., (R-1)$$

$F(p)$ is defined by (5.59) and $\xi_\nu, \nu = 0, 1, 2, 3, ..., k-1$ are the k dominant zeros of the characteristic function (5.48). For $(R, k, m) = (2, 2, 3)$ we have, from (5.60)

$$\sum_{n=0}^{\infty} (n+1) \frac{b^{2n} e^{-b(t-an)} (t-an)^{2n+6}}{(2n+6)!} = \frac{1}{2(b^2 e^{ab})^3} \left[b^2 e^{ab} t (t+4a) + 4\left(1 + a^2 b^2 e^{ab}\right)\right]$$

$$+ \sum_{\nu=0}^{1} \frac{e^{\xi_\nu t}}{(b+\xi_\nu)^2 \left(2(b+\xi_\nu) + ab^2 e^{-a\xi_\nu}\right)^2} \left[t - \frac{2}{b+\xi_\nu} - \frac{2 - a^2 b^2 e^{-a\xi_\nu}}{2(b+\xi_\nu) + ab^2 e^{-a\xi_\nu}}\right]$$

where ξ_ν are the two dominant zeros of $(\xi_\nu + b)^2 - b^2 e^{-a\xi_\nu} = 0$. The degenerate case of $a = 0$, implies that the transcendental function (5.48) reduces to a polynomial in p of degree k. Specifically for $(a, R, k, m) = (0, 2, 2, 3)$ we have the identity

$$\sum_{n=0}^{\infty} (n+1) \frac{(bt)^{2n+6}}{(2n+6)!} = \frac{1}{2}\left[(bt)^2 + 4 + bt \sinh(bt) - 4\cosh(bt)\right].$$

Chapter 6

Fibonacci and related series.

A first order difference-delay system is considered and by the use of Z transform theory generate an infinite sum which by the use of residue theory may be represented in closed form. Related works to this area of study are considered and some central binomial coefficient identities are also given. A development of Fibonacci and related polynomials is undertaken together with products and functional forms.[1]

[1] A shortened version of this chapter has been published in The Fibonacci Quarterly, Vol.36, pp.211-215, June-July 1998.

6.1 Introduction.

In this chapter we consider a difference-delay system and by the use of Z transform theory generate an infinite sum which we shall represent in closed form by the use of residue theory. We shall compare our work to that of Jensen and lay the foundation so that our results may be further generalized in the ensuing chapters. We shall investigate some central binomial coefficient identities, and develop Fibonacci related polynomials, products and functional forms.

6.2 The difference-delay system.

Consider the related Fibonacci difference-delay system

$$\left.\begin{array}{c} f_{n+1} - bf_n - cf_{n-a} = 0,\ n \geq a \\ f_{n+1} - bf_n = 0,\ n < a \end{array}\right\} \tag{6.1}$$

with initial condition $f_0 = 1$, b and c are real constants and a and n are positive integers including zero. The Z transform of a sequence $\{f_n\}$ is a function $F(z)$ of complex variable z defined by $F(z) = Z[f_n] = \sum\limits_{n=0}^{\infty} f_n z^{-n}$ for those values of z for which the infinite series converges. Taking the Z transform of (6.1) and using the initial condition yields, upon rearrangement

$$F(z) = \frac{z}{z - b - cz^{-a}} = \frac{z^{a+1}}{z^{a+1} - bz^a - c}, \tag{6.2}$$

and expanding as a series we have

$$F(z) = \sum_{r=0}^{\infty} \frac{c^r z^{1-ar}}{(z-b)^{1+r}}. \tag{6.3}$$

The inverse Z transform of (6.3) is

$$f_n = \sum_{r=0}^{\infty} \binom{n-ar}{r} \left(\frac{c}{b}\right)^r b^{(n-ar)} U(n-ar) = \sum_{r=0}^{[n/(a+1)]} \binom{n-ar}{r} \left(\frac{c}{b}\right)^r b^{(n-ar)} \tag{6.4}$$

where $U(n-ar)$ is the discrete step function and $[x]$ represents the integer part of x. The inverse Z transform of (6.3) may also be expressed as

$$f_n = \frac{1}{2\pi i}\int_C z^{n-1}F(z)\,dz = \sum_{j=0}^{a} z^n \text{Res}_j\left(\frac{F(z)}{z}\right) \tag{6.5}$$

where C is a smooth Jordan curve enclosing the singularities of (6.2) and the integral is traversed once in an anticlockwise direction around C. It may also be shown that in (6.5) there is no contribution from the integral around the contour. For the restriction

$$\left|\frac{c(a+1)^{a+1}}{b(ab)^a}\right| < 1 \tag{6.6}$$

the characteristic function

$$g(z) = z^{a+1} - bz^a - c \tag{6.7}$$

has $(a+1)$ distinct zeros $\xi_j, j = 0, 1, 2, 3, ..., a$. All the singularities in (6.2) are therefore simple poles such that the residue, Res_j of the poles in (6.2) may be evaluated as follows

$$\text{Res}_j = \lim_{z\to\xi_j}\left[(z-\xi_j)\frac{z^a}{z^{a+1} - bz^a - c}\right] = \frac{\xi_j}{(a+1)\xi_j - ab} \tag{6.8}$$

and hence from (6.4), (6.5) and (6.8) we have

$$f_n = \sum_{r=0}^{[n/(a+1)]}\binom{n-ar}{r}\left(\frac{c}{b}\right)^r b^{(n-ar)} = \sum_{j=0}^{a}\frac{\xi_j^{n+1}}{(a+1)\xi_j - ab}. \tag{6.9}$$

A Tauberian theorem [7] suggests that for n large, from (6.9)

$$\sum_{r=0}^{[n/(a+1)]}\binom{n-ar}{r}\left(\frac{c}{b}\right)^r b^{(n-ar)} \sim \frac{\xi_0^{n+1}}{(a+1)\xi_0 - ab}$$

where ξ_0 is the dominant zero of (6.7), defined as the one with the greatest modulus.

6.3 The infinite sum.

Theorem 26 : *For all values of $a, b, c,$ and n*

$$\sum_{r=0}^{\infty} \binom{n-ar}{r} \left(\frac{c}{b}\right)^r b^{(n-ar)} = \frac{\xi_0^{n+1}}{(a+1)\,\xi_0 - ab} \tag{6.10}$$

in the region of convergence (6.6).

Using (6.9) and (6.10) we may write

$$\sum_{r=0}^{[n/(a+1)]} \binom{n-ar}{r} \left(\frac{c}{b}\right)^r b^{(n-ar)} + \sum_{r=[(n+1)/a]}^{\infty} \binom{n-ar}{r} \left(\frac{c}{b}\right)^r b^{(n-ar)} = \frac{\xi_0^{n+1}}{(a+1)\,\xi_0 - ab}$$

and so

$$\sum_{r=[(n+1)/a]}^{\infty} \binom{-(n+ar)}{r} \left(\frac{c}{b}\right)^r b^{(n-ar)} = -\sum_{j=1}^{a} \frac{\xi_j^{n+1}}{(a+1)\,\xi_j - ab}.$$

Thus

$$\sum_{r=[(n+1)/a]}^{\infty} (-1)^{r+1} \binom{ar+r-1-n}{r} \left(\frac{c}{b}\right)^r b^{(n-ar)} = \sum_{j=1}^{a} \frac{\xi_j^{n+1}}{(a+1)\,\xi_j - ab}.$$

Proof of Theorem 26: Without loss of generality let $c = \alpha b, \alpha \in \Re$, in (6.7) such that

$$b = \frac{\xi_0^{a+1}}{\alpha + \xi_0^a},$$

also, let $n = -\beta a$ and substitute in the left hand side of (6.10) such that

$$\sum_{r=0}^{\infty} \binom{-a(\beta+r)}{r} \alpha^r b^{-a(\beta+r)} = \sum_{r=0}^{\infty} (-1)^r \binom{a\beta+ar+r-1}{r} \alpha^r \left(\frac{\alpha+\xi_0^a}{\xi_0^{a+1}}\right)^{a(\beta+r)}$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{a\beta+ar+r-1}{r} \sum_{k=0}^{a(\beta+r)} \binom{a\beta+ar}{k} \alpha^{a\beta+r(a+1)-k} \xi_0^{ak-a(a+1)(\beta+r)}. \tag{6.11}$$

Now, expand (6.11) term by term and sum the convergent double sum diagonally from the top right hand corner, thus gathering inverse powers of ξ_0, such that

$$\sum_{r=0}^{\infty} \xi_0^{-a(\beta+r)} \sum_{k=0}^{r} (-1)^{r-k} \begin{pmatrix} a(\beta+r-k) \\ a(\beta+r-k)-k \end{pmatrix} \begin{pmatrix} a(\beta+r-k)+r-k-1 \\ r-k \end{pmatrix} \alpha^r$$

and after some lengthy but straightforward algebra we may simplify the double sum to

$$\xi_0^{-a\beta} \left[1 + a \sum_{r=1}^{\infty} (-\alpha)^r (a+1)^{r-1} \xi_0^{-ar} \right]$$

$$= \xi_0^{-a\beta} \left[\frac{\alpha + \xi_0^a}{(a+1)\alpha + \xi_0^a} \right]$$

$$= \xi_0^{-a\beta+1} \left[\frac{1}{(a+1)\xi_0 - a\left(\frac{\xi_0^{a+1}}{\alpha+\xi_0^a}\right)} \right] = \frac{\xi_0^{-a\beta+1}}{(a+1)\xi_0 - ab}$$

which is identical to the right hand side of (6.10), upon replacing $-a\beta = n$. The convergence region (6.6) is obtained by applying the ratio test to the term in the sum (6.10) and is shown in figure 6.1. The theorem is proved.

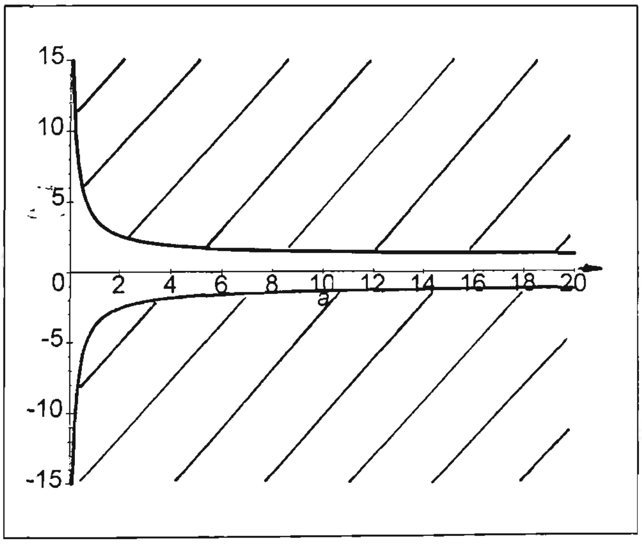

Figure 6.1: Convergence region for the discrete case.

6.4 The Lagrange form.

Jensen's work [63] is related to our sum (6.10). Jensen substitutes $\phi = (z+1)^n$ and $f = (z+1)^\beta$ into Lagrange's formula, theorem 8, and obtains

$$\sum_{j=0}^{\infty} \frac{n}{n+j\beta} \binom{n+j\beta}{j} y^j = (z+1)^n$$

where $y = z(z+1)^{-\beta}$. Similarly, from the Lagrange alternate formula, theorem 9, Jensen obtains

$$\sum_{j=0}^{\infty} \binom{n+j\beta}{j} y^j = \frac{(z+1)^{n+1}}{1-(\beta-1)z} = \frac{(z+1)^n}{1-\beta\left(\frac{z}{z+1}\right)}. \tag{6.12}$$

Jensen also substitutes $\phi = (z+1)^{n+m}$ and $f = (z+1)^\beta$, and obtains

$$\frac{(z+1)^m (z+1)^n}{1-\beta\left(\frac{z}{z+1}\right)} = \sum_{j=0}^{\infty} \binom{n+m+j\beta}{j} y^j \tag{6.13}$$

$$= \sum_{j=0}^{\infty} y^j \left\{ \sum_{j=0}^{n} \frac{n}{n+j\beta} \binom{n+j\beta}{j} \binom{m+(k-j)\beta}{k-j} \right\}, \tag{6.14}$$

equating the coefficients of y^j in (6.13) and (6.14) and substituting $m = m - k\beta$, Jensen obtains the striking result

$$\binom{n+m}{k} = \sum_{j=0}^{n} \frac{n}{n+j\beta} \binom{n+j\beta}{j} \binom{m-j\beta}{k-j}. \tag{6.15}$$

Jensen notes that (6.15) is analogous to Abel's identity, see chapter 2, and is comprised in the more general formula, due to M.I.G. Hagen,

$$\frac{a(p+q-nd)+bnq}{(p+q)(p-nd)q} \binom{p+q}{n} = \sum_{j=0}^{n} \frac{a+bj}{(q+jd)(p-jd)} \binom{q+jd}{j} \binom{p-j\beta}{n-j}.$$

Jensen also used (6.15) and the method of recurrences to prove the novel identity

$$\sum_{j=0}^{n}\binom{a+j\beta}{j}\binom{b-j\beta}{n-j}=\sum_{j=0}^{n}\binom{a+b-j}{n-j}\beta^{j},$$

which Cohen and Sun [32] gave a more general form by the use of general series transformations. Chu Wenchang [27] obtained even more general forms of binomial convolution identities by the use of power series expansions and convolutions. We can now show that our sum (6.10) is the same as Jensen's identity (6.12) and that (6.10) holds for all real parameter values. Let $c = \alpha b, a = -\beta, \alpha b^{\beta} = y$ and $z = \alpha\xi_0^{\beta}$, from (6.7) we have that $\xi_0 = b(z+1)$ and $y = z/(z+1)^{\beta}$, and substituting in (6.10) we obtain Jensen's identity (6.12). If we let $\beta = 0$, (6.12) reduces to the binomial theorem. To show that (6.12) holds for all real parameter values, we first write

$$\sum_{r=0}^{\infty}\binom{n+r\beta}{r}\frac{z^{r}}{(z+1)^{\beta r+n+1}}=\frac{1}{1-(\beta-1)z}$$

and putting the denominator in the left hand side in series form we can write

$$\sum_{r=0}^{\infty}\binom{n+r\beta}{r}z^{r}\sum_{j=0}^{\infty}(-1)^{j}\binom{n+r\beta+j}{j}z^{j};$$

expanding the double series and summing diagonally from top left hand corner we obtain powers of z, namely

$$\sum_{r=0}^{\infty}z^{r}\sum_{\rho=0}^{r}(-1)^{r-\rho}\binom{n+\rho\beta}{\rho}\binom{n+\rho\beta+r-\rho}{r-\rho}$$

and since the inner sum is equivalent to $(\beta-1)^{r}$, the parameters n and β are arbitrary and therefore belong to the set of real numbers. In the next section we shall investigate many interesting cases of the identity (6.10) and an associated related result. We shall investigate the case of $a = 1$.

6.5 Central binomial coefficients.

For the case of $a = 1$, (6.7) produces the dominant zero $\xi_0 = \left(b + \sqrt{b^2 + 4c}\right)/2$ such that for $n = -\alpha \in \Re, c = -x$ and $b = 1$ identity (6.10) may be written as

$$f(\alpha, x) = \sum_{r=0}^{\infty} \binom{2r + \alpha - 1}{r} x^r = \frac{1}{\sqrt{1-4x}} \left(\frac{1 + \sqrt{1-4x}}{2}\right)^{1-\alpha} \tag{6.16}$$

$$= {}_2F_1 \left[\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2} \\ \alpha \end{matrix} \middle| 4x \right]$$

and in particular $f(1, x) = \frac{1}{\sqrt{1-4x}}$ occurs in the book of Wilf [93], $f(\alpha, 1/8) = 2^{(3\alpha-2)/2}\left(1 + \sqrt{2}\right)^{1-\alpha}$ and $f(1, 1/8) = \sqrt{2}$ is a very slow converging series. Replacing x by $-x$ in (6.16) and then adding we obtain

$$g(\alpha, x) = \sum_{r=0}^{\infty} \binom{4r + \alpha - 1}{2r} x^{2r} = {}_4F_3 \left[\begin{matrix} \frac{\alpha}{4}, \frac{\alpha+1}{4}, \frac{\alpha+2}{4}, \frac{\alpha+3}{4} \\ \frac{1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} \end{matrix} \middle| 16x^2 \right], \tag{6.17}$$

so that $g(\alpha, 1/8) = 2^{(2\alpha-1)/2} \left(\frac{\left(1+\sqrt{3/2}\right)^{1-\alpha}}{\sqrt{3}} + \left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)^{1-\alpha} \right)$. Therefore, in general, we may obtain closed form expressions for binomial sums of the type

$$\sum_{r=0}^{\infty} \binom{2(a_1 r + a_2) + \alpha - 1}{a_1 r + a_2} x^{a_1 r + a_2}$$

for constants a_1, a_2 and α. It also follows from (6.16) that for α and $\beta \in \Re$ we have the identity

$$2^{\beta}\, {}_2F_1 \left[\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2} \\ \alpha \end{matrix} \middle| 4x \right] = \left(1 + \sqrt{1-4x}\right)^{\beta} {}_2F_1 \left[\begin{matrix} \frac{\alpha+\beta}{2}, \frac{\alpha+1+\beta}{2} \\ \alpha + \beta \end{matrix} \middle| 4x \right]$$

or

$$2^{\beta} \sum_{r=0}^{\infty} \binom{2r + \alpha - 1}{r} x^r = \left(1 + \sqrt{1-4x}\right)^{\beta} \sum_{r=0}^{\infty} \binom{2r + \alpha + \beta - 1}{r} x^r. \tag{6.18}$$

For $\alpha = 2$ and $\beta = 0$ adding alternative terms in (6.18), we have , after using (6.17) and some minor manipulation

$$\sum_{r=0}^{\infty} \binom{4r+1}{2r} x^{2r} = {}_4F_3 \left[\begin{matrix} \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4} \\ \frac{1}{2}, 1, \frac{3}{2} \end{matrix} \middle| 16x^2 \right].$$

From (6.16) with $\alpha = 1$ and collecting coefficients of x we obtain the common result $\binom{2r}{r} = \binom{r-\frac{1}{2}}{r} 2^{2r}$ and this result will be generalized in the next chapter. The identity (6.16) may be differentiated with respect to x to obtain more identities. If we let the operator $\rho = x\frac{d}{dx} f(\alpha, x)$ we obtain from (6.16)

$$\sum_{r=0}^{\infty} r \binom{2r+\alpha-1}{r} x^r = \frac{x\left(\frac{1+\sqrt{1-4x}}{2}\right)^{-\alpha}}{1-4x} \left\{\alpha + \frac{1}{\sqrt{1-4x}}\right\} \tag{6.19}$$

$$= (1+\alpha)\, x\, {}_2F_1 \left[\begin{matrix} \frac{\alpha+2}{2}, \frac{\alpha+3}{2} \\ \alpha+1 \end{matrix} \middle| 4x \right].$$

Operating on (6.19) by ρ again we have that

$$\begin{aligned} \sum_{r=0}^{\infty} r^2 \binom{2r+\alpha-1}{r} x^r &= (1+\alpha)\, x\, {}_2F_1 \left[\begin{matrix} \frac{\alpha+2}{2}, \frac{\alpha+3}{2} \\ \alpha+1 \end{matrix} \middle| 4x \right] \\ &+ (2+\alpha)(3+\alpha)\, x^2\, {}_2F_1 \left[\begin{matrix} \frac{\alpha+4}{4}, \frac{\alpha+5}{4} \\ \alpha+2 \end{matrix} \middle| 4x \right] \end{aligned}$$

and in particular for $\alpha = 1$ and $x = 1/8$,$\sum_{r=0}^{\infty} r^2 \binom{2r}{r} 8^{-r} = \frac{5}{2\sqrt{2}}$. Thus we may, in general obtain identities of sums of central binomial coefficients;

$$\sum_{r=0}^{\infty} r^k \binom{2r+\alpha-1}{r} x^r$$

for k integer. Similarly , (6.16) may also be integrated to obtain more identities. Integrating the right hand side of (6.16) will necessitate branch cuts and singularities dependent on the value of α. So that putting $\alpha = 1$ will simplify the integral. From (6.16)

$$g_j(x) = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{x^{r+j}}{(r+1)_j} = \underbrace{\int_{t=0}^{x} \cdots \int_{t=0}^{x_1}}_{j\ -\ times} \frac{dt}{\sqrt{1-4t}},$$

where $(r+1)_j$ is Pochhammer's symbol. For $j = 1$, the constant $C_r = \binom{2r}{r} \frac{1}{r+1}$ is known as Catalan numbers. In particular $g_3(x) = \frac{1}{120x^3}\left\{1 - 10x + 30x^2 - (1-4x)^{5/2}\right\}$, $g_3(-x) = \frac{1}{120x^3}\left\{-1 - 10x + 30x^2 + (1+4x)^{5/2}\right\}$, so that $g_3(x) + g_3(-x)$ produces the identity

$$\sum_{r=0}^{\infty} \binom{4r}{2r} \frac{x^{2r}}{(2r+1)(2r+2)(2r+3)} = \frac{1}{240x^3}\left\{(1+4x)^{5/2} - 20x - (1-4x)^{5/2}\right\}.$$

Other values of α may be utilized; if $\alpha = 2$ then from (6.16) we have the identity

$$\sum_{r=0}^{\infty} \binom{2r+1}{r} x^r = {}_2F_1\left[\left.\begin{matrix} 1, \frac{3}{2} \\ 2 \end{matrix}\right| 4x\right] = \frac{2}{1 - 4x + \sqrt{1-4x}}. \tag{6.20}$$

Multiplying both sides of (6.20) by x and then integrating, we have

$$\sum_{r=0}^{\infty} \binom{2r+1}{r} \frac{x^r}{r+2} = \frac{1}{4x^2}\left\{1 - 2x - \sqrt{1-4x}\right\}$$

where the coefficient $MC_r = \binom{2r+1}{r} \frac{1}{r+2} = \frac{2r+2}{r+2} C_r$ may be thought of as modified Catalan numbers , and in particular for $x = 1/4$ we have the slow converging series

$$\sum_{r=0}^{\infty} \binom{2r+1}{r} \frac{4^{-r}}{r+2} = 2.$$

An alternative procedure for integrating (6.16) for $\alpha = 1$, is the following. Take the first term of (6.16) to the right hand side and integrate j-times such that

$$h_j(1,x) = \sum_{r=1}^{\infty} \binom{2r}{r} \frac{x^r}{(r)_j} = \frac{1}{x^{j-1}} \underbrace{\int_{t=0}^{x} \cdots \int_{t=0}^{x_1}}_{j\ -\ times} \left(\frac{1}{\sqrt{1-4t}} - 1 \right) \frac{dt}{t}$$

and consideration needs to be given to the improper integral. In particular

$$h_3(1,x) = \frac{1}{x^2} \left\{ \frac{1}{24} - \frac{x}{2} + \frac{3x^2}{4} + x^2 \ln 2 + \frac{(10x-1)\sqrt{1-4x}}{24} - x^2 \ln\left(1+\sqrt{1-4x}\right) \right\},$$

$$h_3(1,-x) = \frac{1}{x^2} \left\{ \frac{1}{24} + \frac{x}{2} + \frac{3x^2}{4} + x^2 \ln 2 - \frac{(10x+1)\sqrt{1+4x}}{24} - x^2 \ln\left(1+\sqrt{1+4x}\right) \right\}$$

and adding produces the identity

$$\sum_{r=1}^{\infty} \binom{4r}{2r} \frac{x^{2r}}{(2r)(2r+1)(2r+2)} = \frac{1}{2x^2} \left\{ \begin{array}{c} \frac{1}{12} + \frac{3x^2}{2} + x^2 \ln 2 + \frac{(10x-1)\sqrt{1-4x}}{24} - \frac{(10x+1)\sqrt{1+4x}}{24} \\ -x^2 \ln\left(\left(1+\sqrt{1+4x}\right)\left(1+\sqrt{1-4x}\right)\right) \end{array} \right\}.$$

Now, if in (6.16) we put $x = x^2$ and for $\alpha = 1$ integrate, we have

$$f_1(1,x^2) = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{x^{2r+1}}{2r+1} = \frac{\arcsin 2x}{2},$$

and integrating once more

$$f_2(1,x^2) = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{x^{2r+2}}{(2r+1)(2r+2)} = \frac{\sqrt{1-4x^2} + 2x \arcsin 2x - 1}{4}.$$

In particular $f_2\left(1,\frac{1}{2}\right) = \frac{\pi}{2} - 1, f_2\left(1,\frac{i}{2}\right) = 1 - \sqrt{2} - \ln\left(\sqrt{2}-1\right)$ and adding these two cases produces the identity

$$\sum_{r=0}^{\infty} \binom{4r}{2r} \frac{2^{-4r}}{(4r+1)(4r+2)} = \frac{\pi}{4} - \frac{1}{\sqrt{2}} - \frac{1}{2} \ln\left(\sqrt{2}-1\right).$$

Similar identities may be obtained for other values of a. If for example $a = 2, b = 1, c = -2/27$ and $n = -\alpha \in \Re$ then, from (6.10)

$$\sum_{r=0}^{\infty} \binom{3r+\alpha-1}{r} \left(\frac{2}{27}\right)^r = \frac{2+\sqrt{3}}{3}\left(\frac{3}{1+\sqrt{3}}\right)^{\alpha} = {}_3F_2\left[\left.\begin{array}{c} \frac{\alpha}{3}, \frac{\alpha+1}{3}, \frac{\alpha+2}{3} \\ \frac{\alpha}{2}, \frac{\alpha+1}{2} \end{array}\right| \frac{1}{2}\right].$$

6.5.1 Related results.

The previous results that involve the $\arcsin x$ function suggests we may design a related sum of the form $\sum_{r=1}^{\infty} \frac{a_r}{\binom{2r}{r}}$ which may be expressed in closed form. Consider the function $f(x) = \frac{\arcsin x}{\sqrt{1-x^2}}$, then a Taylor series expansion of $f(x)$ about the origin allows us to write

$$f(x) = \frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{r=1}^{\infty} \frac{(2x)^{2r-1}}{r\binom{2r}{r}} \tag{6.21}$$

and $f(1/2) = \frac{\pi}{3\sqrt{3}} = \sum_{r=1}^{\infty} \frac{1}{r\binom{2r}{r}}$, and $f(i/2) = \frac{2}{\sqrt{5}} \ln\left(\frac{\sqrt{5}-1}{2}\right) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r\binom{2r}{r}}$ upon using the relation $\arcsin z = -i \ln\left(iz + \sqrt{1-z^2}\right)$. Multiplying both sides of (6.21) by $2x$ and differentiating any number of times will produce other identities. Differentiating (6.21) once, and simplifying we have

$$f_1(x) = \sum_{r=1}^{\infty} \frac{(2x)^{2r}}{\binom{2r}{r}} = \frac{x}{1-x^2}\left\{x + \frac{\arcsin x}{\sqrt{1-x^2}}\right\}, \tag{6.22}$$

so that, $f_1\left(\frac{1}{\sqrt{2}}\right) = \sum_{r=1}^{\infty} \frac{2^r}{\binom{2r}{r}} = \frac{\pi}{2} + 1$, and $f_1\left(\frac{i}{\sqrt{2}}\right) = \sum_{r=1}^{\infty} \frac{(-2)^r}{\binom{2r}{r}} = \frac{1}{3}\left(\frac{2}{\sqrt{3}} \ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) - 1\right)$.

Differentiating (6.22) with respect to x, and simplifying we obtain

$$f_2(x) = \sum_{r=1}^{\infty} \frac{r(2x)^{2r}}{\binom{2r}{r}} = \frac{x}{2}\left\{\frac{x(1+2x^2)}{(1-x^2)^2} + \frac{2x}{1-x^2} + \frac{(1+2x^2)\arcsin x}{\sqrt{(1-x^2)^5}}\right\}, \tag{6.23}$$

particularly, $f_2\left(\frac{1}{2}\right) = \sum_{r=1}^{\infty} \frac{r}{\binom{2r}{r}} = \frac{2}{3} + \frac{2\pi}{9\sqrt{3}}$, and $f_2\left(\frac{i}{2}\right) = \sum_{r=1}^{\infty} \frac{(-1)^r r}{\binom{2r}{r}} = \frac{2}{25}\left(\frac{2}{\sqrt{5}} \ln\left(\frac{\sqrt{5}-1}{2}\right) - 3\right)$.

At the writing up stage of this thesis, the author discovered that more general identities of the form (6.22) and (6.23) have been given by Chudnovsky and Chudnovsky [28], as an example they give $\sum_{r=1}^{\infty} \frac{2^r}{\binom{2r}{r}} = \frac{\pi}{2} + 1$. In their recent paper, Chudnovsky and Chudnovsky [28] obtain a master theorem from which they can, amongst other manipulations, derive explicit expressions of contiguous generalized hypergeometric functions. The identity (6.21) may also be integrated repeatedly. Integrating (6.21) once we have

$$g_1(x) = \sum_{r=1}^{\infty} \frac{(2x)^{2r}}{r^2 \binom{2r}{r}} = 2(\arcsin x)^2 \tag{6.24}$$

and $g_1(1) = \frac{\pi^2}{2} = 3\zeta(2)$, $g_1(i) = 2\left(\ln\left(\sqrt{2}-1\right)\right)^2$, $g_1\left(\frac{1}{2}\right) = \frac{\pi^2}{18} = \frac{\zeta(2)}{3}$, $g_1\left(\frac{i}{2}\right) = 2\left(\ln\left(\frac{\sqrt{5}-1}{2}\right)\right)^2$, from which upon manipulation we obtain

$$\sum_{r=1}^{\infty} \frac{1}{r^2 \binom{4r}{2r}} = \frac{\pi^2}{9} - 4\left(\ln\left(\frac{\sqrt{5}-1}{2}\right)\right)^2 = \frac{1}{6}\,{}_4F_3\left[\begin{matrix} 1,1,1,\frac{3}{2} \\ \frac{5}{4},\frac{7}{4},2 \end{matrix}\,\middle|\, \frac{1}{16}\right].$$

Integrating (6.24) three more times gives us

$$g_4(x) = \sum_{r=1}^{\infty} \frac{(2x)^{2r+3}}{r^2 \binom{2r}{r} 4(2r+3)(2r+2)(2r+1)} = \frac{x(2x^2+3)(\arcsin x)^2}{3} + \frac{2\sqrt{1-x^2}(11x^2+4)\arcsin x}{9} - \frac{85x^3}{27} - \frac{8x}{9} \tag{6.25}$$

and $g_4(1) = \frac{5\pi^2}{24} - \frac{109}{54}$, so that in general we may obtain identities

$$\sum_{r=1}^{\infty} \frac{2^{2r}}{r^2 \binom{2r}{r} (2r+1)_j} = \alpha\pi^2 + \beta,$$

for $j = 1, 2, 3, ...$ and constants α and β. Comtet [33] obtains

$$\sum_{r=1}^{\infty} \frac{1}{r^4 \binom{2r}{r}} = \frac{17\pi^4}{3240} = \frac{1}{2}\,{}_5F_4\left[\begin{array}{c} 1,1,1,1,1 \\ \frac{3}{2},2,2,2 \end{array} \middle| \frac{1}{4}\right]$$

however there appears to be no closed form expression of $\sum_{r=1}^{\infty} \frac{1}{r^m \binom{2r}{r}}$ for $m > 4$. We can obtain identities of this form, so that from (6.25) with $x = 1/2$, we have

$$\sum_{r=1}^{\infty} \frac{1}{r^2 \binom{2r}{r} (2r+3)(2r+2)(2r+1)} = \frac{7}{18}\zeta(2) + \frac{\sqrt{3}\pi}{2} - \frac{181}{54}.$$

Other identities involving ζ functions and multiple ζ functions are given by Borwein [14]. In the next section we shall investigate the finite sum (6.9). We shall , for $a = 1$, give a trigonometric representation of (6.9), recover and extend some results given by Binz [10], and highlight a number of interesting applications of the Fibonacci sequence.

n	f_n
0	1
1	b
2	b^2+c
3	b^3+2bc
4	$b^4+3b^2c+c^2$
5	$b^5+4b^3c+3bc^2$
6	$b^6+5b^4c+6b^2c^2+c^3$
7	$b^7+6b^5c+10b^3c^2+4bc^3$
8	$b^8+7b^6c+15b^4c^2+10b^2c^3+c^4$
9	$b^9+8b^7c+21b^5c^2+20b^3c^3+5bc^4$
10	$b^{10}+9b^8c+28b^6c^2+35b^4c^3+15b^2c^4+c^5$
11	$b^{11}+10b^9c+36b^7c^2+56b^5c^3+35b^3c^4+6bc^5$
12	$b^{12}+11b^{10}c+45b^8c^2+84b^6c^3+70b^4c^4+21b^2c^5+c^6$

Table 6.1: Polynomials of the finite sum (6.27).

b	c	name	generating function	zeros	solution of (6.26)
1	1	Fibonacci	z^2-z-1	$\frac{1\pm\sqrt{5}}{2}$	$\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}$
1	2	Jacobsthal	z^2-z-2	$2,-1$	$\frac{1}{3}\left(2^n-(-1)^n\right)$
2	1	Pell	z^2-2z-1	$1\pm\sqrt{2}$	$\frac{1}{\sqrt{2}}\left\{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right\}$
3	-2	Fermat	z^2-3z+2	$2,1$	$2^{n+1}-1$
2	-1	Chebyshev	z^2-2z+1	$1,1$	$n+1$

Table 6.2: Special recurrences and solutions.

6.6 Fibonacci, related polynomials and products.

Consider (6.9) for $a=1$, such that from (6.1)

$$f_{n+1}-bf_n-cf_{n-1}=0,\ f_0=1 \tag{6.26}$$

and

$$f_n=\sum_{r=0}^{[n/2]}\binom{n-r}{r}c^r b^{(n-2r)}. \tag{6.27}$$

Some polynomials of f_n are given in table 6.1

The recurrence (6.26), for some parameter values b and c may be identified as shown in table 6.2.

Horadam [61] and [62] has recently written a lucid and pertinent examination of Jacobsthal representation numbers and polynomials, and functional forms. Hendel and Cooke [57] suggest that second order recursions may be represented by finite products involving trigonometric functions. From (6.27) we may thus write

$$f_n = \prod_{j=1}^{n} \left\{ b - 2i\sqrt{c}\cos\left(\frac{\pi j}{n+1}\right) \right\}$$

and so after multiplying corresponding factors

$$f_n = b^{n-2[n/2]} \prod_{j=1}^{[n/2]} \left\{ b^2 + 4c\cos^2\left(\frac{\pi j}{n+1}\right) \right\}.$$

We can see that for $b = 0$ and n even

$$f_n = (4c)^n \prod_{j=1}^{n} \cos^2\left(\frac{\pi j}{2n+1}\right)$$

and since from table 6.1 we note that $f_n = c^n$ we may deduce

$$\prod_{j=1}^{n} \cos\left(\frac{\pi j}{2n+1}\right) = 2^{-n}.$$

The characteristic function (6.7), for $a = 1$, has two zeros $2\xi_{0,1} = b \pm \sqrt{b^2+4c}$ and from (6.27)

$$\sum_{r=0}^{[n/2]} \begin{pmatrix} n-r \\ r \end{pmatrix} c^r b^{(n-2r)} = \frac{1}{\sqrt{b^2+4c}} \left\{ \left(\frac{b+\sqrt{b^2+4c}}{2}\right)^{n+1} - \left(\frac{b-\sqrt{b^2+4c}}{2}\right)^{n+1} \right\} \qquad (6.28)$$

and putting $b = (x-1)^2$ and $c = bx$, we obtain the result obtained by Binz [10]; namely

$$\sum_{r=0}^{[n/2]} \begin{pmatrix} n-r \\ r \end{pmatrix} x^r (x-1)^{2n-2r} = \frac{1}{x^2-1} \left\{ (x(x-1))^{n+1} - (1-x)^{n+1} \right\}.$$

We can integrate, for $0 \le x \le 1$, the result given by Binz , therefore producing the new identity

$$\sum_{r=0}^{[n/2]} \binom{n-r}{r} B(r+1, 2n-2r+1) = \frac{{}_2F_1\left[\begin{matrix}1,1\\ n+2\end{matrix}\middle| -1\right]}{n+1} + \frac{(-1)^n n! \sqrt{\pi}\, {}_2F_1\left[\begin{matrix}1, n+2\\ 2n+3\end{matrix}\middle| -1\right]}{2^{2n+2}\Gamma\left(n+\frac{3}{2}\right)},$$

where $B(x,y)$ is the Beta function and $\Gamma(x)$ is the classical Gamma function. Differentiating (6.28) with respect to c and substituting we obtain, in an easier manner, yet another result quoted by Binz [10]

$$\sum_{r=0}^{[n/2]} r \binom{n-r}{r} x^r (x-1)^{2n-2r} = \frac{x(x-1)^n}{(x+1)^3} \left\{ \begin{matrix} (n+1)\, x \{x^{n-1} + (-1)^n\} + \\ (n-1)\{x^{n+1} + (-1)^n\} \end{matrix} \right\}.$$

We may derive (6.28) from a slightly different viewpoint and also in the process give a solution to a problem posed by Krafft and Schaefer [70]. Consider the two binomial expansions

$$\left(\frac{1+\sqrt{1+x}}{2}\right)^{2\mu} = 4^{-\mu} \sum_{r=0}^{2\mu} \binom{2\mu}{r} (1+x)^{r/2}; \tag{6.29}$$

$$\left(\frac{1-\sqrt{1+x}}{2}\right)^{2\mu} = 4^{-\mu} \sum_{r=0}^{2\mu} (-1)^r \binom{2\mu}{r} (1+x)^{r/2}. \tag{6.30}$$

Subtracting (6.30) from (6.29) we have

$$2^{-(2\mu-1)} \sum_{j=0}^{\mu} \binom{2\mu}{2j+1} (1+x)^j = \frac{\left(\frac{1+\sqrt{1+x}}{2}\right)^{2\mu} - \left(\frac{1-\sqrt{1+x}}{2}\right)^{2\mu}}{\sqrt{1+x}} \tag{6.31}$$

and to identify (6.31) with (6.28) let $2\mu = n+1$ so that

$$2^{-n} \sum_{j=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n+1}{2j+1} (1+x)^j = \frac{\left(\frac{1+\sqrt{1+x}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+x}}{2}\right)^{n+1}}{\sqrt{1+x}} \tag{6.32}$$

where $\lceil \nu \rceil$ is the least integer not smaller than ν. From (6.28) let $b = 1$ and $c = x/4$, hence

$$\sum_{r=0}^{[n/2]} \binom{n-r}{r} \left(\frac{x}{4}\right)^r = \frac{\left(\frac{1+\sqrt{1+x}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+x}}{2}\right)^{n+1}}{\sqrt{1+x}} = 2^{-n} \sum_{j=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n+1}{2j+1} (1+x)^j . \tag{6.33}$$

From the left and right hand sides of (6.33) let $n = 2m+1$ so that

$$\sum_{r=0}^{[(2m+1)/2]} \binom{2m+1-r}{r} \left(\frac{x}{4}\right)^r = 2^{-(2m+1)} \sum_{r=0}^{m} \binom{2m+2}{2r+1} (1+x)^r .$$

Expanding the right hand side and collecting powers of x we have

$$\sum_{r=0}^{[(2m+1)/2]} \binom{2m+1-r}{r} \left(\frac{x}{4}\right)^r = 2^{-(2m+1)} \sum_{r=0}^{m} x^r \left\{ \sum_{j=r}^{m} \binom{2m+2}{2j+1} \binom{j}{r} \right\}$$

and equating coefficients of x we have the novel identity

$$\sum_{j=r}^{m} \binom{2m+2}{2j+1} \binom{j}{r} = 2^{2(m-r)+1} \binom{2m+1-r}{r} .$$

Now, to solve the problem of Krafft and Schaefer, add (6.29) and (6.30) so that

$$2^{-n} \sum_{j=0}^{\lceil \frac{n+1}{2} \rceil} \binom{n+1}{2j} (1+x)^j = \left(\frac{1+\sqrt{1+x}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{1+x}}{2}\right)^{n+1} . \tag{6.34}$$

The ratio of (6.34) and (6.32), after putting $2m = n+1$ is

$$a_m = \frac{\sum_{j=0}^{m} \binom{2m}{2j} (1+x)^j}{\sum_{j=0}^{m-1} \binom{2m}{2j+1} (1+x)^j} = \sqrt{1+x} \left\{ \frac{1 + \left(\frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\right)^{2m}}{1 - \left(\frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\right)^{2m}} \right\}$$

and $\lim_{m\to\infty} a_m = \sqrt{1+x}$, which solves Krafft and Schaefer's problem after replacing $1+x$ with x^*.

It is of some passing interest to note Wilf [93] derived (6.27) for $c = -b$. The derivation, by Wilf, counts the number of words of n letters over an alphabet of b letters that do not contain the substring of a word of a letters. Graham et al. [49] discuss the continuant polynomial, $K_n(x_1, x_2, ..., x_n)$, defined by

$$\left.\begin{aligned} K_0() &= 1, \\ K_1(x_1) &= x_1 \\ K_n(x_1, x_2, ..., x_n) &= K_{n-1}(x_1, x_2, ..., x_{n-1})\, x_n + K_{n-2}(x_1, x_2, ..., x_{n-2}) \end{aligned}\right\}.$$

In particular, a Morse code sequence of length n that has k dashes, has $n-2k$ dots and $n-k$ symbols altogether. These dots and dashes can be arranged in $\binom{n-k}{k}$ ways; therefore if we replace each dot by z and each dash by 1 we get $K_n(z, z, ..., z) = \sum_{r=0}^{n} \binom{n-r}{r} z^{n-2r}$ which is the Fibonacci sequence. There is another intriguing connection of the Fibonacci sequence with the Lambert series. The series $L(x) = \sum_{r=1}^{\infty} \frac{x^r}{1-x^r}$ was presented by Lambert circa 1771, and has been studied extensively. A closed form representation of the Lambert series may be useful, because of its possible importance in prime number theory. For the Fibonacci sequence (6.26) with $b = c = 1$ it may be shown, see Knopp [68], that $\sum_{j=1}^{\infty} \frac{1}{f_{2j}} = \sqrt{5}\left\{L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right\}$. In the next section we will prove a number of functional forms of (6.9), some of which are new.

6.7 Functional forms.

The following lemmas are functional forms of (6.27).

Lemma 27 : *Let f_n be defined by (6.27), $f_{-1} = 0$ and $f_{-n} = (-1)^n c^{1-n} f_{n-2}$, then*

$$\sum_{k=1}^{n} (-c)^k f_{n-2k} = 0. \tag{6.35}$$

Proof: From the left hand side of (6.35)

$$\sum_{k=1}^{n} (-c)^k f_{n-2k} = -cf_{n-2} + c^2 f_{n-4} - c^3 f_{n-6} + ... + (-c)^{n-1} f_{-n+2} + (-c)^n f_{-n}$$

and by the definition

$$= -cf_{n-2} + c^2 f_{n-4} - c^3 f_{n-6} + ... + c^3 f_{n-6} - c^2 f_{n-4} + cf_{n-2} = 0$$

and the proof is complete.

Lemma 28 :

$$b^m f_n = \sum_{j=0}^{m} \binom{m}{j} (-c)^j f_{n+m-2j}, \; for \; m \geq 0. \tag{6.36}$$

Proof: We shall employ an induction argument. For $m = 0,\ f_n = f_n$, for $m = 1$ we obtain (6.26). Consider

$$b^{m+1} f_n = b \left(\sum_{j=0}^{m} \binom{m}{j} (-c)^j f_{n+m-2j} \right)$$

$$\begin{aligned} &= bf_{n+m} - bc \binom{m}{1} f_{n+m-2} + bc^2 \binom{m}{2} f_{n+m-4} + ... \\ &\quad + b(-c)^{m-1} \binom{m}{m-1} f_{n-m+2} + b(-c)^m f_{n-m} \end{aligned}$$

and from (6.26) substitute for bf_n

$$\begin{aligned} &= f_{n+m+1} - cf_{n+m-1} - c\{f_{n+m-1} - cf_{n+m-3}\} + c^2 \binom{m}{2} \{f_{n+m-3} - cf_{n+m-5}\} + \\ &\quad ... + (-c)^{m-1} \binom{m}{m-1} \{f_{n-m+3} - cf_{n-m+1}\} + (-c)^m \binom{m}{m} \{f_{n-m+1} - cf_{n-m-1}\}, \end{aligned}$$

collecting coefficients of $(-c)^j$ gives us

$$f_{n+m+1} + \sum_{j=0}^{m} \binom{m+1}{j} (-c)^j f_{n+m+1-2j} + (-c)^{m+1} f_{n-m-1}$$

$$= \sum_{j=0}^{m+1} \binom{m+1}{j} (-c)^j f_{n+m+1-2j}$$

and the lemma is proved.

Lemma 29 *:*

$$f_u f_v = \sum_{k=0}^{u} (-c)^k f_{u+v-2k}.$$

Proof: From the left hand side

$$f_u f_v = (b f_{u-1} + c f_{u-2}) f_v$$

$$= \sum_{k=0}^{u-1} b (-c)^k f_{u-1+v-2k} + c \sum_{k=0}^{u-2} (-c)^k f_{u-2+v-2k}$$

$$= \sum_{k=0}^{u-1} (-c)^k (f_{u+v-2k} - c f_{u-2+v-2k}) + c \sum_{k=0}^{u-2} (-c)^k f_{u-2+v-2k}$$

$$= \sum_{k=0}^{u-1} (-c)^k f_{u+v-2k} - c \sum_{k=0}^{u-1} (-c)^k f_{u-2+v-2k} + c \sum_{k=0}^{u-2} (-c)^k f_{u-2+v-2k},$$

all of the second and third terms are annihilated except for the $k = u - 1$ term, in which case

$$\sum_{k=0}^{u-1} (-c)^k f_{u+v-2k} + (-c)^u f_{-u+v}$$

$$= \sum_{k=0}^{u} (-c)^k f_{u+v-2k}$$

hence the lemma is proved. Two special cases are

1. for $v = u$, $f_u^2 = \sum_{k=0}^{u} (-c)^{u-k} f_{2k}$ and

2. for $v = u + 1$, $f_u f_{u+1} = \sum_{k=0}^{u} (-c)^{u-k} f_{1+2k}$.

Lemma 30 *:*

$$\sum_{n=0}^{m} f_n f_{m-n} = (m+1) f_m + \sum_{j=1}^{[m/2]} (-c)^j (m + 1 - 2j) f_{m-2j}.$$

Proof: From lemma 29 put $u = m - n, v = n$, hence

$$\sum_{n=0}^{m} f_n f_{m-n} = \sum_{n=0}^{m} \left(\sum_{k=0}^{m-n} (-c)^k f_{m-2k} \right)$$

$$= \sum_{n=0}^{m} \left\{ \sum_{k=0}^{m} (-c)^k f_{m-2k} - \sum_{k=n+1}^{m} (-c)^k f_{m-2k} \right\}$$

$$= \sum_{n=0}^{m} \left\{ f_m + \sum_{k=1}^{m} (-c)^k f_{m-2k} - \sum_{k=n+1}^{m} (-c)^k f_{m-2k} \right\}$$

now apply lemma 27, such that

$$\begin{aligned} \sum_{n=0}^{m} f_n f_{m-n} &= (m+1) f_m - \sum_{n=0}^{m} \sum_{k=n+1}^{m} (-c)^k f_{m-2k} \\ &= (m+1) f_m - \sum_{j=1}^{m} j (-c)^j f_{m-2j} \end{aligned}$$

and reapplying lemma 27

$$\sum_{n=0}^{m} f_n f_{m-n} = (m+1) f_m + \sum_{j=1}^{[m/2]} (-c)^j (m+1-2j) f_{m-2j}$$

hence the lemma is proved.

Lemma 31 :

$$f_n = f_m f_{n-m} + c f_{m-1} f_{n-m-1}$$

Proof: From lemma 29 put $u = m, v = n - m$ hence

$$f_m f_{n-m} = \sum_{k=0}^{m} (-c)^k f_{n-2k} \tag{6.37}$$

also from lemma 29 put $u = m - 1, v = n - m - 1$ hence

$$c f_{m-1} f_{n-m-1} = c \sum_{k=0}^{m-1} (-c)^k f_{n-2-2k}$$

$$= -\sum_{k=0}^{m-1} (-c)^{k+1} f_{n-2-2k}. \qquad (6.38)$$

Adding (6.37) and (6.38), $f_n = f_m f_{n-m} + c f_{m-1} f_{n-m-1}$ hence the lemma is proved.

Lemma 32 :

$$b^m = \sum_{j=0}^{m-1} \binom{m-1}{j} (-c)^j f_{m-2j}.$$

Proof: From lemma 28 put $m = m-1$ and $n = 1$, hence

$$b^{m-1} f_1 = \sum_{j=0}^{m-1} \binom{m-1}{j} (-c)^j f_{m-2j}$$

and since $f_1 = b$, the lemma is proved.

Lemma 33 :

$$f_n^2 = f_{n-m} f_{n+m} + (-c)^{n+1-m} f_{m-1}^2$$

Proof: From lemma 29 put $u = n-m$, $v = n+m$ giving

$$f_{n-m} f_{n+m} = \sum_{k=0}^{n-m} (-c)^k f_{2n-2k}.$$

Put $n - k = n^*$ and rename n^*, giving for $0 \leq m \leq n$

$$f_{n-m} f_{n+m} = \sum_{k=m}^{n} (-c)^{n-k} f_{2k},$$

and specifically for $m = 0$,

$$f_n^2 = \sum_{k=0}^{n} (-c)^{n-k} f_{2k}.$$

Subtracting the last two sums produces

$$\begin{aligned} f_n^2 - f_{n-m} f_{n+m} &= \sum_{k=0}^{n} (-c)^{n-k} f_{2k} - \sum_{k=m}^{n} (-c)^{n-k} f_{2k} \\ &= \sum_{k=0}^{m-1} (-c)^{n-k} f_{2k} \end{aligned}$$

$$= \quad (-c)^{n+1-m} \sum_{k=0}^{m-1} (-c)^{m-1-k} f_{2k};$$

identifying the last sum as f_{m-1}^2 we have the result and hence the proof of the lemma. For $m = 1$, and $c = 1$, we have Cassini's identity, namely $f_n^2 = f_{n-1}f_{n+1} + (-1)^n$.

Lemma 34 *:*

$$\sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} \binom{2n-2j}{n-j} = 1. \tag{6.39}$$

Proof: From(6.27) we have

$$f_p f_q = \sum_{r=0}^{[p/2]} \binom{p-r}{r} c^r b^{p-2r} f_q$$

and using lemma 28

$$f_p f_q = \sum_{r=0}^{[p/2]} \binom{p-r}{r} c^r \left\{ \sum_{j=0}^{p-2r} \binom{p-2r}{j} (-c)^j f_{p+q-2r-2j} \right\}. \tag{6.40}$$

Now put $r + j = \alpha$(constant) and equate coefficients of $f_{p+q-2\alpha}$ in (6.40) and the expression of lemma 32, let $p = n$, and put $n = 2m$, rename the counter and we have the result (6.39). A WZ certificate function, $R(n,k)$, of (6.39) is

$$R(n,k) = \frac{k(2k-3-3n)(2n-k+1)}{(n+1)(k-n-1)^2},$$

which proves that (6.39) is an identity. Other functional identities of the Fibonacci sequence are given by Graham et al. [49].

Chapter 7

A convoluted Fibonacci sequence.

An arbitrary order forced difference-delay system is considered from which finite binomial sums are generated. Z transform theory is then utilized to represent the finite binomial type sums in closed form, moreover Zeilberger's creative telescoping algorithm, Petkovšek's algorithm 'Hyper' and Wilf and Zeilberger's WZ pairs method is used to certify particular instances of the identities.

7.1 Introduction.

In this chapter we generalize the system of chapter 6 and generate finite binomial sums. We utilize Z transform theory to represent the finite binomial sums in closed form and we can also employ Zeilberger's creative telescoping algorithm, Petkovšek's algorithm 'Hyper' and Wilf and Zeilberger's WZ pairs method to certify particular instances. Firstly we consider a homogeneous convoluted Fibonacci sequence and develop the general finite sum and its closed form representation. By considering multiple zeros of an associated characteristic function we develop new identities and certify some of them by the WZ pairs method. Secondly we generalize our results by considering forcing terms of binomial type.

7.2 Technique.

Consider what we shall describe as a generalized, or convoluted, Fibonacci sequence f_n, that satisfies

$$\left.\begin{array}{c} \displaystyle\sum_{j=0}^{R}\binom{R}{R-j}(-c)^{R-j}\sum_{r=0}^{j}\binom{j}{r}(-b)^{j-r}f_{n+r-(R-j)a}=w_n;\ \ n\geq aR \\ \displaystyle\sum_{r=0}^{R}\binom{R}{r}(-b)^{R-r}f_{n+r}=w_n;\ \ n<aR \end{array}\right\} \tag{7.1}$$

with a and R integer, b and c real and w_n is a discrete forcing term. A method of analyzing the solution of system (7.1) is by the use of Z transform techniques. Let $w_n=0$, $f_{R-1}=1$ and all other initial conditions of the system (7.1) be zero. If we now take the Z transform of (7.1), utilize the two Z transform properties

$$Z\left[f_{n+k}\right]=z^k\left[f\left(z\right)-\sum_{n=0}^{k-1}f_nz^{-n}\right]$$

and

$$Z\left[f_{n-k}U_{n-k}\right]=z^{-k}F\left(z\right),$$

where U_{n-k} is the discrete step function, we obtain

$$F(z)\left\{\sum_{j=0}^{R}\binom{R}{j}(z-b)^{j}\left(-cz^{-a}\right)^{R-j}\right\}=z. \tag{7.2}$$

From (7.2)

$$F(z)=\frac{z}{\left(z-b-cz^{-a}\right)^{R}}=\frac{z^{aR+1}}{\left(z^{a+1}-bz^{a}-c\right)^{R}}. \tag{7.3}$$

In series form, (7.3) may be expressed as

$$F(z)=\sum_{r=0}^{\infty}\binom{R+r-1}{r}\frac{c^{r}z^{1-ar}}{(z-b)^{R+r}} \tag{7.4}$$

and we may obtain the inverse Z transform of (7.4) such that

$$f_n(a,b,c,R)=f_n=\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}\left(\frac{c}{b}\right)^{r}b^{n-ar-R+1} \tag{7.5}$$

where $[x]$ represents the integer part of x. The inverse Z transform of (7.3) may also be expressed as

$$f_n=\frac{1}{2\pi i}\oint_C z^{n}\left(\frac{F(z)}{z}\right)dz=\sum_{j=0}^{a}z^{n}\mathrm{Res}_j\left(\frac{F(z)}{z}\right), \tag{7.6}$$

where C is a smooth Jordan curve enclosing the singularities of (7.3) and Res_j is the residue of the poles of (7.3). The residue, Res_j, of (7.6) depend on the zeros of the characteristic function in (7.3), namely

$$g(z)=z^{a+1}-bz^{a}-c. \tag{7.7}$$

Now, $g(z)$ has $a+1$ distinct zeros $\xi_j, j=0,1,2,3,...,a$, for

$$c\neq -a^{a}\left(\frac{b}{a+1}\right)^{a+1}$$

therefore the singularities in (7.3) are all poles of order R. We may now write (7.6) as

$$f_n = \sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}\left(\xi_j\right) \binom{n}{R-1-\mu} \xi_j^{n-R+1+\mu} \tag{7.8}$$

where

$$\mu! Q_{R,\mu}\left(\xi_j\right) = \lim_{z \to \xi_j} \left[\frac{d^{\mu}}{dz^{\mu}} \left\{\left(z-\xi_j\right)^R \frac{F(z)}{z}\right\}\right] \tag{7.9}$$

for each $j = 0, 1, 2, 3, ..., a$, and $F(z)$ is given by (7.3). Combining the expressions in (7.5) and (7.8) we have that

$$\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]} \binom{R+r-1}{r} \binom{n-ar}{R+r-1} \left(\frac{c}{b}\right)^r b^{n-ar-R+1}$$

$$= \sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}\left(\xi_j\right) \binom{n}{R-1-\mu} \xi_j^{n-R+1+\mu} \tag{7.10}$$

and putting $n = n^*(a+1) + R - 1$ in (7.10) and renaming n^* as n, we have an alternate form

$$\sum_{r=0}^{n} \binom{R+r-1}{r} \binom{n(a+1)+R-1-ar}{R+r-1} \left(\frac{c}{b}\right)^r b^{n(a+1)-ar}$$

$$= \sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}\left(\xi_j\right) \binom{n(a+1)+R-1}{R-1-\mu} \xi_j^{n(a+1)+\mu}. \tag{7.11}$$

The case of distinct zeros has been examined in chapter 6, hence we shall briefly investigate the case of multiple zeros. In doing so we shall recover a result given by Wilf [93], and describe a generalization of this result which we believe to be new. The WZ pairs method of Wilf and Zeilberger will be employed to certify particular instances of identities that we shall generate.

7.3 Multiple zeros.

When the characteristic function (7.7) has double (repeated) zeros, which will be the case for $c = -a^a \left(\frac{b}{a+1}\right)^{a+1}$, then (7.3) has poles of order $2R$. In this case we may write from (7.11)

$$\sum_{r=0}^{n} \binom{R+r-1}{r} \binom{n(a+1)+R-1-ar}{R+r-1} \left(\frac{-a^a}{(a+1)^{a+1}}\right)^r = b^{-n(a+1)} \sum_{j=0}^{a} z^n \mathrm{Res}_j \left(\frac{F(z)}{z}\right) \tag{7.12}$$

where the Res_j must take into account the repeated zeros of (7.7). For $a = 1, c = -(b/2)^2$ and, from (7.3),

$$F(z) = \frac{z^{R+1}}{(z-b/2)^{2R}}$$

which has poles of order $2R$ at $z = b/2$. Utilizing (7.8), (7.9) and (7.12) we have

$$f_n(R) = \sum_{r=0}^{n} \binom{R+r-1}{r} \binom{2n+R-1-r}{R+r-1} \left(\frac{-1}{4}\right)^r = 2^{-2n} \sum_{\mu=0}^{R} \binom{R}{\mu} \binom{2n+R-1}{2R-1-\mu}. \tag{7.13}$$

If $R = 1$, then (7.13) reduces to a result given on page 124 of Wilf's book [93], namely

$$\sum_{r=0}^{n} \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r = 2^{-2n}(2n+1) = \prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{2n+1}\right), \tag{7.14}$$

where the trigonometric product is evaluated from the relation in chapter 6. Hence (7.13) is a generalization of (7.14) which we believe to be new. Utilizing Zeilberger's creative telescoping algorithm, described in [74] and available on 'Mathematica', we obtain from the left hand side of (7.13) a recurrence $f_n(R)$ that satisfies

$$4(n+1)(2n+1) f_{n+1}(R) - (n+R)(2n+2R+1) f_n(R) = 0. \tag{7.15}$$

Iterating (7.15), we have that

$$f_n(R) = 2^{-2n} \prod_{j=0}^{n-1} \frac{(R+j)(2R+1+2j)}{(1+j)(1+2j)} \tag{7.16}$$

so that from (7.13) and (7.16) we obtain

$$2^{2n}\sum_{r=0}^{n}\binom{R+r-1}{r}\binom{2n+R-1-r}{R+r-1}\left(\frac{-1}{4}\right)^{r}=\sum_{\mu=0}^{R}\binom{R}{\mu}\binom{2n+R-1}{2R-1-\mu}$$

$$=\prod_{j=0}^{n-1}\frac{(R+j)(2R+1+2j)}{(1+j)(1+2j)}. \qquad (7.17)$$

Further results may be obtained as follows. Differentiate (7.11), for $R=1$, and its trigonometric representation with respect to c, then substitute $c=-(b/2)^2$ and simplify such that

$$f_n'(1)=\sum_{r=1}^{n}r\binom{2n-r}{r}\left(\frac{-1}{4}\right)^{r}=-\prod_{j=1}^{n}\sin^2\left(\frac{\pi j}{2n+1}\right)\sum_{k=1}^{n}\cot^2\left(\frac{\pi k}{2n+1}\right). \qquad (7.18)$$

From 'Mathematica', a recurrence relation for $f_n'(1)$ in (7.18) is

$$4n(2n-1)f_{n+1}'(1)-(n+1)(2n+3)f_n'(1)=0. \qquad (7.19)$$

Iterating (7.19) and using (7.18) we have

$$\sum_{r=1}^{n}r\binom{2n-r}{r}\left(\frac{-1}{4}\right)^{r}=-2^{-2n}\prod_{j=1}^{n-1}\frac{(1+j)(3+2j)}{j(2j-1)} \qquad (7.20)$$

and comparing (7.18) and (7.20), we have

$$2^{-2n}\prod_{j=1}^{n-1}\frac{(1+j)(3+2j)}{j(2j-1)}=\prod_{j=1}^{n}\sin^2\left(\frac{\pi j}{2n+1}\right)\sum_{k=1}^{n}\cot^2\left(\frac{\pi k}{2n+1}\right). \qquad (7.21)$$

To further illustrate the technique, from (7.13) and (7.15) with $R=2$ we obtain

$$\sum_{r=0}^{n}\binom{r+1}{r}\binom{2n+1-r}{r+1}\left(\frac{-1}{4}\right)^{r}=2^{-2n}\left\{\binom{2n+1}{3}+2\binom{2n+1}{2}+\binom{2n+1}{1}\right\}$$

$$=2^{-2n}\prod_{j=0}^{n-1}\frac{(2+j)(5+2j)}{(1+j)(2j+1)}. \qquad (7.22)$$

Writing

$$\sum_{r=0}^{n}\binom{r+1}{r}\binom{2n+1-r}{r+1}\left(\frac{-1}{4}\right)^{r}=\sum_{r=0}^{n}(2n+1-r)\binom{2n-r}{r}\left(\frac{-1}{4}\right)^{r}$$

and using result (7.13) we have that

$$\sum_{r=0}^{n}\binom{r+1}{r}\binom{2n+1-r}{r+1}\left(\frac{-1}{4}\right)^{r}=2^{-2n}\left(2n+1\right)^{2}+\prod_{j=1}^{n}\sin^{2}\left(\frac{\pi j}{2n+1}\right)\sum_{k=1}^{n}\cot^{2}\left(\frac{\pi k}{2n+1}\right). \tag{7.23}$$

From (7.22) and (7.23) the identity

$$2^{-2n}\prod_{j=0}^{n-1}\frac{(2+j)(5+2j)}{(1+j)(2j+1)}-\prod_{j=1}^{n}\sin^{2}\left(\frac{\pi j}{2n+1}\right)\sum_{k=1}^{n}\cot^{2}\left(\frac{\pi k}{2n+1}\right)=2^{-2n}\left(2n+1\right)^{2} \tag{7.24}$$

is obtained and rewriting we have, using (7.21), that

$$(2n+1)^{2}=10\prod_{j=1}^{n-1}\frac{(2+j)(5+2j)}{(1+j)(2j+1)}-\prod_{j=1}^{n-1}\frac{(1+j)(3+2j)}{j(2j-1)}.$$

From (7.21) and (7.22)

$$\frac{n\left(4n^{2}-1\right)}{3}=\prod_{j=1}^{n-1}\frac{(1+j)(3+2j)}{j(2j-1)},$$

$$\frac{(n+1)(2n+1)(2n+3)}{3}=10\prod_{j=1}^{n-1}\frac{(2+j)(5+2j)}{(1+j)(2j+1)}$$

and from (7.18)

$$\sum_{r=1}^{n}r\binom{2n-r}{r}\left(\frac{-1}{4}\right)^{r}=-\frac{2^{-2n}n\left(4n^{2}-1\right)}{3}.$$

Similarly, we can show that

$$f_{n}=\sum_{r=1}^{n}r^{2}\binom{2n-r}{r}\left(\frac{-1}{4}\right)^{r}=\frac{2^{-2n}n\left(8n^{4}-20n^{3}-10n^{2}+5n+2\right)}{15}$$

$$= -2^{-2n} \prod_{j=1}^{n-1} \frac{(j+1)(2j+3)\left(2j^2-j-5\right)}{j(2j-1)\left(2j^2-5j-2\right)}. \tag{7.25}$$

The left hand side of (7.25) satisfies the recurrence

$$4n(2n-1)\left(2n^2-5n-2\right) f_{n+1} + (n+1)(2n+3)\left(2n^2-n-5\right) f_n = 0$$

and hence

$$\frac{n\left(8n^4-20n^3-10n^2+5n+2\right)}{15} = -\prod_{j=1}^{n-1} \frac{(j+1)(2j+3)\left(2j^2-j-5\right)}{j(2j-1)\left(2j^2-5j-2\right)}.$$

Similarly

$$\sum_{r=1}^{n} r^3 \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r = \frac{2^{-2n} n\left(16n^6-112n^5+112n^4+140n^3-21n^2-28n-2\right)}{105}$$

$$= -2^{-2n} \prod_{j=1}^{n-1} \frac{(j+1)(2j+3)\left(4j^4-12j^3-31j^2+18j+35\right)}{j(2j-1)\left(4j^4-28j^3+29j^2+28j+2\right)}, \text{ and}$$

$$\prod_{j=1}^{n-1} \frac{(j+1)(2j+3)\left(4j^4-12j^3-31j^2+18j+35\right)}{j(2j-1)\left(4j^4-28j^3+29j^2+28j+2\right)} = \frac{n\left(16n^6-112n^5+112n^4+140n^3-21n^2-28n-2\right)}{105}.$$

In general

$$\frac{2^{2n}}{n} \sum_{r=1}^{n} r^m \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r$$

can be expressed as a polynomial in n of degree $2m$ for m integer. By the WZ package on 'Mathematica' the identity (7.25) may be verified by the certificate function

$$V(n,r) = \frac{2(r-1)(r-1-2n)\left(\begin{array}{c} 12n^4(r-1) - 8n^3\left(r^2-r+3\right) + n^2\left(4r^2-15r-13\right) \\ +2rn(6r-7) + 6r^2 - 5r + 1 \end{array}\right)}{r(2r-1-2n)(r-1-n)(n+1)(2n+3)\left(2n^2-n-5\right)}.$$

Similarly for the identity (7.13), for particular values of R, and by the use of the WZ package we may obtain a rational certificate function, $V(n,r,R)$ that certifies the identity, in particular

$$V(n,r,1)=\frac{2r(2n+1-r)(4r-5-6n)}{(2n+3)(2n+1-2r)(n+1-r)} \text{ and}$$

$$V(n,r,4)=\frac{2r(2n+4-r)(4nr+10r-6n^2-23n-14)}{(n+4)(n+9)(2n+1-2r)(n+1-r)}.$$

7.4 More sums.

Since (7.7) has at most three real zeros we may obtain further results as follows. Consider multiple zeros of (7.7) for $a=2$ and $c=-4(b/3)^3$ such that $g(z)=\left(z-\frac{2b}{3}\right)^2\left(z+\frac{b}{3}\right)$ and therefore (7.3) and (7.9) may be modified such that

$$F(z)=\frac{z^{2R+1}}{\left(\left(z-\frac{2b}{3}\right)^2\left(z+\frac{b}{3}\right)\right)^R},$$

$$\mu!Q_{2R,\mu}\left(\frac{2b}{3}\right)=\lim_{z\to\frac{2b}{3}}\left[\frac{d^\mu}{dz^\mu}\left\{\left(z-\frac{2b}{3}\right)^{2R}\frac{F(z)}{z}\right\}\right], \mu=0,1,2,...,2R-1,$$

$$\nu!P_{R,\nu}\left(-\frac{b}{3}\right)=\lim_{z\to-\frac{b}{3}}\left[\frac{d^\nu}{dz^\nu}\left\{\left(z+\frac{b}{3}\right)^{R}\frac{F(z)}{z}\right\}\right], \nu=0,1,2,...,R-1$$

hence from (7.11)

$$\sum_{r=0}^{n}\binom{R+r-1}{r}\binom{3n+R-1-2r}{R+r-1}\left(\frac{-4}{27}\right)^r b^{3n}=$$

$$\sum_{j=0}^{a}\sum_{\mu=0}^{2R-1}Q_{2R,\mu}\left(\frac{2b}{3}\right)\binom{3n+R-1}{2R-1-\mu}\left(\frac{2b}{3}\right)^{3n+\mu}+\sum_{j=0}^{a}\sum_{\nu=0}^{R-1}P_{R,\nu}\left(-\frac{b}{3}\right)\binom{3n+R-1}{R-1-\mu}\left(-\frac{b}{3}\right)^{3n+\mu}. \tag{7.26}$$

For $R=1$ and $R=2$, we have respectively from (7.26) that

$$f_n(1)=\sum_{r=0}^{n}\binom{3n-2r}{r}\left(\frac{-4}{27}\right)^r=3^{-(3n+2)}\left\{2^{3n+1}(9n+4)+(-1)^n\right\} \text{ and} \tag{7.27}$$

$$f_n(2)=\sum_{r=0}^{n}(r+1)\binom{3n+1-2r}{r+1}\left(\frac{-4}{27}\right)^r=3^{-(3n+2)}\left\{\begin{array}{c}2^{3n+2}\binom{3n+1}{3}+\frac{2^{3n+4}}{27}+\\ \frac{2^{3n+5}}{3}\binom{3n+1}{2}+2^{3n+3}(3n+1)\\ +\frac{(-1)^n}{9}\left(3n+\frac{11}{3}\right)\end{array}\right\} \quad (7.28)$$

$$=(3n+1)\ {}_3F_2\left[\begin{array}{c}\frac{1-3n}{3},\frac{2-3n}{3},-n\\ \frac{-1-3n}{2},\frac{-3n}{2}\end{array}\middle|1\right].$$

From 'Hyper', in 'Mathematica' a recurrence relation for (7.27) and (7.28) is, respectively

$$729(3n+4)f_{n+2}(1)-27(21n+52)f_{n+1}(1)-8(3n+7)f_n(1)=0, f_0(1)=1, f_1(1)=\frac{23}{27}$$

and

$$729(3n+5)(3n+4)^2 f_{n+2}(2)-27\left(189n^3+1440n^2+3399n+2348\right)f_{n+1}(2)$$
$$-8(3n+7)(3n+8)(3n+10)f_n(2)=0, f_0(2)=1, f_1(2)=\frac{100}{27}.$$

7.5 Other forcing terms.

We can now consider the system (7.1) with non zero forcing terms. Consider a forcing term of the form, (other forms may also be taken).

$$w_n=\binom{n}{m+R-1}b^{n+1-R-m}$$

with all initial conditions zero and m a positive integer, again the results of the previous section are applicable. For the purpose of demonstration let $a=1$ and $c=-(b/2)^2$ so that from (7.12)

$$\sum_{r=0}^{n}\binom{R+r-1}{r}\binom{2n+R+m-1-r}{R+m+r-1}\left(\frac{-1}{4}\right)^r=$$

$$b^{-2n}\sum_{\mu=0}^{2R-1}Q_{2R,\mu}\left(\frac{b}{2}\right)\binom{n}{2R-1-\mu}\left(\frac{b}{2}\right)^{2n-R+m+\mu}+\sum_{\nu=0}^{m-1}P_{m,\nu}(b)\binom{n}{m-1-\mu}b^{R+\mu}$$

where

$$\mu! Q_{2R,\mu}\left(\frac{b}{2}\right) = \lim_{z\to\frac{b}{2}}\left[\frac{d^\mu}{dz^\mu}\left\{\frac{z^R}{(z-b)^R}\right\}\right]$$

and

$$\nu! P_{m,\nu}(b) = \lim_{z\to b}\left[\frac{d^\nu}{dz^\nu}\left\{\frac{z^R}{\left(z-\frac{b}{2}\right)^{2R}}\right\}\right].$$

In the case that $R=1, m=1$ and 2 respectively we obtain

$$\sum_{r=0}^{n}\binom{2n+1-r}{r+1}\left(\frac{-1}{4}\right)^r = 4-2^{-2n}(2n+3) = 4-3.2^{-2n}\prod_{j=0}^{n-1}\frac{5+2j}{3+2j}$$

and

$$\begin{aligned}\sum_{r=0}^{n}\binom{2n+2-r}{r+2}\left(\frac{-1}{4}\right)^r &= 4(2n-1)+2^{-2n}(2n+5)\\ &= 4\prod_{j=1}^{n-1}\frac{2j+1}{2j-1}+7.2^{-2n}\prod_{j=1}^{n-1}\frac{7+2j}{5+2j}.\end{aligned}$$

For constants α_j and positive integer m we have that

$$f_n = \sum_{r=0}^{n}\binom{2n+m-r}{r+m}\left(\frac{-1}{4}\right)^r = (-1)^m 2^{-2n}(2n+2m+1)+4\sum_{j=0}^{n-1}\alpha_j n^j$$

and for $m=0$ reduces to identity (7.14); moreover a recurrence for the left hand side is

$$4(2n+2m+1)f_{n+1}-(2n+2m+3)f_n = \frac{4m(6n+2m+5)}{(2n+m+2)}\binom{2n+m+2}{2n+m+2}, f_0=1.$$

In the next chapter we shall develop new identities for the infinite representation of the sum (7.5).

Chapter 8

Sums of Binomial variation.

In this chapter the results of chapters six and seven will be generalized. By residue theory and induction the author proves that infinite generated sums may be represented in closed form which depend on k dominant zeros of an associated polynomial characteristic function. A connection between the infinite series and generalized hypergeometric functions is also demonstrated.[1].

[1] A modification of this chapter has been submitted for publication .

8.1 Introduction.

In this chapter we shall generalize the identities of chapters six and seven. We shall consider the infinite form of the sum (7.5) and develop a procedure for representing it in closed form. We will prove that the closed form representation will depend on a dominant zero of an associated characteristic function. We will also demonstrate a connection of the infinite binomial sums with generalized hypergeometric functions and some of its identities. We can then develop these ideas further and generate infinite binomial type sums which we may represent in closed form depending on k dominant zeros of an associated characteristic function. Particular cases of our identities may be certified by the WZ pairs method of Wilf and Zeilberger. We shall illustrate our theoretical results with some numerical examples. In the appendix we will investigate some properties of zeros of polynomial characteristic functions.

8.2 One dominant zero.

If we consider the system (7.5) of chapter seven, with $c = b$ to make the following algebra more manageable, we obtain

$$F(z) = \frac{z}{(z - b - bz^{-a})^R} = \frac{z^{aR+1}}{(g(z))^R} \tag{8.1}$$

with

$$g(z) = z^{a+1} - bz^a - b. \tag{8.2}$$

Now $g(z)$ has $a+1$ distinct zeros ξ_j, $j = 0, 1, 2, ..., a$, for

$$\left| \frac{(a+1)^{a+1}}{(ab)^a} \right| < 1, \tag{8.3}$$

and from residue consideration

$$\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]} \binom{r+R-1}{r} \binom{n-ar}{r+R-1} b^{n-ar-R+1} = \sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_j) \binom{n}{R-1-\mu} \xi_j^{n-R+1+\mu}. \tag{8.4}$$

If we define the dominant zero ξ_0 of $g(z)$ in (8.2) as the one with the greatest modulus, we conjecture (and shortly prove) that

$$\sum_{r=0}^{\infty} T_r(R,n,a,b) = \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_0) \binom{n}{R-1-\mu} \xi_0^{n-R+1+\mu} \tag{8.5}$$

where

$$\mu! Q_{R,\mu}(\xi_0) = \lim_{z\to\xi_0} \left[\frac{d^\mu}{dz^\mu} \left\{ (z-\xi_0)^R \frac{F(z)}{z} \right\} \right], \mu = 0,1,2,...,R-1, \tag{8.6}$$

$F(z)$ is given by (8.1) and

$$T_r(R,n,a,b) = \binom{r+R-1}{r} \binom{n-ar}{r+R-1} b^{n-ar-R+1}. \tag{8.7}$$

We may also note,from (8.4) and (8.5), that

$$\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]} T_r(R,n,a,b) + \sum_{r=\left[\frac{n+2-R}{a+1}\right]}^{\infty} T_r(R,n,a,b) = \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_0) \binom{n}{R-1-\mu} \xi_0^{n-R+1+\mu}$$

and therefore

$$\sum_{r=\left[\frac{n+2-R}{a+1}\right]}^{\infty} T_r(R,n,a,b) = -\sum_{j=1}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_j) \binom{n}{R-1-\mu} \xi_j^{n-R+1+\mu}.$$

The conjecture (8.5) will be proved later, firstly we give a recurrence for the evaluation of (8.6) and a recurrence for (8.5).

8.2.1 Recurrences.

A recurrence relation for the evaluation of $Q_{R,\mu}(\xi_0)$ in (8.6) is now given.

Lemma 35 *:*

$$(\mu+1) Q_{R,\mu+1}(\xi_0) = \frac{R(\xi_0 - b)}{\xi_0^2} \sum_{k=0}^{\mu} \frac{(-1)^k (k+1)}{\xi_0^k} \binom{a+k+1}{a-1} Q_{R+1,\mu-k}(\xi_0), \tag{8.8}$$

$\mu = 0, 1, 2, ..., R-1$, *with* $Q_{R,0}(\xi_0) = \left(\frac{\xi_0}{(a+1)\xi_0 - ab}\right)^R$.

Proof: Putting $\mu = 0$ into (8.6) we obtain the expression for $Q_{R,0}(\xi_0)$. Also from the definition,

$$(\mu+1)!Q_{R,\mu+1}(\xi_0) = \lim_{z\to\xi_0}\left[\frac{d^{\mu+1}}{dz^{\mu+1}}\left\{\left(\frac{z^a(z-\xi_0)}{g(z)}\right)^R\right\}\right]$$

$$= R\lim_{z\to\xi_0}\left[\frac{d^{\mu}}{dz^{\mu}}\left\{\left(\frac{z^a(z-\xi_0)}{g(z)}\right)^{R+1}\frac{h(z)}{(z-\xi_0)^2}\right\}\right]$$

where

$$h(z) = \frac{(a(z-\xi_0)+z)\,g(z) - z(z-\xi_0)\,g'(z)}{z^{a+1}}$$

and $g'(z) = z^{a-1}((a+1)z - ab)$. It can be seen that $h(\xi_0) = 0$ and $h'(\xi_0) = 0$, hence expanding $h(z)$ in a Taylor series about $z = \xi_0$ we find that

$$\frac{h(z)}{(z-\xi_0)^2} = \sum_{j=2}^{\infty}\frac{(-1)^j(\xi_0-b)(j-1)(z-\xi_0)^{j-2}}{\xi_0^j}\begin{pmatrix} a+j-1 \\ a-1 \end{pmatrix} = B(z).$$

We now have

$$(\mu+1)!Q_{R,\mu+1}(\xi_0) = R\lim_{z\to\xi_0}\left[\frac{d^{\mu}}{dz^{\mu}}\left\{\left(\frac{z^a(z-\xi_0)}{g(z)}\right)^{R+1}B(z)\right\}\right]$$

$$= R\lim_{z\to\xi_0}\sum_{k=0}^{\mu}\begin{pmatrix} \mu \\ k \end{pmatrix}\left(\frac{z^{a(R+1)}(z-\xi_0)^{R+1}}{g^{R+1}(z)}\right)^{(\mu-k)}B^{(k)}(z)$$

where $B^{(k)}(z) = \frac{d^k}{dz^k}B(z)$, and

$$\lim_{z\to\xi_0}\left[\frac{d^k}{dz^k}B(z)\right] = \lim_{z\to\xi_0}\left[\sum_{j=2}^{\infty}\frac{(-1)^j(\xi_0-b)(j-1)_k(z-\xi_0)^{j-2-k}}{\xi_0^j}\begin{pmatrix} a+j-1 \\ a-1 \end{pmatrix}\right]$$

$$= \frac{(-1)^k(\xi_0-b)(k+1)!}{\xi_0^{k+2}}\begin{pmatrix} a+k-1 \\ a-1 \end{pmatrix}$$

μ	$\mu!Q_{R,\mu}(\xi_0)$
0	$\frac{\xi_0^R}{A^R}$
1	$\frac{aR(A-b)}{2A^{R+1}}\xi_0^{R-1} = \frac{aR(a+1)(\xi_0-b)}{2A^{R+1}}\xi_0^{R-1}$
2	$\frac{aR\xi_0^{R-2}}{12A^{R+2}}\left[A^2(3aR-a-8)-2bA(3aR+a-4)+3(R+1)ab^2\right]$
3	$\frac{aR\xi_0^{R-3}}{8A^{R+3}}\left[\begin{array}{c} A^3\left\{a^2R(R-1)+2a(1-4R)+12\right\} \\ +bA^2\left\{-a^2R(3R+1)+2a(8R+3)-12\right\} \\ +b^2A\left\{a^2(3R+2)(R+1)-8a(R+1)\right\} \\ -a^2b^3(R+1)(R+2) \end{array}\right]$

Table 8.1: The Q values of the discrete case.

R	closed form (8.5)
1	$\xi_0^{n+1}\left[\frac{1}{A}\right]$
2	$\frac{\xi_0^{n+1}}{A^2}\left[n+\frac{a(a+1)(\xi_0-b)}{A}\right]$
3	$\frac{\xi_0^{n+1}}{A^3}\left[\binom{n}{2}+\frac{3na(A-b)}{2A}+\frac{a\left(2(a-1)A^2-bA(5a-2)+3ab^2\right)}{2A^2}\right]$
4	$\frac{\xi_0^{n+1}}{A^4}\left[\begin{array}{c} \binom{n}{3}+\frac{2a(A-b)}{A}\binom{n}{2}+\frac{na\left((11a-8)A^2-2bA(13a-4)+15ab^2\right)}{6A^2} \\ +\frac{a}{12A^3}\left(\begin{array}{c} A^3\left(12a^2-30a+12\right)-bA^2\left(52a^2-70a+12\right) \\ +b^2Aa(70a-40)-30a^2b^3 \end{array}\right) \end{array}\right]$

Table 8.2: Closed form of the discrete case.

and $(j-1)_k$ is Pochhammer's symbol. Hence we can write

$$(\mu+1)!Q_{R,\mu+1}(\xi_0) = R\sum_{k=0}^{\mu}\binom{\mu}{k}(\mu-k)!Q_{R+1,\mu-k}(\xi_0)\frac{(-1)^k(\xi_0-b)(k+1)!}{\xi_0^{k+2}}\binom{a+k-1}{a-1}$$

and upon simplification we obtain (8.8) hence, the lemma 35 is proved. We can now list some values of $Q_{R,\mu}(\xi_0)$ in table 8.1, where for ease, $A=(a+1)\xi_0-ab$.

Using the values of $Q_{R,\mu}(\xi_0)$ in table 8.1, some closed form expressions of (8.5) are listed in table 8.2.

The following lemma gives a recurrence relation for the left hand side of the identity (8.5).

Lemma 36 : *Let*

$$S_R = \sum_{r=0}^{\infty} T_r(R,n,a,b) \tag{8.9}$$

where $T_r(R,n,a,b)$ is given by (8.7). A recurrence relation of (8.9) is

$$(a+1)\,b\frac{d}{db}S_R - abRS_{R+1} - (n+1-R)\,S_R = 0. \tag{8.10}$$

Proof:

$$\frac{d}{db}S_R = -\frac{a}{b}\sum_{r=0}^{\infty} rT_r(R,n,a,b) + \left(\frac{n+1-R}{b}\right)S_R$$

and

$$\begin{aligned} S_{R+1} &= \sum_{r=0}^{\infty}\begin{pmatrix} r+R \\ r \end{pmatrix}\begin{pmatrix} n-ar \\ r+R \end{pmatrix} b^{n-ar-R} \\ &= -\frac{(a+1)}{bR}\sum_{r=0}^{\infty} rT_r(R,n,a,b) + \left(\frac{n+1-R}{bR}\right)S_R. \end{aligned}$$

From the left hand side of (8.10)

$$\begin{aligned} &(a+1)\,b\left[-\frac{a}{b}\sum_{r=0}^{\infty} rT_r(R,n,a,b) + \left(\frac{n+1-R}{b}\right)S_R\right] - \\ &abR\left[-\frac{(a+1)}{bR}\sum_{r=0}^{\infty} rT_r(R,n,a,b) + \left(\frac{n+1-R}{bR}\right)S_R\right] - (n+1-R)\,S_R \\ = {}& 0, \end{aligned}$$

which is the right hand side of (8.10) and the proof is complete.

8.2.2 Proof of conjecture.

The proof of the conjecture (8.5) will involve an induction argument on the parameter R. For the basis, $R=1$, (8.5) was proved in chapter six. Now we give an induction argument for the right hand side of (8.5).

$$S_{R+1} = \frac{1}{abR}\left\{(a+1)\,b\frac{d}{db}S_R - (n+1-R)\,S_R\right\}, \tag{8.11}$$

also, $\frac{d}{db}S_R = \frac{\xi_0^2}{bA}\frac{d}{d\xi_0}S_R$ and

$$\frac{d}{db}S_R = \frac{\xi_0^2}{bA}\sum_{\mu=0}^{R-1}\begin{pmatrix} n \\ R-1-\mu \end{pmatrix}\xi_0^{n-R+1+\mu}\left\{\frac{d}{d\xi_0}Q_{R,\mu}(\xi_0) + \left(\frac{n+1+\mu-R}{\xi_0}\right)Q_{R,\mu}(\xi_0)\right\}.$$

Substituting into the right hand side of (8.11), we have

$$S_{R+1} = \frac{1}{abR}\left\{\begin{array}{c} \frac{(a+1)b\xi_0^2}{bA}\sum\limits_{\mu=0}^{R-1}\begin{pmatrix} n \\ R-1-\mu \end{pmatrix}\xi_0^{n-R+1+\mu}\left(\begin{array}{c}\frac{d}{d\xi_0}Q_{R,\mu}(\xi_0) + \\ \left(\frac{n+1+\mu-R}{\xi_0}\right)Q_{R,\mu}(\xi_0)\end{array}\right) \\ -(n+1-R)\sum\limits_{\mu=0}^{R-1}\begin{pmatrix} n \\ R-1-\mu \end{pmatrix}\xi_0^{n-R+1+\mu}Q_{R,\mu}(\xi_0) \end{array}\right\}$$

$$= \frac{1}{abRA}\sum_{\mu=0}^{R-1}\begin{pmatrix} n \\ R-1-\mu \end{pmatrix}\xi_0^{n-R+1+\mu}\left\{\begin{array}{c}(A+ab)\,\xi_0\frac{d}{d\xi_0}Q_{R,\mu}(\xi_0) + \mu A Q_{R,\mu}(\xi_0) \\ +ab\,(n+1+\mu-R)\,Q_{R,\mu}(\xi_0)\end{array}\right\}$$

$$= \frac{1}{abRA}\left\{\begin{array}{c}\sum\limits_{\mu=0}^{R-1} ab\,(R-\mu)\,\xi_0\begin{pmatrix} n \\ R-\mu \end{pmatrix}\xi_0^{n-R+\mu}Q_{R,\mu}(\xi_0) + \sum\limits_{\mu=0}^{R-1}\begin{pmatrix} n \\ R-1-\mu \end{pmatrix}\xi_0^{n-R+1+\mu} \\ \left((A+ab)\,\xi_0\frac{d}{d\xi_0}Q_{R,\mu}(\xi_0) + \mu A Q_{R,\mu}(\xi_0)\right)\end{array}\right\}$$

$$= \frac{1}{abRA}\left\{\begin{array}{c} abR\xi_0\begin{pmatrix} n \\ R \end{pmatrix}\xi_0^{n-R}Q_{R,0}(\xi_0) + \sum\limits_{\mu=1}^{R-1} ab\,(R-\mu)\begin{pmatrix} n \\ R-\mu \end{pmatrix}\xi_0^{n-R+1+\mu}Q_{R,\mu}(\xi_0) \\ +\sum\limits_{\mu=0}^{R-1}\begin{pmatrix} n \\ R-1-\mu \end{pmatrix}\xi_0^{n-R+1+\mu}\left((A+ab)\,\xi_0\frac{d}{d\xi_0}Q_{R,\mu}(\xi_0) + \mu A Q_{R,\mu}(\xi_0)\right)\end{array}\right\}.$$

In the second sum rename $\mu^* = \mu + 1$ (and let $\mu^* = \mu$), so that we may write

$$\begin{pmatrix} n \\ R \end{pmatrix}\frac{\xi_0}{A}\xi_0^{n-R}Q_{R,0}(\xi_0) + \frac{1}{abRA}\left\{\begin{array}{c}\sum\limits_{\mu=1}^{R-1} ab\,(R-\mu)\begin{pmatrix} n \\ R-\mu \end{pmatrix}\xi_0^{n-R+1+\mu}Q_{R,\mu}(\xi_0) + \\ \sum\limits_{\mu=1}^{R-1}\begin{pmatrix} n \\ R-\mu \end{pmatrix}\xi_0^{n-R+\mu}\left(\begin{array}{c}(A+ab)\,\xi_0\frac{d}{d\xi_0}Q_{R,\mu-1}(\xi_0) \\ +(\mu-1)\,AQ_{R,\mu-1}(\xi_0)\end{array}\right)\end{array}\right\}$$

$$= \binom{n}{R} \xi_0^{n-R} Q_{1,0}(\xi_0) Q_{R,0}(\xi_0) + \frac{1}{abRA} \sum_{\mu=1}^{R-1} \binom{n}{R-\mu} \xi_0^{n-R+\mu} \left\{ \begin{array}{c} ab(R-\mu)\xi_0 Q_{R,\mu}(\xi_0) \\ +(A+ab)\xi_0 \frac{d}{d\xi_0} Q_{R,\mu-1}(\xi_0) \\ +(\mu-1) A Q_{R,\mu-1}(\xi_0) \end{array} \right\}$$

$$= \binom{n}{R} \xi_0^{n-R} Q_{R+1,0}(\xi_0) + \frac{1}{abRA} \sum_{\mu=1}^{R-1} \binom{n}{R-\mu} \xi_0^{n-R+\mu} \left\{ \begin{array}{c} ab(R-\mu)\xi_0 Q_{R,\mu}(\xi_0) \\ +(A+ab)\xi_0 \frac{d}{d\xi_0} Q_{R,\mu-1}(\xi_0) \\ +(\mu-1) A Q_{R,\mu-1}(\xi_0) \end{array} \right\}.$$

Using the relationship (8.42) in appendix A reduces the previous line to

$$\binom{n}{R} \xi_0^{n-R} Q_{R+1,0}(\xi_0) + \sum_{\mu=1}^{R} \binom{n}{R-\mu} \xi_0^{n-R+\mu} Q_{R+1,\mu}(\xi_0)$$

$$= \sum_{\mu=0}^{R} \binom{n}{R-\mu} \xi_0^{n-R+\mu} Q_{R+1,\mu}(\xi_0)$$

which completes the proof of the conjecture.

The degenerate case, for $a = 0$, of the identity (8.5) can be noted. Firstly, $\xi_0 = 2b$ and $\mu! Q_{R,\mu}(2b) = 1$ for $\mu = 0$ and zero otherwise, hence (8.5) reduces to $\sum_{r=0}^{n+1-R} \binom{n+1-R}{r} = 2^{n+1-R}$, which is not Gosper summable, as defined by Petkovšek et al. [74], however by the WZ pairs method a rational certificate function is $V(n,r) = \frac{r}{2(r-n-2+R)}$.

It is of passing interest only, to note that the dominant zero ξ_0, may be set to unity, in which case the closed forms in table 8.2 of the identity (8.5) would be simplified. If we put, from (8.2), $g_U(z) = z^{a+1} - bz^a - 1 + b$, then for $1 < b < \left(1 + \frac{1}{a}\right)$, $g_U(z)$ has a unit distinct dominant zero ξ_0. From (8.5) for $R = 1, 2$ and 3, $n = -\alpha \in \Re$ and putting $A_U = a(1-b) + 1$, we have explicitly that

$$\sum_{r=0}^{\infty} (-1)^r \binom{(a+1)r + \alpha - 1}{r} \left(\frac{1-b}{b^{a+1}}\right)^r = \frac{b^\alpha}{A_U},$$

$$\sum_{r=0}^{\infty} (-1)^r \binom{r+1}{r} \binom{(a+1)r + \alpha}{r+1} \left(\frac{1-b}{b^{a+1}}\right)^r = \frac{b^{\alpha+1}}{A_U^2} \left\{\alpha - \frac{a(a+1)(1-b)}{A_U}\right\},$$

$$\sum_{r=0}^{\infty}(-1)^r\begin{pmatrix} r+2 \\ r \end{pmatrix}\begin{pmatrix} (a+1)r+\alpha+1 \\ r+2 \end{pmatrix}\left(\frac{1-b}{b^{a+1}}\right)^r = \frac{b^{\alpha+2}}{A_U^3}\left\{\begin{array}{c} \frac{\alpha(\alpha+1)}{2} - \frac{3a\alpha(\alpha+1)(1-b)}{2A_U} \\ \frac{a\left(2(a-1)A_U^2-(5a-2)bA_U+3ab^2\right)}{2A_U^2} \end{array}\right\}$$

and in general all other R value identities can be obtained from (8.5).

8.2.3 Hypergeometric functions.

Let $T_r(R,n,a,b)$ be defined by (8.7) and $T_0(R,n,a,b) = \begin{pmatrix} n \\ R-1 \end{pmatrix} b^{n+1-R}$. The ratio of consecutive terms

$$\frac{T_{r+1}(R,n,a,b)}{T_r(R,n,a,b)} = \frac{\prod_{j=0}^{a}\left(r+\frac{j+R-n-1}{a+1}\right)}{(r+1)\prod_{j=0}^{a-1}\left(r+\frac{j-n}{a}\right)} s(a,b)$$

is a rational function in r and therefore the series $S_R(n,a,b)$ of (8.9) may be expressed as a generalized hypergeometric function

$$T_0\ {}_{a+1}F_a\left[\begin{array}{c} \frac{R-n-1}{a+1}, \frac{R-n}{a+1}, \frac{R-n+1}{a+1}, \ldots, \frac{R+a-n-1}{a+1} \\ -\frac{n}{a}, \frac{1-n}{a}, \frac{2-n}{a}, \ldots, \frac{a-1-n}{a} \end{array}\middle| s(a,b)\right] \tag{8.12}$$

$$= T_0\sum_{k=0}^{\infty}\frac{\left(\frac{R-n-1}{a+1}\right)_k\left(\frac{R-n}{a+1}\right)_k\left(\frac{R-n+1}{a+1}\right)_k\cdots\left(\frac{R+a-n-1}{a+1}\right)_k}{\left(-\frac{n}{a}\right)_k\left(\frac{1-n}{a}\right)_k\left(\frac{2-n}{a}\right)_k\cdots\left(\frac{a-1-n}{a}\right)_k}\frac{s^k(a,b)}{k!}$$

where $(x)_m$ is Pochhammer's symbol and

$$s(a,b) = -\frac{(a+1)^{a+1}}{(ab)^a}.$$

Some particular results of (8.12) are worthy of a mention, since it may be shown that (8.12) reduces to known hypergeometric functions. For $a=1$ and $\alpha=-n\in\Re\backslash J^-$ then

$$T_0\ {}_2F_1\left[\begin{array}{c} \frac{R+\alpha-1}{2}, \frac{R+\alpha}{2} \\ \alpha \end{array}\middle| s(1,b)\right] \tag{8.13}$$

$$= T_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k! b^k} \frac{\prod_{j=0}^{2k-1} (R + \alpha - 1 + j)}{\prod_{j=0}^{k-1} (\alpha + j)} \right],$$

the Gauss hypergeometric series (8.13) may also be written as

$$\frac{T_0}{B\left(\frac{R+\alpha}{2}, \frac{\alpha-R}{2}\right)} \int_{t=0}^{1} (1-t)^{(\alpha-R-2)/2} t^{(R+\alpha-2)/2} (1 - s(1,b)\, t)^{(1-R-\alpha)/2} dt$$

which is valid for $|s(1,b)| < 1$ and $B(x,y)$ is the Beta function. The difference in the two top terms of the hypergeometric function (8.13) is one-half, hence there exists a quadratic transformation, see [1], connected with the Legendre function, P_ν^μ. Using the identity on page 562 of Abramowitz and Stegun [1] we may write (8.13) as

$$T_0 2^{\alpha-1} \Gamma(\alpha) \{-s(1,b)\}^{(1-\alpha)/2} \{1 - s(1,b)\}^{-R/2} P_{R-1}^{1-\alpha} \left\{ \{1 - s(1,b)\}^{-1/2} \right\},$$

where $s(1,b) \in (-\infty, 0)$ and $\Gamma(x)$ is the Gamma function. We may write the identity (8.5) as

$$\sum_{r=0}^{\infty} \binom{R+r-1}{r} \binom{-\alpha-r}{R+r-1} b^{-\alpha-r-R+1} = \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_0) \binom{-\alpha}{R-1-\mu} \xi_0^{n-R+1+\mu}$$

$$= \frac{T_0 \Gamma(\alpha)}{2^{1-\alpha}} \left\{ \frac{4}{b} \right\}^{(1-\alpha)/2} \left\{ 1 + \frac{4}{b} \right\}^{-R/2} P_{R-1}^{1-\alpha} \left\{ \left\{ 1 + \frac{4}{b} \right\}^{-1/2} \right\}$$

for $b > 4, 2\xi_0 = b + \sqrt{b^2 + 4b}$ and $Q_{R,\mu}(\xi_0)$ is defined by (8.6). Other specific cases of (8.13) are as follows.

(i). For $b = 4$, $s(1,4) = -1$, $\alpha = 3/2$, and from page 557 of Abramowitz and Stegun [1]

$$T_0\, {}_2F_1 \left[\begin{matrix} \frac{2R+1}{4}, \frac{2R+3}{4} \\ \frac{3}{2} \end{matrix} \middle| -1 \right] = \frac{T_0 2^{-(2R+1)/4} \Gamma\left(\frac{3}{2}\right) \sqrt{\pi}}{\frac{2R-1}{4}} \left\{ \begin{matrix} \frac{1}{\Gamma\left(\frac{R}{4}+\frac{1}{8}\right)\Gamma\left(\frac{7}{8}-\frac{R}{4}\right)} \\ -\frac{1}{\Gamma\left(\frac{R}{4}+\frac{5}{8}\right)\Gamma\left(\frac{3}{8}-\frac{R}{4}\right)} \end{matrix} \right\},$$

since the parameters in the hypergeometric function

$$\frac{2R+1}{4} - \frac{2R+3}{4} + \frac{3}{2} = 1,$$

then from Kummer's identity we have

$$T_0\,{}_2F_1\left[\begin{array}{c}\frac{2R+1}{4},\frac{2R+3}{4}\\ \frac{3}{2}\end{array}\bigg| -1\right] = \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_0)\binom{-3/2}{R-1-\mu}\xi_0^{\mu-R-1/2}$$

$$= \sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{-3/2-r}{R+r-1}4^{-1/2-r-R}$$

$$= \frac{T_0\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{R}{4}+\frac{11}{8}\right)}{\Gamma\left(\frac{R}{2}+\frac{7}{4}\right)\Gamma\left(\frac{9}{8}-\frac{R}{4}\right)}.$$

Here the dominant zero $\xi_0 = 2\left(1+\sqrt{2}\right)$and some values of the infinite sum are

$R=1$	$R=2$	$R=3$	$R=4$
$\frac{1}{8\left(1+\sqrt{2}\right)^{1/2}}$	$\frac{-\left(2+\sqrt{2}\right)}{2(4^3)\left(1+\sqrt{2}\right)^{1/2}}$	$\frac{3\left(2+\sqrt{2}\right)}{2(4^5)\sqrt{2}\left(1+\sqrt{2}\right)^{1/2}}$	$\frac{-5}{4^7\sqrt{2}\left(1+\sqrt{2}\right)^{1/2}}$

(ii). For $b=4$, $s(1,4)=-1$, $\alpha=1/2$, and from page 557 of Abramowitz and Stegun [1] we have

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{-1/2-r}{R+r-1}4^{1/2-r-R} = T_0\,{}_2F_1\left[\begin{array}{c}\frac{2R-1}{4},\frac{2R+1}{4}\\ \frac{1}{2}\end{array}\bigg| -1\right]$$

$$= \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_0)\binom{-1/2}{R-1-\mu}\xi_0^{\mu-R+1/2}$$

$$= \frac{T_0\pi 2^{(1-2R)/2}}{\Gamma\left(\frac{R}{4}+\frac{3}{8}\right)\Gamma\left(\frac{5}{8}-\frac{R}{4}\right)}$$

and some values of the infinite sum are

$R=1$	$R=2$	$R=3$	$R=4$
$\frac{\left(1+\sqrt{2}\right)^{1/2}}{4}$	$\frac{\left(1+\sqrt{2}\right)^{1/2}\left(\sqrt{2}-2\right)}{4^3}$	$\frac{3\sqrt{2}\left(1+\sqrt{2}\right)^{1/2}\left(\sqrt{2}-2\right)}{2(4^5)}$	$\frac{5\sqrt{2}\left(1+\sqrt{2}\right)^{1/2}}{4^7}$

8.2.4 Forcing terms.

As in chapter seven, if in the dynamical system we consider forcing terms of the type

$$w_n = \binom{n}{m+R-1} b^{n+1-R-m},$$

then the result is the identity

$$\sum_{r=0}^{\infty} \binom{r+R-1}{r} \binom{n-ar}{r+R+m-1} b^{n-ar-R-m+1} = \sum_{\mu=0}^{R-1} Q_{R,\mu}(\xi_0) \binom{n}{R-1-\mu} \xi_0^{n-R+1+\mu}$$

$$+ \sum_{\nu=0}^{m-1} P_{m,\nu}(b) \binom{n}{m-1-\nu} b^{n-m+1+\nu}$$

where

$$\mu! Q_{R,\mu}(\xi_0) = \lim_{z \to \xi_0} \left[\frac{d^\mu}{dz^\mu} \left\{ (z-\xi_0)^R \frac{F(z)}{z} \right\} \right],$$

$$\nu! P_{m,\nu}(b) = \lim_{z \to b} \left[\frac{d^\nu}{dz^\nu} \left\{ (z-b)^m \frac{F(z)}{z} \right\} \right]$$

and

$$F(z) = \frac{z^{aR+1}}{(z-b)^m (z^{a+1} - bz^a - b)^R}$$

where ξ_0 is the dominant zero of (8.2). If $R = 2$ and $m = 1$, then

$$\sum_{r=0}^{\infty} \binom{r+1}{r} \binom{n-ar}{r+2} \left(\frac{1}{b^a}\right)^r = b^{2a} + \frac{b^{2-n}\xi_0^{1+n} \left(n\left((a+1)\xi_0 - ab\right) + a(a+1)(\xi_0 - b)\right)}{(\xi_0 - b)\left((a+1)\xi_0 - ab\right)},$$

for the degenerate case of $a = 0$, we obtain the interesting Binomial convolution identity

$$\sum_{r=0}^{n+1-m-R} \binom{r+R-1}{r} \binom{n}{r+R+m-1} = \sum_{\nu=0}^{m-1} (-1)^R \binom{R+\nu-1}{\nu} \binom{n}{m-1-\nu}$$

$$+ \sum_{\mu=0}^{R-1} (-1)^\mu \binom{m+\mu-1}{\mu} \binom{n}{R-1-\mu} 2^{n-R+1+\mu}$$

$$= \frac{{}_2F_1\left[\begin{matrix} R, R+m-n-1 \\ R+m \end{matrix} \middle| -1\right]}{(n+1)\, B\,(R+m, n+2-R-m)}.$$

which for specific values of m and R may be certified by the WZ pairs method of Wilf and Zeilberger.

8.2.5 Products of central binomial coefficients.

In chapter six we obtained identities of central binomial coefficients, we can carry out a similar but brief examination here for the identity (8.5). From chapter six, (6.10), and (8.5) putting $R = 2, a = 1, n = -\alpha, b = 1$ and $c = -x$ then

$$f(\alpha, x) = \sum_{r=0}^{\infty} \binom{r+1}{r} \binom{2r+\alpha}{r+1} x^r = \frac{\left(\frac{1+\sqrt{1-4x}}{2}\right)^{1-\alpha}}{1-4x} \left\{\alpha - 1 + \frac{1}{\sqrt{1-4x}}\right\} \tag{8.14}$$

$$= \alpha\, {}_2F_1\left[\begin{matrix} \frac{\alpha+1}{2}, \frac{\alpha+2}{2} \\ \alpha \end{matrix} \middle| 4x\right],$$

$$f(\alpha, -x) = \sum_{r=0}^{\infty} \binom{r+1}{r} \binom{2r+\alpha}{r+1} (-x)^r = \frac{\left(\frac{1+\sqrt{1+4x}}{2}\right)^{1-\alpha}}{1+4x} \left\{\alpha - 1 + \frac{1}{\sqrt{1+4x}}\right\} \tag{8.15}$$

and adding (8.14) and (8.15) we have

$$2\sum_{r=0}^{\infty} \binom{2r+1}{2r} \binom{4r+\alpha}{2r+1} x^{2r} = \alpha\, {}_4F_3\left[\begin{matrix} \frac{\alpha+1}{4}, \frac{\alpha+2}{4}, \frac{\alpha+3}{4}, \frac{\alpha+4}{4} \\ \frac{1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} \end{matrix} \middle| 16x^2\right] \tag{8.16}$$

$$= \frac{\left(\frac{1+\sqrt{1-4x}}{2}\right)^{1-\alpha}}{1-4x} \left\{\alpha - 1 + \frac{1}{\sqrt{1-4x}}\right\} + \frac{\left(\frac{1+\sqrt{1+4x}}{2}\right)^{1-\alpha}}{1+4x} \left\{\alpha - 1 + \frac{1}{\sqrt{1+4x}}\right\}.$$

From (8.14)

$$f(1, x) = \sum_{r=0}^{\infty} \binom{r+1}{r} \binom{2r+1}{r+1} x^r = \frac{1}{(1-4x)^{3/2}} = \sum_{r=0}^{\infty} \binom{r+1/2}{r} (4x)^r$$

and collecting coefficients of x^r we have the result

$$\binom{r+1}{r}\binom{2r+1}{r+1} = \binom{r+1/2}{r} 2^{2r} \tag{8.17}$$

which we shall generalize shortly. Integrating (8.16) we have

$$\sum_{r=0}^{\infty} \binom{2r+1}{2r}\binom{4r+\alpha}{2r+1} \frac{x^{2r+1}}{2r+1} = \alpha x\, {}_4F_3\left[\begin{matrix} \frac{\alpha+1}{4}, \frac{\alpha+2}{4}, \frac{\alpha+3}{4}, \frac{\alpha+4}{4} \\ \frac{3}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} \end{matrix} \middle| 16x^2 \right]$$

dividing by x and integrating, and performing this operation again we have

$$\sum_{r=0}^{\infty} \binom{2r+1}{2r}\binom{4r+\alpha}{2r+1} \frac{x^{2r+1}}{(2r+1)^3} = \alpha x\, {}_6F_5\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{\alpha+1}{4}, \frac{\alpha+2}{4}, \frac{\alpha+3}{4}, \frac{\alpha+4}{4} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} \end{matrix} \middle| 16x^2 \right].$$

Differentiating (8.16) will produce other identities, as well as putting a higher value of R. Returning briefly to the relation (8.17) we will demonstrate that this is a special case of a more general relation. From (8.14) with general R and $\alpha = 1$ we have

$$\sum_{r=0}^{\infty} \binom{r+R-1}{r}\binom{2r+R-1}{r+R-1} x^r = {}_2F_1\left[\begin{matrix} \frac{R}{2}, \frac{R+1}{2} \\ 1 \end{matrix} \middle| 4x \right]$$

$$= \frac{\sum_{j=0}^{\left[\frac{R-1}{2}\right]} \lambda_{R,j} x^j}{(1-4x)^{R-1/2}} \quad \text{and} \tag{8.18}$$

$$\sum_{r=0}^{\infty} \binom{r+R-3/2}{r} (4x)^r = \frac{1}{(1-4x)^{R-1/2}}. \tag{8.19}$$

From (8.18) and (8.19) adjust and then equate the coefficients of x^j gives

$$\binom{r+R-1}{r}\binom{2r+R-1}{r+R-1} = \sum_{j=0}^{\left[\frac{R-1}{2}\right]} \binom{r+R-3/2-j}{r-j} 2^{2(r-j)} \lambda_{R,j} \tag{8.20}$$

$R\backslash j$	0	1	2	3	4	5	6	7
1	1							
2	1							
3	1	2						
4	1	6						
5	1	12	6					
6	1	20	30					
7	1	30	90	20				
8	1	42	210	140				
9	1	56	420	560	70			
10	1	72	756	1680	630			
11	1	90	1260	4200	3150	252		
12	1	110	1980	9240	11550	2772		
13	1	132	2970	18480	34650	16632	924	
14	1	156	4290	34320	90090	72072	12012	
15	1	182	6006	60060	210210	252252	84084	3432
16	1	210	8190	100100	450450	756756	420420	51480

Table 8.3: The constant lambda of (8.18).

where the constants $\lambda_{R,j}$ are the coefficients of x in the expansion of

$$(1-4x)^{R-1/2}\ {}_2F_1\left[\begin{array}{c}\frac{R}{2},\frac{R+1}{2}\\ 1\end{array}\middle|\, 4x\right]=\sum_{j=0}^{[\frac{R-1}{2}]}\lambda_{R,j}x^j,$$

some values of $\lambda_{R,j}$ are given in the table 8.3.

From (8.20) for $R=1$, we obtain the common result $\binom{2r}{r}=\binom{r-1/2}{r}2^{2r}$, for $R=2$ we obtain the result (8.17) and for $R=6$,

$$\binom{r+5}{r}\binom{2r+5}{r+5}=\binom{r+9/2}{r}2^{2r}+20\binom{r+7/2}{r-1}2^{2r-2}+30\binom{r+5/2}{r-2}2^{2r-4}.$$

8.3 Multiple dominant zeros.

In this section we shall generalize the identity (8.5). We shall prove that a generated infinite series may be represented in closed form that depend on k dominant zeros of an associated polynomial characteristic function. Consider the delay system

$$\left.\begin{array}{l} \sum_{j=0}^{k} \binom{k}{j} (-b)^{k-j} f_{n+j} + c f_{n-ak} = w_n;\ n \geq ak \\ \sum_{j=0}^{k} \binom{k}{j} (-b)^{k-j} f_{n+j} = w_n;\ n < ak \end{array}\right\} \tag{8.21}$$

with $c + b^k = 0$ and all initial conditions at rest, except for $f_{k-1} = 1$. As in the first section, we set $w_n = 0$ and take the Z transform of (8.21) such that

$$F(z) = \frac{z}{(z-b)^k - (bz^{-a})^k} = \frac{z^{ak+1}}{(z^a(z-b))^k - b^k} \tag{8.22}$$

and upon inversion

$$f_n = \sum_{r=0}^{\left[\frac{n-k+1}{(a+1)k}\right]} \binom{n-akr}{kr+k-1} b^{n-akr-k+1}. \tag{8.23}$$

Also from (8.22) we may write

$$\begin{aligned} f_n &= \frac{1}{2\pi i} \oint_C z^n \left(\frac{F(z)}{z}\right) dz \\ &= \sum_{j=0}^{k-1} \sum_{\nu=0}^{a} z^n \mathrm{Res}_{j,\nu} \left(\frac{F(z)}{z}\right), \end{aligned}$$

C is a smooth Jordan curve enclosing the singularities of (8.22) and $\mathrm{Res}_{j,\nu}$ is the residue at the poles. The characteristic function

$$g_k(z) = (z^a(z-b))^k - b^k \tag{8.24}$$

with restriction

$$\left|\left\{\frac{(a+1)^{a+1}}{(ab)^{a}}\right\}^{k}\right| < 1 \tag{8.25}$$

has exactly $(a+1)\,k$ distinct zeros $\xi_{j,\nu}$ for $\nu = 0,1,2,...,k-1$ and $\nu = 0,1,2,...,a$. This statement will be clarified in appendix B. From (8.22), then

$$f_n = \sum_{j=0}^{k-1}\sum_{\nu=0}^{a} Q\left(\xi_{j,\nu}\right)\xi_{j,\nu}^{n} \text{ where} \tag{8.26}$$

$$\begin{aligned} Q\left(\xi_{j,\nu}\right) &= \lim_{z\to\xi_{j,\nu}}\left[\left(z-\xi_{j,\nu}\right)\frac{F(z)}{z}\right] \\ &= \frac{\xi_{j,\nu}}{k\left(\xi_{j,\nu}-b\right)^{k-1}\left((a+1)\,\xi_{j,\nu}-ab\right)} \end{aligned} \tag{8.27}$$

and from (8.23), (8.26) and (8.27)

$$\sum_{r=0}^{\left[\frac{n-k+1}{(a+1)k}\right]}\binom{n-akr}{kr+k-1} b^{n-akr-k+1} = \sum_{j=0}^{k-1}\sum_{\nu=0}^{a}\frac{\xi_{j,\nu}^{n+1}}{k\left(\xi_{j,\nu}-b\right)^{k-1}\left((a+1)\,\xi_{j,\nu}-ab\right)}. \tag{8.28}$$

If we let $\xi_{j,0}, j = 0,1,2,...,k-1$ be the k dominant zeros of (8.24) then we have the following theorem.

8.3.1 The k theorem.

Theorem 37 *Let*

$$T_r\left(k,n,a,b\right) = \binom{n-akr}{kr+k-1} b^{n-akr-k+1} \text{ and} \tag{8.29}$$

$$S\left(k,n,a,b\right) = \sum_{r=0}^{\infty} T_r\left(k,n,a,b\right) \tag{8.30}$$

which is convergent for all values of k, n, a and b in the region of convergence (8.25), then

$$S\left(k,n,a,b\right) = \sum_{j=0}^{k-1}\frac{\xi_{j,0}^{n+1}}{k\left(\xi_{j,0}-b\right)^{k-1}\left((a+1)\,\xi_{j,0}-ab\right)}. \tag{8.31}$$

The series (8.28) with (8.31) gives us

$$\sum_{r=0}^{\left[\frac{n-k+1}{(a+1)k}\right]} T_r(k,n,a,b) + \sum_{r=\left[\frac{n+ak+1}{(a+1)k}\right]}^{\infty} T_r(k,n,a,b) = \sum_{j=0}^{k-1} \frac{\xi_{j,0}^{n+1}}{k(\xi_{j,0}-b)^{k-1}((a+1)\xi_{j,0}-ab)}$$

and hence

$$\sum_{r=\left[\frac{n+ak+1}{(a+1)k}\right]}^{\infty} T_r(k,n,a,b) = -\sum_{j=0}^{k-1}\sum_{\nu=1}^{a} \frac{\xi_{j,\nu}^{n+1}}{k(\xi_{j,\nu}-b)^{k-1}((a+1)\xi_{j,\nu}-ab)}$$

Proof of theorem 37. The characteristic function (8.24) may be expressed as the product of factors such that

$$g_k(z) = \prod_{j=0}^{k-1}\left(z^{a+1} - bz^a - be^{2\pi ij/k}\right) = \prod_{j=0}^{k-1} q_j(z). \tag{8.32}$$

For each of the j factors in (8.32) we may write

$$F_j(z) = \frac{z^{a+1}}{z^{a+1} - bz^a - be^{2\pi ij/k}} \tag{8.33}$$

for $j = 0,1,2,...,k-1$. The characteristic function $q_j(z)$ in (8.32) has exactly $a+1$ distinct zeros for each j, of which $\alpha_{j,0}$ shall indicate the dominant zero, the one with the largest modulus, which may be complex. All the singularities in (8.33) are simple and therefore for each j, $F_j(z)$ has simple poles. Utilizing the result (6.10), from (8.33) and evaluating its residue, we may write

$$\sum_{r=0}^{\infty} e^{2\pi ijr/k} \binom{n-ar}{r} b^{n-ar} = \frac{\alpha_{j,0}^{n+1}}{(a+1)\alpha_{j,0} - ab} \tag{8.34}$$

for each $j = 0,1,2,...,k-1$, and note that (8.34) may in fact be a complex number. Summing (8.34) over j gives

$$\sum_{j=0}^{k-1}\sum_{r=0}^{\infty} e^{2\pi ijr/k} \binom{n-ar}{r} b^{n-ar} = \sum_{j=0}^{k-1} \frac{\alpha_{j,0}^{n+1}}{(a+1)\alpha_{j,0} - ab}$$

Rescaling the left hand side by $r = (r^* + 1)\,k$, (and then replacing r^* by r) results in , after changing the order of summation

$$\sum_{r=-1}^{\infty}\sum_{j=0}^{k-1} e^{2\pi i j(r+1)} \begin{pmatrix} n - ak\,(r+1) \\ k\,(r+1) \end{pmatrix} b^{n-ak(r+1)} = \sum_{j=0}^{k-1} \frac{\alpha_{j,0}^{n+1}}{(a+1)\,\alpha_{j,0} - ab},$$

$$\sum_{r=0}^{\infty} \begin{pmatrix} n - ak\,(r+1) \\ k\,(r+1) \end{pmatrix} b^{n-ak(r+1)} = \sum_{j=0}^{k-1} \frac{\alpha_{j,0}^{n+1}}{k\left((a+1)\,\alpha_{j,0} - ab\right)} - b^{n}. \tag{8.35}$$

Now make the substitution $n - ak = m$ in (8.35) giving upon simplification

$$\sum_{r=0}^{\infty} \begin{pmatrix} m - akr \\ kr + k \end{pmatrix} b^{-akr} = \sum_{j=0}^{k-1} \frac{b^{-m}\alpha_{j,0}^{m+ak+1}}{k\left((a+1)\,\alpha_{j,0} - ab\right)} - b^{ak}. \tag{8.36}$$

Newton's forward difference formula of a function $h\,(x_j) = h_j$ at x_j is defined as

$$\Delta^k h_j = \Delta^{k-1} h_{j+1} - \Delta^{k-1} h_j, k = 1, 2, 3, ...$$

and taking the first difference of (8.36) with respect to m results in, from the left hand side

$$\sum_{r=0}^{\infty} \begin{pmatrix} m + 1 - akr \\ kr + k \end{pmatrix} b^{-akr} - \sum_{r=0}^{\infty} \begin{pmatrix} m - akr \\ kr + k \end{pmatrix} b^{-akr} = \sum_{r=0}^{\infty} \begin{pmatrix} m - akr \\ kr + k - 1 \end{pmatrix} b^{-akr}. \tag{8.37}$$

Similarly from the right hand side of (8.36)

$$\sum_{j=0}^{k-1} \frac{\left(b^{-(m+1)}\alpha_{j,0}^{m+ak+2} - b^{-m}\alpha_{j,0}^{m+ak+1}\right)}{kA_{\alpha}} = \sum_{j=0}^{k-1} \frac{b^{-m}\alpha_{j,0}^{m+ak+1}}{kA_{\alpha}} \left(\frac{\alpha_{j,0} - b}{b}\right) \tag{8.38}$$

where $A_{\alpha} = (a+1)\,\alpha_{j,0} - ab$. From the characteristic function $q_j\,(z)$ in (8.32) $\alpha_{j,0}^{a}\,(\alpha_{j,0} - b) - be^{2\pi i j/k} = 0$ we may write (8.38) as

$$\sum_{j=0}^{k-1} \frac{b^{-m+k-1}\alpha_{j,0}^{m+1}}{kA_{\alpha}\,(\alpha_{j,0} - b)^{k-1}}$$

and combining with (8.37) gives, after simplification

$$\sum_{r=0}^{\infty}\left(\begin{array}{c} m-akr \\ kr+k-1 \end{array}\right) b^{m-akr-k+1}=\sum_{j=0}^{k-1}\frac{\alpha_{j,0}^{m+1}}{k\left(\alpha_{j,0}-b\right)^{k-1}\left(\left(a+1\right)\alpha_{j,0}-ab\right)}. \qquad (8.39)$$

Since the dominant zeros $\alpha_{j,0}, j=0,1,2,...,k-1$ of $q_j(z)$ are the same as the dominant zeros $\xi_{j,0}$ of $g_k(z)$ in (8.24), (this statement will be proved in the appendix B), then upon renaming m as n in (8.39) the theorem 37 is proved since (8.39) and (8.31) are identical. Note that putting $k=1$ in (8.31) yields the result (8.5) for $R=1$.

8.3.2 Numerical results and special cases.

In the following numerical results, the dominant zeros $\xi_{j,0}$ are evaluated from $g_k(z)$ in (8.24). It may also be noted that for $k \geq 3$ the dominant zeros occur in complex conjugate pairs. The numerical results are given to four significant digits.

k	n	a	b	$\xi_{j,0}$	identity (8.31)
2	3	2	10	$\xi_{0,0}=$ 9.8979 $\xi_{1,0}=$ 10.0981	299.9899
3	3	1	-10	$\xi_{0,0}=$ -8.8730 $\xi_{1,0}=$ -10.5329+.7826i $\xi_{2,0}=$ -10.5329-.7826i	-30.0005
3	3	2	10	$\xi_{0,0}=$ 10.0981 $\xi_{1,0}=$ 9.9511+.0883i $\xi_{2,0}=$ 9.9511-.0883i	29.9998.

The degenerate case, $a=0$, of (8.31) yields the result

$$\sum_{r=0}^{\left[\frac{m-k+1}{k}\right]}\left(\begin{array}{c} m \\ kr+k-1 \end{array}\right)=\sum_{j=0}^{k-1}\frac{\left(1+e^{2\pi ij/k}\right)^{m}}{ke^{2\pi ij(k-1)/k}}=\frac{2^m}{k}\sum_{j=0}^{k-1}e^{\pi ij(m+2)/k}\cos^m\left(\frac{\pi j}{k}\right), \qquad (8.40)$$

and for $k=4$, we have

$$\sum_{r=0}^{\left[\frac{m-3}{4}\right]} \binom{m}{4r+3} = \frac{1}{4}\left(2^m - 2^{\frac{m}{2}+1}\sin\frac{m\pi}{4}\right).$$

Using the WZ pairs method of Wilf and Zeilberger a rational function proof certificate, $V_k(m,r)$, for $k=1$ and 2 of (8.40) is respectively

$$V_1(m,r) = \frac{r}{2(r-1-m)} \text{ and } V_2(m,r) = \frac{(r-1)(2r-1)}{m(2r-2-m)}.$$

8.3.3 The Hypergeometric connection.

Consider the term $T_r(k,n,a,b)$ of (8.29) with $T_0(k,n,a,b) = \binom{n}{k-1} b^{n-k+1}$, the ratio of consecutive terms

$$\frac{T_{r+1}(k,n,a,b)}{T_r(k,n,a,b)} = \frac{\prod_{j=0}^{(a+1)k-1}\left(r+\frac{j+k-n-1}{(a+1)k}\right)}{(r+1)\prod_{j=0}^{ak-1}\left(r+\frac{j-n}{ak}\right)\prod_{j=0}^{k-2}\left(r+\frac{2k-j-1}{k}\right)} s(a,b,k)$$

is a rational function in r and therefore the series $S(k,n,a,b)$ of (8.30) may be expressed as a generalized hypergeometric function

$$T_0\ {}_{(a+1)k}F_{(a+1)k-1}\left[\begin{matrix} \frac{k-n-1}{(a+1)k}, \frac{k-n}{(a+1)k}, \ldots, \frac{k-n-1+(a+1)k-1}{(a+1)k} \\ -\frac{n}{ak}, \frac{1-n}{ak}, \ldots, \frac{ak-1-n}{ak}, \frac{2k-1}{k}, \frac{2k-2}{k}, \ldots, \frac{2k-1-(k-2)}{k} \end{matrix} \middle| s(a,b,k)\right]$$

where

$$s(a,b,k) = \left(-\frac{(a+1)^{a+1}}{(ab)^a}\right)^k.$$

A simple example shows that, from (8.30) and (8.31) for $k=2, a=1, b=x^{-1}$ and $n=-\alpha\in\Re$, we have two distinct dominant zeros of (8.24), $\xi_{0,0}=\frac{1}{2x}\left(1+\sqrt{1+4x}\right)$, $\xi_{1,0}=\frac{1}{2x}\left(1+\sqrt{1-4x}\right)$

and therefore

$$\sum_{r=0}^{\infty}\binom{4r+\alpha}{2r+1}x^{2r} = \alpha\,{}_4F_3\left[\begin{matrix}\frac{\alpha+1}{4},\frac{\alpha+2}{4},\frac{\alpha+3}{4},\frac{\alpha+4}{4}\\ \frac{\alpha}{2},\frac{\alpha+1}{2},\frac{3}{2}\end{matrix}\middle|\,16x^2\right]$$
$$= \sum_{j=0}^{1}\frac{\xi_{j,0}^{1-\alpha}}{2\left(x\xi_{j,0}+1\right)\left(2x\xi_{j,0}-1\right)}.$$

Specifically with $\alpha = 1$,

$$\sum_{r=0}^{\infty}\binom{4r+1}{2r+1}x^{2r} = {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{3}{4},1,\frac{5}{4}\\ \frac{1}{2},1,\frac{3}{2}\end{matrix}\middle|\,16x^2\right]$$

$$= \frac{\sqrt{1-4x}+\sqrt{1+4x}-2}{\left(\sqrt{1+4x}-(1+4x)\right)\left(\sqrt{1-4x}-(1-4x)\right)}$$

$$= \frac{1}{4x}\left(\frac{1}{\sqrt{1-4x}}-\frac{1}{\sqrt{1+4x}}\right) = \sum_{r=0}^{\infty}\binom{4r+1}{2r}x^{2r}$$

which confirms the result obtained in chapter six. Again the identities may be differentiated and integrated to produce more results.

8.4 Non-zero forcing terms.

If we consider the system (8.21) with all initial conditions at rest and with a forcing term of the form $w_n = \binom{n}{m-1} b^{n-m+1}$ for m integer and follow the procedure of the previous section we obtain the identity

$$\sum_{r=0}^{\infty}\binom{r+R-1}{r}\binom{n-akr}{kr+Rk+m-1}b^{n-akr-kR-m+1} = \sum_{\nu=0}^{m-1}P_{m,\nu}\left(b\right)\binom{n}{m-1-\nu}b^{n-m+1+\nu}$$

$$+\sum_{\mu=0}^{R-1}\sum_{j=0}^{k-1}Q_{R,\mu}\left(\xi_{j,0}\right)\binom{n}{R-1-\mu}\xi_{j,0}^{n-R+1+\mu} \qquad (8.41)$$

in the region of convergence (8.25) where

$$\nu! P_{m,\nu}(b) = \lim_{z\to b}\left[\frac{d^{\nu}}{dz^{\nu}}\left\{(z-b)^m \frac{F(z)}{z}\right\}\right],$$

$$\mu! Q_{R,\mu}(\xi_{j,0}) = \lim_{z\to \xi_{j,0}}\left[\frac{d^{\mu}}{dz^{\mu}}\left\{(z-\xi_{j,0})^R \frac{F(z)}{z}\right\}\right]$$

and

$$F(z) = \frac{z^{akR+1}}{(z-b)^m\left((z^a(z-b))^k - b^k\right)^R}.$$

For $R=1, k=1, n=-\alpha\in\Re$ and $m=2$, from (8.41) we have

$$\sum_{r=0}^{\infty}(-1)^r\binom{(a+1)r+\alpha+1}{r+2}\left(\frac{1}{b^a}\right)^r = (\alpha-a)\,b^a - b^{2a} + \frac{b^{2+\alpha}\xi_{0,0}^{1-\alpha}}{(\xi_{0,0}-b)^2\left((a+1)\,\xi_{0,0}-ab\right)}$$

where $\xi_{0,0}$ is the dominant zero which satisfies (8.24), and specifically for $(a,\alpha)=(1,\alpha)$, $2\xi_{0,0} = b+\sqrt{b^2+4b}$, so that

$$\sum_{r=0}^{\infty}(-1)^r\binom{2r+\alpha+1}{r+2}\left(\frac{1}{b}\right)^r = (\alpha-1)\,b - b^2 + \frac{b\,(2b)^{\alpha}\left(b+\sqrt{b^2+4b}\right)^{1-\alpha}}{(b+2)\sqrt{b^2+4b}-(b^2+4b)}.$$

Many other identities of this form may be attained by various manipulations, one such result is

$$\sum_{r=0}^{\infty}\binom{r+1}{r}\binom{4r+4}{2r+2}x^{-2r} = \frac{x}{2}\left(\left(\frac{x}{x-4}\right)^{\frac{3}{2}} - \left(\frac{x}{x+4}\right)^{\frac{3}{2}}\right).$$

8.5 Appendix A: A recurrence for Q's.

In this appendix we shall demonstrate how we arrived at the recurrence

$$\begin{aligned} abRAQ_{R+1,\mu}(\xi_0) &= ab(R-\mu)\xi_0 Q_{R,\mu}(\xi_0) + (\mu-1)AQ_{R,\mu-1}(\xi_0) \\ &+ (A+ab)\xi_0 \frac{d}{d\xi_0} Q_{R,\mu-1}(\xi_0) \end{aligned} \tag{8.42}$$

that was used in the proof of the conjecture in section 8.2.2.

From (8.6)

$$\mu! Q_{R,\mu}(\xi_0) = \lim_{z\to\xi_0}\left[\frac{d^\mu}{dz^\mu}\left\{(z-\xi_0)^R \frac{z^{aR}}{g^R(z)}\right\}\right] \tag{8.43}$$

where $g(z)$ is defined by (8.2) and ξ_0 is the dominant zero of $g(z)$. From (8.43) we can differentiate with respect to b such that

$$\mu!\frac{d}{db} Q_{R,\mu}(\xi_0) = \lim_{z\to\xi_0}\left[\frac{d^\mu}{dz^\mu}\left\{\frac{\xi_0^2 (z-\xi_0)^{R-1} z^{aR}}{Ag^R(z)} + \frac{(z-\xi_0)^R z^{aR}}{g^R(z)} - \frac{(z-\xi_0)^R z^{a(R+1)+1}}{g^{R+1}(z)}\right\}\right] \tag{8.44}$$

where $A = (a+1)\xi_0 - ab$. Simplifying (8.44) by adjusting the third term, we obtain

$$\begin{aligned} \mu!\frac{d}{db} Q_{R,\mu}(\xi_0) &= \frac{-R}{b}\mu! Q_{R,\mu}(\xi_0) + \frac{R}{b}\mu! Q_{R+1,\mu}(\xi_0) \\ &-\frac{R\xi_0}{bA}\lim_{z\to\xi_0}\left[\frac{d^\mu}{dz^\mu}\left\{\frac{(z-\xi_0)^{R-1} z^{a(R+1)}}{g^{R+1}(z)}\left(\xi_0 g(z) z^{-a} - A(z-\xi_0)\right)\right\}\right]. \end{aligned} \tag{8.45}$$

Let $h(z) = \xi_0 g(z) z^{-a} - A(z-\xi_0)$, a Taylor series expansion about the dominant zero ξ_0, gives us

$$h(z) = (z-\xi_0)^2 \sum_{j=2}^{\infty} (-1)^j \binom{a+j-1}{j} (b-\xi_0)\xi_0^{1-j} (z-\xi_0)^{j-2}$$

and substituting into (8.45) we have

$$\begin{aligned} \mu!\frac{d}{db} Q_{R,\mu}(\xi_0) &= \frac{-R}{b}\mu! Q_{R,\mu}(\xi_0) + \frac{R}{b}\mu! Q_{R+1,\mu}(\xi_0) \\ &-\frac{R\xi_0}{bA}\lim_{z\to\xi_0}\left[\frac{d^\mu}{dz^\mu}\left\{\frac{(z-\xi_0)^{R+1} z^{a(R+1)}}{g^{R+1}(z)} B_j\right\}\right] \end{aligned} \tag{8.46}$$

where

$$B_j = \frac{h(z)}{(z-\xi_0)^2} = \sum_{j=2}^{\infty} (-1)^j \begin{pmatrix} a+j-1 \\ j \end{pmatrix} (b-\xi_0)\,\xi_0^{1-j}\,(z-\xi_0)^{j-2}. \tag{8.47}$$

Expanding (8.46) by the Liebniz differentiation rule

$$\mu!\frac{d}{db}Q_{R,\mu}(\xi_0) = \frac{-R}{b}\mu!Q_{R,\mu}(\xi_0) + \frac{R}{b}\mu!Q_{R+1,\mu}(\xi_0)$$

$$-\frac{R\xi_0}{bA}\sum_{k=0}^{\mu}\begin{pmatrix} \mu \\ k \end{pmatrix}(\mu-k)!Q_{R+1,\mu-k}(\xi_0)\lim_{z\to\xi_0} B_j^{(k)} \tag{8.48}$$

and after evaluating $\lim_{z\to\xi_0} B_j^{(k)}$ from (8.47) and substituting into (8.48) we obtain

$$\frac{b}{R}\frac{d}{db}Q_{R,\mu}(\xi_0) = Q_{R+1,\mu}(\xi_0) - Q_{R,\mu}(\xi_0) - \frac{(b-\xi_0)}{A}\sum_{k=0}^{\mu}(-1)^k\begin{pmatrix} a+k+1 \\ k+2 \end{pmatrix}\xi_0^{-k}Q_{R+1,\mu-k}(\xi_0). \tag{8.49}$$

Now (8.49) and (8.8) suggest that the $Q(\xi_0)$ terms may be related by an expression of the form

$$Q_{R+1,\mu+1}(\xi_0) = c_1\frac{d}{db}Q_{R,\mu}(\xi_0) + c_2\mu Q_{R,\mu}(\xi_0) + c_3(R-\mu-1)Q_{R,\mu+1}(\xi_0) \tag{8.50}$$

for $\mu = 0, 1, 2, ..., R-1$. The constants $c_1, c_2,$ and c_3 can be evaluated by forming three simultaneous equations and using the $Q(\xi_0)$ values, given in table 8.1, we may write (8.50) as

$$Q_{R+1,\mu+1}(\xi_0) = \frac{a+1}{aR}\frac{d}{db}Q_{R,\mu}(\xi_0) + \frac{\mu}{abR}Q_{R,\mu}(\xi_0) + \frac{\xi_0(R-\mu-1)}{AR}Q_{R,\mu+1}(\xi_0)$$

which upon rearrangement gives

$$\frac{d}{db}Q_{R,\mu}(\xi_0) = \frac{\xi_0}{bA(A+ab)}\left\{\begin{array}{c} abARQ_{R+1,\mu+1}(\xi_0) - \mu AQ_{R,\mu}(\xi_0) \\ -ab\xi_0(R-\mu-1)Q_{R,\mu+1}(\xi_0) \end{array}\right\}. \tag{8.51}$$

Now, since

$$\frac{d}{d\xi_0}Q_{R,\mu}(\xi_0) = \frac{Ab}{\xi_0^2}\frac{d}{db}Q_{R,\mu}(\xi_0)$$

substituting into (8.51) and rearranging we obtain

$$abRAQ_{R+1,\mu+1}(\xi_0) = ab(R-\mu-1)\,\xi_0 Q_{R,\mu+1}(\xi_0) + \mu A Q_{R,\mu}(\xi_0)$$

$$+ (A+ab)\,\xi_0 \frac{d}{d\xi_0} Q_{R,\mu}(\xi_0)$$

which is the required relation (8.42) after replacing μ with $\mu - 1$.

8.6 Appendix B: Zeros.

Some properties of the zeros of the characteristic functions (8.2);

$$g(z) = z^{a+1} - bz^a - b$$

and (8.24);

$$g_k(z) = (z^a(z-b))^k - b^k$$

will be discussed in this appendix. Let b be a real constant, a and $k \in N$ and z is a complex variable.

Theorem 38 *(i). The function (8.2) has at least one and at most two real zeros and a dominant zero, the one with the greatest modulus,* ξ_0 *such that* $\xi_0 > b$ *for* $b > 0$ *and* $|\xi_0| > \left|\frac{ab}{a+1}\right|$ *for* $b < 0$ *and the restriction (8.3); namely* $\left|\frac{(a+1)^{a+1}}{(ab)^a}\right| < 1$.

(ii) The function (8.24) has at least one and at most four real zeros.

Proof: (i). The characteristic function (8.2) with restriction (8.3) has $a+1$ distinct zeros, for the derivative of $g(z)$ cannot vanish coincidentally with $g(z)$. The fact that a related function to (8.2) has distinct zeros appears to have been reported first by Bailey [5]. Let $G(z) = z^a(z-b)$, hence $G(z) = b$, and the turning point of $G(z)$, away from the origin occurs at $z = \frac{ab}{a+1}$. Now consider the graphs of $G(z)$. For $b > 0$,

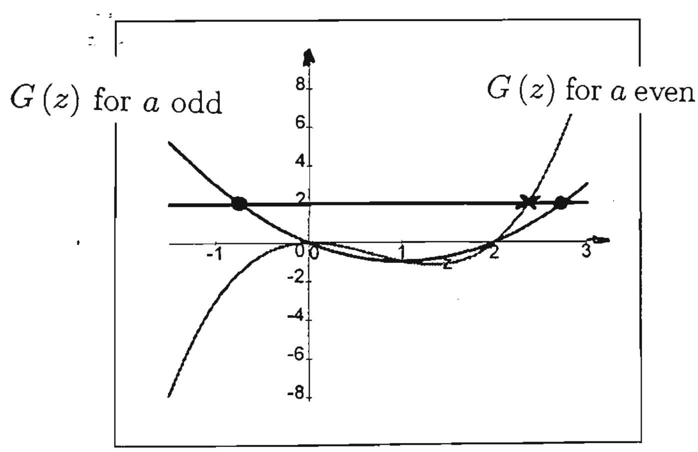

Figure 8.1: The graph of $G(z)$ for a odd or even and $b > 0$.

For $b < 0$,

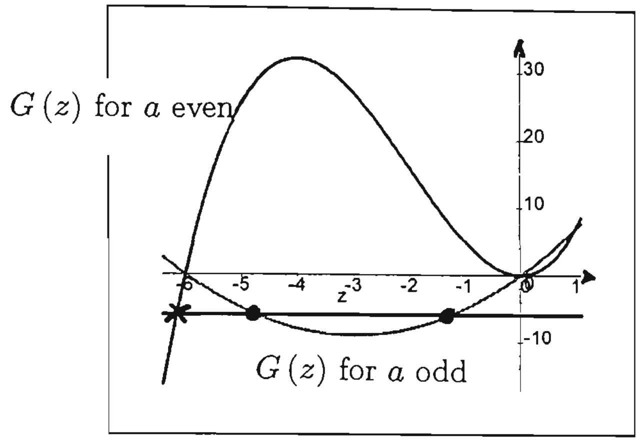

Figure 8.2: The graph of $G(z)$ for a odd or even and $b < 0$.

The two graphs of $G(z)$ indicate therefore that (8.2) has at least one and at most two real zeros. In both cases of $b > 0$ and $b < 0$ it will be shown in the next theorem that the dominant zero, ξ_0, the one with the greatest modulus, of (8.2) is always real, such that $\xi_0 > b$ for $b > 0$ and $|\xi_0| > \left|\frac{ab}{a+1}\right|$ for $b < 0$ and the restriction (8.3) with a real.

(ii). In a similar fashion it may be seen that $g_k(z)$ has at least one and at most four real zeros. Let $G_k(z) = (z^a(z-b))^k$, hence $G_k(z) = b^k$ and the turning point of $G_k(z)$ away from the origin occurs at $z = \frac{ab}{a+1}$. Now consider the graph of $G_k(z)$ for $b > 0$ (the case of $b < 0$ follows in a similar fashion).

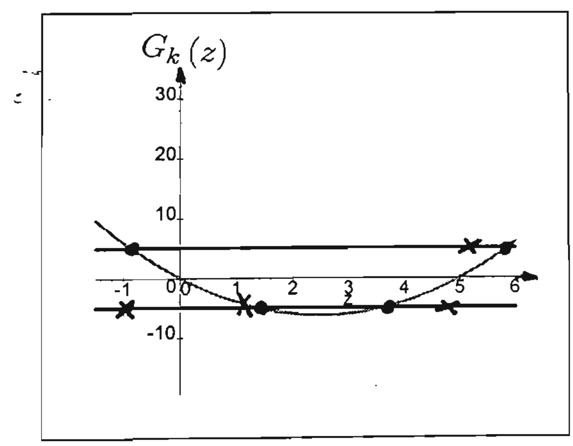

Figure 8.3: The graph of $G_k(z)$ for $b > 0$.

Theorem 39 *The characteristic function (8.32)*

$$z^{a+1} - bz^a - be^{2\pi ij/k} = q_j(z)$$

has a zeros on the region $C : |z| \leq \left|\frac{ab}{a+1}\right|$ *for each* $j = 0, 1, 2, ..., k-1$ *with restriction (8.3).*

The study of the zeros of (8.32), (8.24) and (8.2) is important in the area of queuing theory, and severeal papers have been devoted to this study, see for example, Chaudhry, Harris and Marchal [25] and Zhao [97]. Their studies have concentrated, amongst other things, on robustness of methods for locating zeros inside a unit circle. In this thesis the location of dominant zeros of (8.2), (8.24) and (8.32) is of prime importance.

Proof: Let $A(z) = -bz^a$. Then $A(z)$ has a zeros in the region C and $|A(z)| \leq b\left(\frac{ab}{a+1}\right)^a$. Now

$$\begin{aligned} |q_j(z) - A(z)| &= \left|z^{a+1} - be^{2\pi ij/k}\right| \\ &\leq \left(\frac{ab}{a+1}\right)^{a+1} + b \\ &= b\left(1 + b^a\left(\frac{a}{a+1}\right)^{a+1}\right). \end{aligned}$$

By Rouche's theorem, see Tagaki [85], it is required that $|q_j(z) - A(z)| \leq |A(z)|$ hence

$$\left(1 + \left(\frac{a}{a+1}\right)\left(\frac{ab}{a+1}\right)^a\right) \leq \left(\frac{ab}{a+1}\right)^a$$

and

$$1 \leq \left(\frac{1}{a+1}\right)\left(\frac{ab}{a+1}\right)^a$$

which is satisfied since (8.3) applies, therefore the theorem is proved. Also from (8.32), letting $j = 0$ gives the characteristic function (8.2). Theorem 38 now follows since at least one zero of (8.2) must be real, it is evident that $\xi_0 > b$ for $b > 0$ and $|\xi_0| > \left|\frac{ab}{a+1}\right|$ for $b < 0$. Note that the restriction (8.3) is imperitive for theorem 39 to apply. If for example $a = 1, b = 1/2$ and $k = 1$ which indicates that (8.3) is not satisfied, then $q_0(z) = z^2 - z/2 - 1/2$ gives the two zeros as $z = \{-1/2, 1\}$, neither of which are in the region $C : |z| \leq \frac{1}{4}$.

Theorem 40 *The characteristic function $g_k(z)$ has ak zeros in the region $C : |z| \leq \left|\frac{ab}{a+1}\right|$ with restriction (8.3).*

Proof: Let $B(z) = (q_j(z))^k = \left(z^{a+1} - bz^a - be^{2\pi ij/k}\right)^k$. Utilizing theorem 39, $B(z)$ has therefore ak zeros in the region C and therefore k zeros have modulus bigger than $\left|\frac{ab}{a+1}\right|$. In the region C,

$$\begin{aligned} |B(z)| &= \left|\left(z^{a+1} - bz^a - be^{2\pi ij/k}\right)^k\right| \\ &\leq \left\{\left(\frac{ab}{a+1}\right)^{a+1} + b\left(\frac{ab}{a+1}\right)^a + b\right\}^k \\ &= b^k\left\{\left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right) + 1\right\}^k. \end{aligned} \tag{8.52}$$

Now

$$|g_k(z) - B(z)| = \left|(G(z))^k - b^k - \left(z^{a+1} - bz^a - be^{2\pi ij/k}\right)^k\right|$$

for every $j = 0, 1, 2, ..., k-1$, and $G(z) = z^a(z-b)$. Furthermore, let $c_j = be^{2\pi ij/k}$, such that

$$\begin{aligned} \left|(G(z))^k - b^k - (G(z) - c_j)^k\right| &= \left|-b^k - \sum_{r=1}^{k}\binom{k}{r}(-1)^r c_j^r G^{k-r}(z)\right| \\ &\leq b^k + \sum_{r=1}^{k}\binom{k}{r} b^r\left[\left(\frac{ab}{a+1}\right)^a\left(\frac{ab}{a+1} + b\right)\right]^{k-r} \\ &= b^k\left[1 + \sum_{r=1}^{k}\binom{k}{r}\left\{\left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right)\right\}^{k-r}\right] \\ &= b^k\left[1 + (1+M)^k - M^k\right] \end{aligned} \tag{8.53}$$

where $M = \left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right) > 0$. By Rouches theorem, it is required that $|g_k(z) - B(z)| \leq |B(z)|$ and upon using (8.52) and (8.53) we have that

$$b^k\left[1 + (1+M)^k - M^k\right] \leq b^k\left[(1+M)^k\right]$$

$$1 \leq \left\{\left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right)\right\}^k$$

which is satisfied by virtue of restriction (8.3). Therefore the characteristic function (8.24) has ak zeros in the region $C : |z| \leq \left|\frac{ab}{a+1}\right|$ and k zeros with modulus bigger than $\left|\frac{ab}{a+1}\right|$. The theorem is proved. Consider as an example, $a = 3, b = 10$ and $k = 6$ such that restriction (8.3) is satisfied and $C : |z| \leq 7.5$. The zeros of $q_j(z)$, are listed below, showing that one dominant zero appears from each of the $q_j(z)$, for $j = 0, 1, 2, 3, 4$ and 5.

$q_0(z)$	10.0100	-0.9696	0.4798-0.8944i	0.4798+0.8944i
$q_1(z)$	10.0051+0.0086i	-0.9157-0.3231i	0.7697-0.6786i	0.1412+0.9933i
$q_2(z)$	9.9951+0.0087i	-0.7589-0.6127i	0.9669-0.3668i	-0.2031+0.9708i
$q_3(z)$	9.9900	1.0372	-0.5136+0.8375i	-0.5136-0.8375i
$q_4(z)$	9.9951-0.0087i	-0.7589+0.6127i	0.9669+0.3668i	-0.2031-0.9708i
$q_5(z)$	10.0051-0.0086i	-0.9157+0.3231i	0.7697+0.6786i	0.1412-0.9933i

The dominant zeros of $q_j(z)$ are listed in the first column and all have modulus bigger than 7.5. These dominant zeros are exactly the same k dominant zeros of (8.24). It appears that the zeros, $\alpha_j(a, b)$ of function (8.32) can be related for $b > 0$ and $b < 0$. It may be shown that the following relationships hold.

(i). For all values of k and a even, $\alpha_j(a, b) = -\alpha_j(a, -b)$,

(ii). For k odd and a odd, $\alpha_j(a, b) \neq -\alpha_j(a, -b)$, and

(iii). For k even and a odd,

$$\alpha_j(a, b) = \begin{cases} -\alpha_{j+\frac{k}{2}}(a, -b); \text{ for } j < \frac{k}{2} \\ -\alpha_0(a, -b); \text{ for } j = \frac{k}{2} \\ -\alpha_{j-\frac{k}{2}}(a, -b); \text{ for } j > \frac{k}{2} \end{cases}$$

where $j = 0, 1, 2, ..., k - 1$.

Conclusion and suggestions for further work.

This thesis deals with the classical quest for closed form expressions for certain classes of infinite series and in particular, with identities for classical hypergeometric series and their generalisations. The methods utilized in this thesis, for 'identity proving' are those of function theoretic methods and computer-assisted techniques of symbolic manipulation algorithms of 'certification' of hypergeometric identities. In this thesis the author also develops new interesting classes of 'binomial' series identities and their non-hypergeometric generalisations.

A number of suggestions are now indicated in which further investigations may be undertaken to expand on some of the ideas presented in this thesis.

(1). It is possible to obtain many other multiple infinite sum identities by considering higher order differential-delay and difference-delay systems.

(2). It is feasible that the approach and methods employed in this thesis may be extended to prove identities of a 'continuous nature'. In this respect it may be possible to generalise the known identity;

$$\int_{t=x}^{\infty} \frac{e^{(t-x)} (t-x)^{\beta t-1}\, dt}{\Gamma(\beta t)} = \begin{cases} \frac{1}{1-\beta}, & x > 0 \\ \frac{\beta}{1-\beta}, & x = 0 \end{cases}.$$

(3). Some of the results of chapter six may also be derived by considering a simple Markovian queue of bulk service variation of the $M/M^{(a)}/1$ system in which service is in fixed batches of size a, irrespective of whether or not the server has to wait for a full batch of size a. Renewal processes therefore provide a rich source of material for the investigation of representation of series in closed form.

(4). If we pose the conjecture: Given that

$$\sum_{n=0}^{\infty} F(t-an)\, H(t-an) \sim X(t,a)$$

then

$$\sum_{n=0}^{\infty} F(t-an) = X(t,a)$$

where $H(x)$ is the Heaviside function,

$$\sum_{n=0}^{\infty} F_n(p) = \sum_{n=0}^{\infty} T_r[F(t-an)H(t-an)],$$

$T_r[F(t-an)H(t-an)]$ represents the transform of $F(t-an)H(t-an)$ and $X(t,a)$ is a function dependant on t and a. The transform may be the appropriate one depending on the model under investigation. It would be a worthwhile research project to investigate:

(a). What general class of functions $F(t-an)$ do we require for the conjecture to be valid?.

(b). What form does $X(t,a)$ take and under what general conditions does the conjecture hold?.

(5). It may be seen that some partial sums of the constants in table 8.3 form known sequences, which should provide a basis for future research.

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