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AN INEQUALITY OF OSTROWSKI'S TYPE FOR CUMULATIVE DISTRIBUTION FUNCTIONS

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Abstract. The main aim of this paper is to establish an Ostrowski type inequality for the cumulative distribution function of a random variable taking values in a finite interval $[a,b]$. An application for a Beta random variable is given.

1 Introduction

In [1], S.S. Dragomir and S. Wang proved the following version of Ostrowski's inequality for differentiable mappings whose derivatives belong to $L_1(a,b)$:

Theorem 1.1. *Let $f:[a,b]\to\mathbf{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbf{R}$ belongs to $L_1(a,b)$. Then we have the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_1 \tag{1.1}$$

for all $x\in[a,b]$.

Note that the classical Ostrowski's integral inequality states that (see e.g. [3, p.468]):

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.2}$$

for all $x\in[a,b]$ provided $f'\in L_\infty(a,b)$.

In the above paper [1], the authors have applied inequality (1.1) to Numerical Integration obtaining estimations for the error bounds of general Riemann's quadrature formulae in terms of $\|f'\|_1$.

Applications of Ostrowski's inequality for the same problems in Numerical Integration have been pointed out by the same authors in [2].

The main aim of the present work is to establish an Ostrowski like inequality for the cumulative distribution function and expectation of a random variable.

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2 The Results

Let X be a random variable taking values in the finite interval $[a,b]$, with cumulative distribution function $F(x) = \Pr(X \leq x)$.

The following theorem holds

Theorem 2.1. *Let X and F be as above. Then we have the inequality*

$$\left|\Pr(X \leq x) - \frac{b - E(X)}{b-a}\right| \tag{2.1}$$

$$\leq \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X \leq x) + \int_a^b sgn(t-x)F(t)\,dt\right]$$

$$\leq \frac{1}{b-a}\left[(b-x)\Pr(X \geq x) + (x-a)\Pr(X \leq x)\right]$$

$$\leq \frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{(b-a)}$$

for all $x \in [a,b]$. All the inequalities in (2.1) *are sharp and the constant $\frac{1}{2}$ is the best possible.*

Proof. Consider the kernel $p : [a,b]^2 \to \mathbf{R}$ given by

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b] \end{cases}. \tag{2.2}$$

Then the Riemann-Stieltjes integral $\int_a^b p(x,t)\,dF(t)$ exists for any $x \in [a,b]$ and the formula of integration by parts for Riemann-Stieltjes integral gives:

$$\int_a^b p(x,t)\,dF(t) = \int_a^x (t-a)\,dF(t) + \int_x^b (t-b)\,dF(t) \tag{2.3}$$

$$= (t-a)F(t)\big|_a^x - \int_a^x F(t)\,dt + (t-b)F(t)\big|_x^b - \int_x^b F(t)\,dt$$

$$= (b-a)F(x) - \int_a^b F(t)\,dt.$$

On the other hand, the integration by parts formula for Riemann-Stieltjes integral also gives:

$$E(X) := \int_a^b t dF(t) = tF(t)|_a^b - \int_a^b F(t)\,dt \tag{2.4}$$

$$= bF(b) - aF(a) - \int_a^b F(t)\,dt = b - \int_a^b F(t)\,dt.$$

Now, using (2.3) and (2.4), we get the equality

$$(b-a)F(x) + E(X) - b = \int_a^b p(x,t)\,dF(t) \tag{2.5}$$

for all $x \in [a,b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where

$$\nu(\Delta_n) := \max\left\{x_{i+1}^{(n)} - x_i^{(n)} : i = 0, ..., n-1\right\}.$$

If $p : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and $\nu : [a,b] \to \mathbf{R}$ is monotonous nondecreasing, then the Riemann-Stieltjes integral $\int_a^b p(x)\,d\nu(x)$ exists and

$$\left|\int_a^b p(x)\,d\nu(x)\right| = \left|\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)\right]\right| \tag{2.6}$$

$$\le \lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)\right)$$

$$= \int_a^b |p(x)|\,d\nu(x).$$

Using (2.6) we have:

$$\left|\int_a^b p(x,t)\,dF(t)\right| = \left|\int_a^x (t-a)\,dF(t) + \int_x^b (t-b)\,dF(t)\right| \tag{2.7}$$

$$\le \left|\int_a^x (t-a)\,dF(t)\right| + \left|\int_x^b (t-b)\,dF(t)\right| \le \int_a^x |t-a|\,dF(t) + \int_x^b |t-b|\,dF(t)$$

$$= \int_a^x (t-a)\,dF(t) + \int_x^b (b-t)\,dF(t)$$

$$= (t-a)\,F(t)|_a^x - \int_a^x F(t)\,dt - (b-t)\,F(t)|_x^b + \int_x^b F(t)\,dt$$

$$= \left[[2x-(a+b)]\,F(x) - \int_a^x F(t)\,dt + \int_x^b F(t)\,dt \right]$$

$$= [2x-(a+b)]\,F(x) + \int_a^b sgn\,(t-x)\,F(t)\,dt.$$

Using the identity (2.5) and the inequality (2.7), we deduce the first part of (2.1).

We know that

$$\int_a^b sgn\,(t-x)\,F(t)\,dt = -\int_a^x F(t)\,dt + \int_x^b F(t)\,dt.$$

As $F(\cdot)$ is monotonous nondecreasing on $[a,b]$, we can state that

$$\int_a^x F(t)\,dt \geq (x-a)\,F(a) = 0$$

and

$$\int_x^b F(t)\,dt \leq (b-x)\,F(b) = b-x$$

and then

$$\int_a^b sgn\,(t-x)\,F(t)\,dt \leq b-x \quad \text{for all } x \in [a,b].$$

Consequently, we have the inequality

$$[2x-(a+b)]\,F(x) + \int_a^b sgn\,(t-x)\,F(t)\,dt$$

$$\leq [2x-(a+b)]\, F(x)+(b-x)=(b-x)\,(1-F(x))+(x-a)\, F(x)$$

$$=(b-x)\Pr(X\geq x)+(x-a)\Pr(X\leq x)$$

and the second part of (2.1) is proved.

Finally,

$$(b-x)\Pr(X\geq x)+(x-a)\Pr(X\leq x)$$

$$\leq \max\{b-x,x-a\}\left[\Pr(X\geq x)+\Pr(X\leq x)\right]$$

$$=\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|$$

and the last part of (2.1) is also proved.

Now, assume that the inequality (2.1) holds with a constant $c>0$ instead of $\frac{1}{2}$, i.e.,

$$\left|\Pr(X\leq x)-\frac{b-E(X)}{b-a}\right| \tag{2.8}$$

$$\leq \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X\leq x)+\int_a^b sgn(t-x)\, F(t)\, dt\right]$$

$$\leq \frac{1}{b-a}\left[(b-x)\Pr(X\geq x)+(x-a)\Pr(X\leq x)\right]$$

$$\leq c+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}$$

for all $x\in[a,b]$.

Choose the random variable X such that $F:[0,1]\to\mathbf{R}$,

$$F(x):=\begin{cases}0 & \text{if } x=0\\ 1 & \text{if } x\in(0,1]\end{cases}.$$

Then we have:

$$E(X)=0,\qquad \int_0^1 sgn(t)\, F(t)\, dt=1$$

and by (2.8), for $x=0$, we get

$$1\leq c+\frac{1}{2}$$

which shows that $c=\frac{1}{2}$ is the best possible value. ∎

Remark 2.1. *Taking into account the fact that*

$$\Pr(X \geq x) = 1 - \Pr(X \leq x),$$

then from (2.1) *we get the equivalent inequality*

$$\left|\Pr(X \geq x) - \frac{E(X) - a}{b - a}\right| \tag{2.9}$$

$$\leq \frac{1}{b-a}\left[\left[2x - (a+b)\right]\Pr(X \leq x) + \int_a^b sgn(t - x) F(t)\, dt\right]$$

$$\leq \frac{1}{b-a}\left[(b - x)\Pr(X \geq x) + (x - a)\Pr(X \leq x)\right]$$

$$\leq \frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}$$

for all $x \in [a, b]$.

Remark 2.2. *The following particular cases are also interesting:*

$$\left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{b - E(X)}{b-a}\right| \leq \int_a^b sgn\left(t - \frac{a+b}{2}\right) F(t)\, dt \leq \frac{1}{2} \tag{2.10}$$

and

$$\left|\Pr\left(X \geq \frac{a+b}{2}\right) - \frac{E(X) - a}{b-a}\right| \leq \int_a^b sgn\left(t - \frac{a+b}{2}\right) F(t)\, dt \leq \frac{1}{2}. \tag{2.11}$$

The following corollary could be useful in practice

Corollary 2.2. *Under the above assumptions, we have*

$$\frac{1}{b-a}\left[\frac{a+b}{2} - E(X)\right] \tag{2.12}$$

$$\leq \Pr\left(X \leq \frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\frac{a+b}{2} - E(X)\right] + 1.$$

Proof. From the inequality (2.10), we get

$$-\frac{1}{2} + \frac{b - E(X)}{b-a} \leq \Pr\left(X \leq \frac{a+b}{2}\right) \leq \frac{1}{2} + \frac{b - E(X)}{b-a}.$$

But

$$-\frac{1}{2} + \frac{b - E(X)}{b-a} = \frac{-b + a + 2b - 2E(X)}{2(b-a)} = \frac{1}{b-a}\left[\frac{a+b}{2} - E(X)\right]$$

and

$$\frac{1}{2}+\frac{b-E(X)}{b-a}=1+\frac{b-E(X)}{b-a}-\frac{1}{2}=1+\frac{2b-2E(X)-b+a}{2(b-a)}$$

$$=1+\frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]$$

and the inequality is proved. ∎

Remark 2.3. *Let* $1\geq\varepsilon\geq0$, *and assume that*

$$E(X)\geq\frac{a+b}{2}+(1-\varepsilon)(b-a) \tag{2.13}$$

then

$$\Pr\left(X\leq\frac{a+b}{2}\right)\leq\varepsilon. \tag{2.14}$$

Indeed, if (2.13) holds, then by the right-hand side of (2.12) we get

$$\Pr\left(X\leq\frac{a+b}{2}\right)\leq\frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]+1\leq\frac{(\varepsilon-1)(b-a)}{b-a}+1=\varepsilon.$$

Remark 2.4. *Also, if*

$$E(X)\leq\frac{a+b}{2}-\varepsilon(b-a) \tag{2.15}$$

then, by the right-hand side of (2.12),

$$\Pr\left(X\leq\frac{a+b}{2}\right)\geq\left[\frac{a+b}{2}-E(X)\right]\cdot\frac{1}{b-a}\geq\frac{\varepsilon(b-a)}{(b-a)}=\varepsilon$$

i.e.,

$$\Pr\left(X\leq\frac{a+b}{2}\right)\geq\varepsilon\qquad(\varepsilon\in[0,1]). \tag{2.16}$$

The following corollary is also interesting:

Corollary 2.3. *Under the above assumptions of Theorem 2.1, we have the inequality:*

$$\frac{1}{b-x}\int_a^b\left[\frac{1+sgn(t-x)}{2}\right]F(t)\,dt\geq\Pr(X\geq x) \tag{2.17}$$

$$\geq\frac{1}{x-a}\int_a^b\left[\frac{1-sgn(t-x)}{2}\right]F(t)\,dt$$

for all $x\in(a,b)$.

Proof. From the inequality (2.1) we have:

$$\Pr(X \leq x) - \frac{b - E(X)}{b - a}$$

$$\leq \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X \leq x) + \int_a^b sgn(t-x)F(t)\,dt\right]$$

which is equivalent to:

$$(b-a)\Pr(X \leq x) - [2x-(a+b)]\Pr(X \leq x)$$

$$\leq b - E(X) + \int_a^b sgn(t-x)F(t)\,dt,$$

i.e.,

$$2(b-x)\Pr(X \leq x) \leq b - E(X) + \int_a^b sgn(t-x)F(t)\,dt.$$

As (see the Proof of Theorem 2.1):

$$b - E(X) = \int_a^b F(t)\,dt$$

then from the above inequality we deduce the first part of (2.17).

The second part of (2.17) follows by a similar argument from

$$\Pr(X \leq x) - \frac{b - E(X)}{b - a}$$

$$\geq -\frac{1}{b-a}\left[[2x-(a+b)]\Pr(X \leq x) + \int_a^b sgn(t-x)F(t)\,dt\right]$$

and we shall omit the details. ∎

Remark 2.5. *If we put* $x = \frac{a+b}{2}$ *in* (2.17), *then we get*

$$\frac{1}{b-a}\int_a^b \left[1 + sgn\left(t - \frac{a+b}{2}\right)\right]F(t)\,dt \geq \Pr\left(X \geq \frac{a+b}{2}\right) \tag{2.18}$$

$$\geq \frac{1}{b-a}\int_a^b \left[1 - sgn\left(t - \frac{a+b}{2}\right)\right]F(t)\,dt.$$

3 Applications for a Beta Random Variable

A Beta random variable X with parameters (p,q) has the probability density function

$$f(x;p,q) := \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}; \qquad 0<x<1$$

where $\Omega = \{(p,q) : p,q>0\}$ and $B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$.

Let us compute the expectation of X.

We have

$$E(X) = \frac{1}{B(p,q)} \int_0^1 x \cdot x^{p-1}(1-x)^{q-1}\,dx = \frac{B(p+1,q)}{B(p,q)},$$

i.e.,

$$E(X) = \frac{p}{p+q}.$$

The following proposition holds:

Proposition 3.1. *Let X be a Beta random variable with parameters (p,q). Then we have the inequalities:*

$$\left|\Pr(X \le x) - \frac{q}{p+q}\right| \le \frac{1}{2} + \left|x - \frac{1}{2}\right|$$

and

$$\left|\Pr(X \ge x) - \frac{p}{p+q}\right| \le \frac{1}{2} + \left|x - \frac{1}{2}\right|$$

for all $x \in [0,1]$ and particularly:

$$\left|\Pr\left(X \le \frac{1}{2}\right) - \frac{q}{p+q}\right| \le \frac{1}{2}$$

and

$$\left|\Pr\left(X \ge \frac{1}{2}\right) - \frac{p}{p+q}\right| \le \frac{1}{2}$$

respectively.

The proof follows by Theorem 2.1 applied for a Beta random variable, X.

References

[1] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_1 norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[2] S.S. DRAGOMIR and S.WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.,* **11**(1998), 105-109.

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic, Dordrecht, 1994.

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