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*Some Ostrowski Type Inequalities for N-Time Differentiable Mappings and Applications*

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## SOME OSTROWSKI TYPE INEQUALITIES FOR $N$ -TIME DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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ABSTRACT. Some generalizations of the Ostrowski inequality for  $n$ -time differentiable mappings are given. Applications in Numerical Integration and for power series expansions are also presented.

### 1 INTRODUCTION

In 1938, Ostrowski (see for example [3, p.468]) proved the following integral inequality:

Let  $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$  be a differentiable mapping on  $I^\circ$  ( $I^\circ$  is the interior of  $I$ ), and let  $a, b \in I^\circ$  with  $a < b$ . If  $f' : (a, b) \rightarrow \mathbf{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$ , then we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all  $x \in [a, b]$ .

The constant  $\frac{1}{4}$  is sharp in the sense that it can not be replaced by a smaller one.

For applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [2] by S.S. Dragomir and S. Wang.

In 1976, G.V. Milovanović and J.E. Pečarić (see for example [3, p. 468]), proved the following generalization of Ostrowski's result:

Let  $f : [a, b] \rightarrow R$  be an  $n$ -times differentiable function,  $n \geq 1$ , and such that  $\|f^{(n)}\|_\infty := \sup_{t \in (a, b)} |f^{(n)}(t)| < \infty$ . Then

$$\left| \frac{1}{n} \left( f(x) + \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right|$$

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$$\leq \frac{\|f^{(n)}\|_{\infty}}{n(n+1)!} \cdot \frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a}$$

for all  $x \in [a, b]$ .

In [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following Ostrowski type inequality for twice differentiable mappings:

Let  $f : [a, b] \rightarrow \mathbf{R}$  be a twice differentiable mapping on  $(a, b)$  and  $f'' : (a, b) \rightarrow \mathbf{R}$  is bounded, i.e.,  $\|f''\|_{\infty} = \sup_{t \in (a, b)} |f''(t)| < \infty$ . Then we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right|$$

$$\leq \left[ \frac{1}{24} (b-a)^2 + \frac{1}{2} \left(x - \frac{a+b}{2}\right)^2 \right] \|f''\|_{\infty} \leq \frac{(b-a)^2}{6} \|f''\|_{\infty}$$

for all  $x \in [a, b]$ .

In this paper we establish another generalization of Ostrowski inequality for  $n$ -time differentiable mappings which naturally generalizes the result from [1] and apply it in Numerical Integration and for power series expansions of functions on an interval.

## 2 INTEGRAL IDENTITIES

The following lemma holds:

**Lemma 2.1.** *Let  $f : [a, b] \rightarrow \mathbf{R}$  be a mapping such that  $f^{(n-1)}$  is absolutely continuous on  $[a, b]$ . Then for all  $x \in [a, b]$  we have the identity:*

$$(2.1) \quad \int_a^b f(t) dt = \sum_{k=0}^{n-1} \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x)$$

$$+ (-1)^n \int_a^b K_n(x, t) f^{(n)}(t) dt$$

where the kernel  $K_n : [a, b]^2 \rightarrow \mathbf{R}$  is given by

$$(2.2) \quad K_n(x, t) := \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a, x] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (x, b] \end{cases}, \quad x \in [a, b]$$

and  $n$  is a natural number,  $n \geq 1$ .

*Proof.* The proof is by mathematical induction.

For  $n = 1$ , we have to prove the equality

$$(2.3) \quad \int_a^b f(t) dt = (b-a)f(x) - \int_a^b K_1(x, t) f^{(1)}(t) dt$$

where

$$K_1(x, t) := \begin{cases} t-a & \text{if } t \in [a, x] \\ t-b & \text{if } t \in (x, b] \end{cases}.$$

Integrating by parts, we have:

$$\begin{aligned} \int_a^b K_1(x, t) f^{(1)}(t) dt &= \int_a^x (t-a) f'(t) dt + \int_x^b (t-b) f'(t) dt \\ &= (t-a)f(t)|_a^x - \int_a^x f(t) dt + (t-b)f(t)|_x^b - \int_x^b f(t) dt \\ &= (x-a)f(x) + (b-x)f(x) - \int_a^b f(t) dt = (b-a)f(x) - \int_a^b f(t) dt \end{aligned}$$

and the identity (2.3) is proved.

Assume that (2.1) holds for " $n$ " and let us prove it for " $n+1$ ". That is, we have to prove the equality

$$(2.4) \quad \int_a^b f(t) dt = \sum_{k=0}^n \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \\ + (-1)^{n+1} \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt$$

where, obviously,

$$K_{n+1}(t) := \begin{cases} \frac{(t-a)^{n+1}}{(n+1)!} & \text{if } t \in [a, x] \\ \frac{(t-b)^{n+1}}{(n+1)!} & \text{if } t \in (x, b] \end{cases}.$$

We have

$$\int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt = \int_a^x \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(t) dt + \int_x^b \frac{(t-b)^{n+1}}{(n+1)!} f^{(n+1)}(t) dt$$

and integrating by parts gives

$$\begin{aligned} \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt &= \frac{(t-a)^{n+1}}{(n+1)!} f^{(n)}(t) \Big|_a^x - \frac{1}{n!} \int_a^x (t-a)^n f^{(n)}(t) dt \\ &\quad + \frac{(t-b)^{n+1}}{(n+1)!} f^{(n)}(t) \Big|_x^b - \frac{1}{n!} \int_x^b (t-b)^n f^{(n)}(t) dt \\ &= \frac{(x-a)^{n+1} + (-1)^{n+2} (b-x)^{n+1}}{(n+1)!} f^{(n)}(x) - \int_a^b K_n(x, t) f^{(n)}(t) dt. \end{aligned}$$

That is

$$\begin{aligned} \int_a^b K_n(x, t) f^{(n)}(t) dt &= \frac{(x-a)^{n+1} + (-1)^{n+2} (b-x)^{n+1}}{(n+1)!} f^{(n)}(x) \\ &\quad - \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt. \end{aligned}$$

Now, using the mathematical induction hypothesis, we get

$$\begin{aligned} \int_a^b f(t) dt &= \sum_{k=0}^{n-1} \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \\ &\quad + \frac{(b-x)^{n+1} + (-1)^n (x-a)^{n+1}}{(n+1)!} f^{(n)}(x) - (-1)^n \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt \\ &= \sum_{k=0}^n \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \\ &\quad + (-1)^{n+1} \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt. \end{aligned}$$

That is, the identity (2.4); and the theorem is thus proved. ■

**Corollary 2.2.** *With the above assumptions, we have the representation*

$$(2.5) \quad \int_a^b f(t) dt = \sum_{k=0}^{n-1} \left[ \frac{1 + (-1)^k}{(k+1)!} \right] \frac{(b-a)^{k+1}}{2^{k+1}} f^{(k)} \left( \frac{a+b}{2} \right) \\ + (-1)^n \int_a^b M_n(t) f^{(n)}(t) dt$$

where

$$M_n(t) := \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a, \frac{a+b}{2}] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}.$$

The proof follows by Lemma 2.1 by choosing  $x = \frac{a+b}{2}$ .

**Corollary 2.3.** *With the above assumptions, we have the representation:*

$$(2.6) \quad \int_a^b f(t) dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \\ + \int_a^b T_n(t) f^{(n)}(t) dt$$

where

$$T_n(t) := \frac{1}{n!} \left[ \frac{(b-t)^n + (-1)^n (t-a)^n}{2} \right], \quad t \in [a, b].$$

*Proof.* Firstly, choose in (2.1)  $x = a$  to get

$$\int_a^b f(t) dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + (-1)^n \int_a^b \frac{(t-b)^n}{n!} f^{(n)}(t) dt \\ = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + \int_a^b \frac{(b-t)^n}{n!} f^{(n)}(t) dt.$$

Also, if we put  $x = b$  in (2.1), we get

$$\int_a^b f(t) dt = \sum_{k=0}^{n-1} (-1)^k \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(b) + (-1)^n \int_a^b \frac{(t-a)^n}{n!} f^{(n)}(t) dt.$$

Summing the above two identities and dividing by 2, we get

$$\int_a^b f(t) dt = \sum_{k=0}^n \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \\ + \int_a^b T_n(t) f^{(n)}(t) dt$$

and the corollary is proved. ■

The following Taylor-like formula with integral remainder also holds:

**Corollary 2.4.** *Let  $g : [a, y] \rightarrow \mathbf{R}$  be a mapping such that  $g^{(n)}$  is absolutely continuous on  $[a, y]$ . Then for all  $x \in [a, y]$ , we have the identity*

$$(2.7) \quad g(y) = g(a) + \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k (x-a)^{k+1}]}{(k+1)!} g^{(k+1)}(x) \\ + (-1)^n \int_a^y K_n(y, t) g^{(n+1)}(t) dt.$$

The proof is obvious by Lemma 2.1 choosing  $f = g'$ , and  $b = y$ .

**Remark 2.1.** *If we choose  $n = 1$  in (2.1), we get the identity*

$$(2.8) \quad \int_a^b f(t) dt = (b-a) f(x) - \int_a^b K_1(x, t) f'(t) dt$$

for all  $x \in [a, b]$ , where

$$K_1(x, t) := \begin{cases} t-a & \text{if } t \in [a, x] \\ t-b & \text{if } t \in (x, b] \end{cases}$$

which is the identity employed by S.S. Dragomir and S. Wang to prove an Ostrowski type inequality in paper [2].

If in (2.5) we choose  $n = 1$ , then we get

$$(2.9) \quad \int_a^b f(t) dt = (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b M_1(t) f'(t) dt$$

where

$$M_1(t) = \begin{cases} t-a & \text{if } t \in [a, \frac{a+b}{2}] \\ t-b & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}$$

which gives the mid-point type inequality useful in Numerical Analysis.

Also, if we put  $n = 1$  in (2.6), we get the trapezoid identity

$$(2.10) \quad \int_a^b f(t) dt = \frac{f(a) + f(b)}{2} (b - a) + \int_a^b T_1(t) f'(t) dt$$

where

$$T_1(t) = \frac{a+b}{2} - t, \quad t \in [a, b].$$

Finally, if in the Taylor-like formula (2.7) we put  $n = 1$ , we get

$$g(y) = g(a) + (y - a) g'(x) - \int_a^y K_1(y, t) g^{(2)}(t) dt$$

where  $x \in [a, y]$ .

**Remark 2.2.** *If we choose  $n = 2$  in (2.1), we get the identity:*

$$(2.11) \quad \int_a^b f(t) dt = (b - a) f(x) - \left(x - \frac{a+b}{2}\right) f'(x) + \int_a^b K_2(x, t) f''(t) dt$$

where

$$K_2(x, t) := \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a, x] \\ \frac{(t-b)^2}{2} & \text{if } t \in (x, b] \end{cases}$$

and  $x \in [a, b]$ , which is the identity employed by P. Cerone, S.S. Dragomir and J. Roumeliotis to prove some Ostrowski type inequalities for twice differentiable mappings in the paper [1].

If in (2.5) we choose  $n = 2$ , then we get

$$(2.12) \quad \int_a^b f(t) dt = (b - a) f\left(\frac{a+b}{2}\right) + \int_a^b M_2(t) f''(t) dt$$

where

$$M_2(t) = \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a, \frac{a+b}{2}] \\ \frac{(t-b)^2}{2} & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}$$

which is the classical mid-point identity.



Also, if we put  $n = 2$  in (2.6), we get the identity

$$(2.13) \quad \int_a^b f(t) dt = \frac{f(a) + f(b)}{2} (b - a) + \frac{(b - a)^2}{2} \cdot \frac{f'(a) - f'(b)}{2} \\ + \int_a^b T_2(t) f''(t) dt$$

where

$$T_2(t) = \frac{1}{2} \cdot \frac{(b - t)^2 + (t - a)^2}{2}, \quad t \in [a, b].$$

Finally, if we put  $n = 2$  in (2.7), we get

$$(2.14) \quad g(y) = g(a) + (y - a) g'(x) - (y - a) \left( x - \frac{a + y}{2} \right) g''(x) \\ + \int_a^y K_2(y, t) g^{(3)}(t) dt,$$

where  $K_2$  is as above and  $a \leq x \leq y$ .

### 3 SOME INTEGRAL INEQUALITIES FOR $\|\cdot\|_\infty$ -NORM

The following theorem holds:

**Theorem 3.1.** *Let  $f : [a, b] \rightarrow \mathbf{R}$  be a mapping such that  $f^{(n-1)}$  is absolutely continuous on  $[a, b]$  and  $f^{(n)} \in L_\infty[a, b]$ . Then for all  $x \in [a, b]$ , we have the inequality:*

$$(3.1) \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \left[ (x-a)^{n+1} + (b-x)^{n+1} \right] \leq \frac{\|f^{(n)}\|_\infty (b-a)^{n+1}}{(n+1)!}$$

where

$$\|f^{(n)}\|_\infty := \sup_{t \in [a, b]} |f^{(n)}(t)| < \infty.$$

*Proof.* Using the identity (2.1), we have:

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right|$$

$$\begin{aligned} &= \left| \int_a^b K_n(x, t) f^{(n)}(t) dt \right| \leq \|f^{(n)}\|_\infty \int_a^b |K_n(x, t)| dt \\ &= \|f^{(n)}\|_\infty \left[ \int_a^x \frac{(t-a)^n}{n!} dt + \int_x^b \frac{(b-t)^n}{n!} dt \right] \\ &= \frac{\|f^{(n)}\|_\infty}{(n+1)!} \left[ (x-a)^{n+1} + (b-x)^{n+1} \right], \end{aligned}$$

and the first part of inequality (3.1) is proved.

To prove the second inequality in (3.1), we observe that

$$(3.2) \quad (x-a)^{n+1} + (b-x)^{n+1} \leq (b-a)^{n+1},$$

for all  $x \in [a, b]$ . ■

Taking into account the fact that the mapping  $h_n : [a, b] \rightarrow \mathbf{R}$ ,  $h_n(x) := (x-a)^{n+1} + (b-x)^{n+1}$ , has the property

$$\inf_{x \in [a, b]} h_n(x) = h_n\left(\frac{a+b}{2}\right) = \frac{(b-a)^{n+1}}{2^n},$$

then the best inequality we can get from (3.1) is the one for which  $x = \frac{a+b}{2}$ .

In this way, we can state the following corollary.

**Corollary 3.2.** *Assume that  $f$  is as in Theorem 3.1. Then we have the inequality:*

$$(3.3) \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \left[ \frac{1 + (-1)^k}{(k+1)!} \right] \frac{(b-a)^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{a+b}{2}\right) \right| \leq \frac{\|f^{(n)}\|_\infty (b-a)^{n+1}}{2^n (n+1)!}.$$

Another result which generalizes the trapezoid inequality, is the following one:

**Corollary 3.3.** *With the above assumptions, we have the inequality:*

$$(3.4) \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \leq \frac{(b-a)^{n+1}}{(n+1)!} \|f^{(n)}\|_\infty \times \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1} - 1}{2^{2r}} & \text{if } n = 2r + 1 \end{cases}.$$

*Proof.* Using the identity (2.6), we get

$$\begin{aligned} & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \\ &= \left| \int_a^b T_n(t) f^{(n)}(t) dt \right| \leq \|f^{(n)}\|_{\infty} \int_a^b |T_n(t)| dt. \end{aligned}$$

If  $n = 2r$ , then

$$\begin{aligned} (3.5) \quad & \int_a^b |T_n(t)| dt = \frac{1}{(2r)!} \int_a^b \left[ \frac{(b-t)^{2r} + (t-a)^{2r}}{2} \right] dt \\ &= \frac{1}{(2r)!} \cdot \frac{1}{2} \left[ \frac{(b-a)^{2r+1}}{(2r+1)} + \frac{(b-a)^{2r+1}}{(2r+1)} \right] = \frac{(b-a)^{2r+1}}{(2r+1)!}. \end{aligned}$$

For  $n = 2r+1$ , put  $h_{2r+1}(t) := (b-t)^{2r+1} - (t-a)^{2r+1}$ ,  $t \in [a, b]$ . Observe that  $h_{2r+1}(t) = 0$  iff  $t = \frac{a+b}{2}$  and  $h_{2r+1}(t) > 0$  if  $t \in [a, \frac{a+b}{2})$  and  $h_{2r+1}(t) < 0$  if  $t \in (\frac{a+b}{2}, b]$ .

Then

$$\begin{aligned} & \int_a^b |T_{2r+1}(t)| dt \\ &= \int_a^{\frac{a+b}{2}} \left[ (b-t)^{2r+1} - (t-a)^{2r+1} \right] dt + \int_{\frac{a+b}{2}}^b \left[ (t-a)^{2r+1} - (b-t)^{2r+1} \right] dt \\ &= 2 \frac{(b-a)^{2r+2}}{2r+2} - \frac{4 \left(\frac{b-a}{2}\right)^{2r+2}}{2r+2} \\ &= \frac{1}{2r+2} \left[ 2(b-a)^{2r+2} - \frac{(b-a)^{2r+2}}{2^{2r}} \right] = \frac{(b-a)^{2r+2}}{2r+2} \left( 2 - \frac{1}{2^{2r}} \right) \\ &= \frac{(b-a)^{2r+2}}{2r+2} \cdot \frac{2^{2r+1} - 1}{2^{2r}}. \end{aligned}$$

Using (3.4) we get the desired inequality (3.3). ■

The following inequality in terms of  $\|\cdot\|_{\infty}$  - norm for the Taylor like expansion (2.7) also holds:

**Corollary 3.4.** *Let  $g$  be as in Corollary 2.4. Then we have the inequality:*

$$(3.6) \quad \left| g(y) - g(a) - \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k (x-a)^{k+1}]}{(k+1)!} g^{(k+1)}(x) \right| \\ \leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)!} [(y-x)^{n+1} + (x-a)^{n+1}] \leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)!} (y-a)^{n+1}$$

for all  $a \leq x \leq y$ .

**Remark 3.1.** *It is well known that for the classical Taylor expansion around  $a$  we have the inequality*

$$(3.7) \quad \left| g(y) - \sum_{k=0}^n \frac{(y-a)^k}{k!} g^{(k)}(a) \right| \leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)!} (y-a)^{n+1}$$

for all  $y \geq a$ . It is clear now that the above approximation (3.6) around the arbitrary point  $x \in [a, y]$  provides a better approximation for the mapping  $g$  at the point  $y$  than the classical Taylor expansion around the point  $a$ .

If in (3.6) we choose  $x = \frac{a+y}{2}$ , then we get

$$(3.8) \quad \left| g(y) - g(a) - \sum_{k=1}^n \frac{[1 + (-1)^{k-1}]}{k!} \frac{(y-a)^k}{2^k} g^{(k)}\left(\frac{a+y}{2}\right) \right| \\ \leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)! 2^n} (y-a)^{n+1}.$$

The above inequality (3.8) shows that for  $g \in C^{\infty}[a, b]$  the series

$$g(a) + \sum_{k=0}^{\infty} \frac{[1 + (-1)^k]}{(k+1)!} \frac{(y-a)^{k+1}}{2^{k+1}} g^{(k+1)}\left(\frac{a+y}{2}\right)$$

converges more rapidly to  $g(y)$  than the usual one

$$\sum_{k=0}^{\infty} \frac{(y-a)^k}{k!} g^{(k)}(a)$$

which comes from Taylor's expansion. Further, it should be noted that (3.8) only involves odd derivatives of  $g$ .

**Remark 3.2.** *If in the inequality (3.1) we choose  $n = 1$  we get*

$$\left| \int_a^b f(t) dt - (b-a) f(x) \right| \leq \frac{(x-a)^2 + (b-x)^2}{2} \|f'\|_{\infty}.$$

As a simple calculation shows that

$$\frac{1}{2} [(x-a)^2 + (b-x)^2] = \frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2}\right)^2$$

consequently we obtain the Ostrowski inequality:

$$(3.9) \quad \left| \int_a^b f(t) dt - (b-a) f(x) \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)^2 \|f'\|_\infty$$

for all  $x \in [a, b]$ .

If in (3.2) we put  $n = 1$ , we get the mid-point inequality

$$(3.10) \quad \left| \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) (b-a) \right| \leq \frac{1}{4} (b-a)^2 \|f'\|_\infty.$$

From the inequality (3.4), for  $n = 1$ , we get the trapezoid inequality

$$(3.11) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b-a) \right| \leq \frac{1}{2} (b-a)^2 \|f'\|_\infty.$$

Also, from (3.6) we deduce

$$(3.12) \quad |g(y) - g(a) - (y-a)g'(x)| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+y}{2}\right)^2}{(y-a)^2} \right] \|g''\|_\infty$$

for all  $a \leq x \leq y$ .

**Remark 3.3.** If in the inequality (3.1) we choose  $n = 2$ , then we get

$$\begin{aligned} & \left| \int_a^b f(t) dt - (b-a) f(x) + \left(x - \frac{a+b}{2}\right) (b-a) f'(x) \right| \\ & \leq \frac{1}{6} [(x-a)^3 + (b-x)^3] \|f''\|_\infty \end{aligned}$$

for all  $x \in [a, b]$ .

Now, observe that

$$\begin{aligned} (x-a)^3 + (b-x)^3 &= (b-a) [(x-a)^2 + (b-x)^2 - (x-a)(b-x)] \\ &= (b-a) [(x-a+b-x)^2 - 3(x-a)(b-x)] \\ &= (b-a) [(b-a)^2 + 3[x^2 - (a+b)x + ab]] \\ &= (b-a) \left[ (b-a)^2 + 3 \left[ \left(x - \frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 \right] \right] \\ &= (b-a) \left[ \left(\frac{b-a}{2}\right)^2 + 3 \left(x - \frac{a+b}{2}\right)^2 \right] \end{aligned}$$

and then we recapture the result obtained in [1], namely

$$(3.13) \quad \left| \int_a^b f(t) dt - (b-a)f(x) + \left(x - \frac{a+b}{2}\right)(b-a)f'(x) \right| \\ \leq \left[ \frac{1}{24} + \frac{1}{2} \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)^3 \|f''\|_\infty.$$

If we put  $n = 2$  in (3.3), we get the classical mid-point inequality

$$(3.14) \quad \left| \int_a^b f(t) dt - (b-a)f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{24} (b-a)^3 \|f''\|_\infty.$$

Now, if in (3.4) we put  $n = 2$ , then we get the inequality:

$$(3.15) \quad \left| \int_a^b f(t) dt - \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^2}{2} \cdot \frac{f'(a)-f'(b)}{2} \right| \\ \leq \frac{(b-a)^3}{6} \|f''\|_\infty.$$

Finally, if we put  $n = 2$  in (3.6), then we get the inequality:

$$\left| g(y) - g(a) - (y-a)g'(x) + (y-a)\left(x - \frac{a+y}{2}\right)g''(x) \right| \\ \leq \left[ \frac{1}{24} + \frac{1}{2} \cdot \frac{\left(x - \frac{a+y}{2}\right)^2}{(y-a)^2} \right] (y-a)^3 \|g'''\|_\infty.$$

#### 4 APPLICATIONS FOR NUMERICAL INTEGRATION

Consider the partition  $I_m : a = x_0 < x_1 < \dots < x_{m-1} < x_m = b$  of the interval  $[a, b]$  and the intermediate points  $\xi = (\xi_0, \dots, \xi_{m-1})$  where  $\xi_j \in [x_j, x_{j+1}]$   $j = 0, \dots, m-1$ . Define the formula

$$F_{m,k}(f, I_m, \xi) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\left[ (x_{j+1} - \xi_j)^{k+1} + (-1)^k (\xi_j - x_j)^{k+1} \right]}{(k+1)!} f^{(k)}(\xi_j)$$

which can be regarded as a perturbation of Riemann's sum

$$\Gamma(f, I_m, \xi) = \sum_{j=0}^{m-1} f(\xi_j) h_j$$

where  $h_j := x_{j+1} - x_j$ ,  $j = 0, \dots, m-1$ .

The following theorem holds.

**Theorem 4.1.** Let  $f : [a, b] \rightarrow \mathbf{R}$  be a mapping such that  $f^{(n-1)}$  is absolutely continuous on  $[a, b]$  and  $I_m$  a partitioning of  $[a, b]$  as above. Then we have the quadrature formula

$$(4.1) \quad \int_a^b f(x) dx = F_{m,k}(f, I_m, \boldsymbol{\xi}) + R_{m,k}(f, I_m, \boldsymbol{\xi})$$

where  $F_{m,k}$  is defined above and the remainder  $R_{m,k}$  satisfies the estimation:

$$(4.2) \quad |R_{m,k}(f, I_m, \boldsymbol{\xi})| \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \sum_{j=0}^{m-1} [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}]$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \sum_{j=0}^{m-1} h_j^{n+1}$$

for all  $\boldsymbol{\xi}$  as above.

*Proof.* Apply Theorem 3.1 on the interval  $[x_j, x_{j+1}]$  to get

$$\left| \int_{x_j}^{x_{j+1}} f(t) dt - \sum_{k=0}^{n-1} \left[ \frac{[(x_{j+1} - \xi_j)^{k+1} + (-1)^k (\xi_j - x_j)^{k+1}]}{(k+1)!} \right] f^{(k)}(\xi_j) \right|$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}]$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} h_j^{n+1}$$

for all  $j \in \{0, \dots, m-1\}$ .

Summing over  $j$  from 0 to  $m-1$  and using the generalized triangle inequality, we deduce the desired estimation (4.2). ■

As an interesting particular case, we can consider the following perturbed mid-point formula

$$M_{m,k}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \left[ \frac{1 + (-1)^k}{(k+1)!} \right] \frac{h_j^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{x_j + x_{j+1}}{2}\right),$$

which in effect involves only even  $k$ .

We state the following result concerning the estimation of the remainder term.

**Corollary 4.2.** Let  $f$  and  $I_m$  be as in Theorem 4.1. Then we have

$$(4.3) \quad \int_a^b f(t) dt = M_{m,k}(f, I_m) + R_{m,k}(f, I_m)$$

and the remainder term  $R_{m,k}$  satisfies the estimation

$$(4.4) \quad |R_{m,k}(f, I_m)| \leq \frac{\|f^{(n)}\|_\infty}{2^n (n+1)!} \sum_{j=0}^{m-1} h_j^{n+1}.$$

We can consider the following perturbed version of the trapezoid formula:

$$T_{m,k}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{h_j^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})}{2} \right].$$

By the use of Corollary 3.3, we have the following approximation of the integral  $\int_a^b f(t) dt$  in terms of  $T_{m,k}(f, I_m)$ :

**Corollary 4.3.** *Let  $f$  and  $I_m$  be as in Theorem 4.1. Then we have*

$$(4.5) \quad \int_a^b f(t) dt = T_{m,k}(f, I_m) + \tilde{R}_{m,k}(f, I_m)$$

and the remainder  $\tilde{R}_{m,k}(f, I_m)$  satisfies the inequality

$$|\tilde{R}_{m,k}(f, I_m)| \leq \frac{C_n}{(n+1)!} \|f^{(n)}\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}$$

where

$$C_n := \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1} - 1}{2^{2r}} & \text{if } n = 2r + 1 \end{cases}.$$

**Remark 4.1.** a). *If we choose  $n = 1$  in the above quadrature formulae (4.1) and (4.3), we recapture some results from the paper [2].*

b). *If we put  $n = 2$ , then by the above Theorem 4.1 and Corollary 4.2, we recover some results from the paper [1].*

We omit the details.

## 5 APPLICATIONS FOR SOME PARTICULAR MAPPINGS

a) Consider  $g : \mathbf{R} \rightarrow \mathbf{R}$ ,  $g(x) = e^x$ . Then  $g^{(n)}(x) = e^x$ ,  $n \in \mathbf{N}$  and

$$\|g^{(n+1)}\|_\infty = \sup_{t \in [a, y]} |g^{(n+1)}(t)| = e^y.$$

Using inequality (3.6), we have

$$(5.1) \quad \left| e^y - e^a - e^x \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k (x-a)^{k+1}]}{(k+1)!} \right|$$



$$\leq \frac{e^y}{(n+1)!} \left[ (y-x)^{n+1} + (x-a)^{n+1} \right] \leq \frac{e^y}{(n+1)!} (y-a)^{n+1}$$

for all  $a \leq x \leq y$ .

Particularly, if we choose  $a = 0$ , then we get

$$(5.2) \quad \left| e^y - 1 - e^x \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k x^{k+1}]}{(k+1)!} \right|$$

$$\leq \frac{e^y}{(n+1)!} \left[ (y-x)^{n+1} + x^{n+1} \right] \leq \frac{e^y}{(n+1)!} y^{n+1}.$$

Moreover, if we choose  $x = \frac{y}{2}$ , then we get

$$(5.3) \quad \left| e^y - 1 - e^{\frac{y}{2}} \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{(k+1)!} \cdot \frac{y^{k+1}}{2^{k+1}} \right| \leq \frac{e^y y^{n+1}}{2^n (n+1)!}$$

for all  $y \geq 0$ .

b) Consider  $g : (0, \infty) \rightarrow \mathbf{R}$ ,  $g(x) = \ln x$ . Then

$$g^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{x^n}, \quad n \geq 1, x > 0$$

and

$$\|g^{(n+1)}\|_{\infty} = \sup_{t \in [a, y]} \left| \frac{(-1)^n n!}{t^{n+1}} \right| = \frac{n!}{a^{n+1}}, \quad a > 0.$$

Using the inequality (3.6) we can state:

$$\left| \ln y - \ln a - \sum_{k=0}^{n-1} \frac{(y-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \cdot \frac{(-1)^k k!}{x^{k+1}} \right|$$

$$\leq \frac{n!}{(n+1)! a^{n+1}} \left[ (y-x)^{n+1} + (x-a)^{n+1} \right] \leq \frac{n!}{(n+1)! a^{n+1}} (y-a)^{n+1}$$

which is equivalent to

$$(5.4) \quad \left| \ln \left( \frac{y}{a} \right) - \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot \frac{(x-a)^{k+1} + (-1)^k (y-x)^{k+1}}{x^{k+1}} \right|$$

$$\leq \frac{(y-x)^{n+1} + (x-a)^{n+1}}{(n+1) a^{n+1}} \leq \frac{1}{(n+1) a^{n+1}} (y-a)^{n+1}.$$

Now, if we choose in (5.4)  $y = z + 1$ ,  $x = w + 1$ ,  $a = 1$ ,  $z \geq w \geq 0$ , then we get

$$(5.5) \quad \left| \ln(z+1) - \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot \frac{w^{k+1} + (-1)^k (z-w)^{k+1}}{(w+1)^{k+1}} \right|$$

$$\leq \frac{(z-w)^{n+1} + w^{n+1}}{n+1} \leq \frac{1}{(n+1)} z^{n+1}.$$

Finally, if we choose in (5.4),  $y = ua, x = wa$  with  $u \geq w > 1$ , then we have

$$\left| \ln u - \sum_{k=0}^{n-1} \frac{1}{k+1} \frac{(w-1)^{k+1} + (-1)^k (u-w)^{k+1}}{w^{k+1}} \right|$$

$$\leq \frac{(u-w)^{n+1} + (w-1)^{n+1}}{n+1} \leq \frac{(u-1)^{n+1}}{(n+1)}.$$

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