Spectral Radii of Operators and High-Power Operator Inequalities

SPECTRAL RADII OF OPERATORS AND HIGH-POWER OPERATOR INEQUALITIES

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Abstract. For some different types of operators on a Hilbert space, we present new high-power operator inequalities, and their corresponding operator inequalities involving spectral radii of operators. We prove that each such operator inequality is equivalent to the Cauchy-Schwarz inequality. In particular, we show that Halmos' two operator inequalities, Reid's inequality, and many others hold easily. We obtain a new generalized Löwner inequality, and a short proof of the classical Löwner-Heinz inequality is given.

1. Introduction

The Cauchy-Schwarz inequality is a powerful inequality which states that the relatio

$$|(x, y)| \leq \|x\| \, \|y\| \tag{1.1}$$

holds for every x and y in a pre-Hilbert space. Every inequality in this space is either derived from the Cauchy-Schwarz inequality, or equivalent to it. For recent developments on inequalities related to (1.1) see [1] an the references therein.

In this paper we use capital letters to denote bounded linear operators on a complex Hilbert space H, and I denotes the identity operator. A positive operator T is written as $T \geq 0$, the zero operator. We shall consider four types of high-power operator inequalities, and their corresponding operator inequalities involving spectral radii of operators. Four types are: a positive operator, two arbitrary operators, mixed operators, and two selfadjoint operators. Indeed, our results are motivated by Halmos' two operator inequalities in [2, p. 51 and 244]. He proved that if $T \geq 0$, S is arbitrary and TS is selfadjoint operators, then, for every $x \in H$, the following high-power operator inequality holds

$$|(TSx, x)|^{2^n} \leq (TS^{2^n}x, x)(Tx, x)^{2^n - 1} \tag{1.2}$$

for $n \geq 0$. From this he concluded that the inequality involving spectral radius

$$|(TSx, x)| \leq r(S)(Tx, x) \tag{1.3}$$

holds, where $r(S)$ means the spectral radius of S. It is a stronger version of a result due to Reid [6]; Reid had $\|S\|$ instead of $r(S)$ (that $r(S) \leq \|S\|$ is known [2, p. 45]). Actually, we prove that some generalizations of inequalities (1.2), (1.3), Reid's inequality, and many others are all equivalent to the Cauchy-Schwarz inequality. In particular, it is shown that inequalities (1.2), (1.3), Reid's inequality, and many

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others hold easily. We also obtain a new generalized Löwner inequality, and a proof that the Cauchy-Schwarz inequality implies the classical Löwner-Heinz inequality, which is essential in operator inequalities on H. Finally we pose a question.

2. Results

First of all, recall that the inequality $|(Tx,y)|^2 \le (Tx,x)(Ty,y)$ holds for $T \ge 0$ and for all $x, y \in H$ (consider the unique positive square root of T). In fact, it is known to be equivalent to the Cauchy-Schwarz inequality, and this is crucial in the proof of our results. Next, we need a well known relation: $r(S) = \lim_n \|S^n\|^{1/n}$ for any operator S [2, Problem 74].

Theorem 1. *Let $T \ge 0$, and S and C be arbitrary operators. Also let TS, TC, A and B be all selfadjoint operators. If n is a positive integer, then for all $x, y \in H$, $y \ne x$, the following are equivalent to one another and to the Cauchy-Schwarz inequality (1.1):*

$$|(Tx,y)|^{2^n} \le (T^{1+2^{n-1}}x,x)(Tx,x)^{2^{n-1}-1}\|y\|^{2^n} \quad \textit{for } n \ge 1; \quad \textit{and} \tag{2.1}$$

$$|(Tx,y)|^2 \le r(T)(Tx,x)\|y\|^2; \tag{2.2}$$

$$|(Sx,Cy)|^{2^n} \le ((S^*S)^{2^{n-1}}x,x)((C^*C)^{2^{n-1}}y,y)\|x\|^{2^n-2}\|y\|^{2^n-2} \quad \textit{for } n \ge 1; \quad \textit{and} \tag{2.3}$$

$$|(Sx,Cy)|^2 \le r(S^*S)r(C^*C)\|x\|^2\|y\|^2; \tag{2.4}$$

$$|(TSx,Cy)|^{2^n} \le (TS^{2^n}x,x)(Tx,x)^{2^{n-1}-1}(TC^{2^n}y,y)(Ty,y)^{2^{n-1}-1} \quad \textit{for } n \ge 1; \quad \textit{and} \tag{2.5}$$

$$|(TSx,Cy)| \le r(S)r(C)(Tx,x)^{\frac{1}{2}}(Ty,y)^{\frac{1}{2}}; \tag{2.6}$$

$$|(Ax,By)|^{2^n} \le (A^{2^{n-1}+2}x,x)(B^{2^{n-1}+2}y,y)\|Ax\|^{2^{n-1}-2}\|By\|^{2^{n-1}-2} \times \|x\|^{2^{n-1}}\|y\|^{2^{n-1}2^{n-1}-1} \quad \textit{for } n \ge 2; \quad \textit{and} \tag{2.7}$$

$$|(Ax,By)|^2 \le r(A)r(B)\|Ax\|\|By\|\|x\|\|y\|. \tag{2.8}$$

Proof. It is trivial to show that any one of statements (2.2), (2.4), (2.6) and (2.8) implies (1.1); just letting $T = S = C = A = B = I$ will suffice.

(1.1)⇒(2.1). We shall prove it inductively as follows, and start with $n = 1$ first.

$$|(Tx,y)|^2 \le (T^2x,x)\|y\|^2.$$

As

$$(T^2x,x)^2 \le (TTx,Tx)(Tx,x) = (T^3x,x)(Tx,x),$$

we have

$$|(Tx,y)|^4 \le (T^3x,x)(Tx,x)\|y\|^4$$

for $n = 2$. Since

$$(T^{1+2^{n-1}}x,x)^2 \le (TT^{2^{n-1}}x,T^{2^{n-1}}x)(Tx,x) = (T^{1+2^n}x,x)(Tx,x),$$

we obtain

$$\begin{aligned} |(Tx,y)|^{2^{n+1}} &\le \left[|(Tx,y)|^{2^n}\right]^2 \\ &\le \left[(T^{1+2^{n-1}}x,x)(Tx,x)^{2^{n-1}-1}\|y\|^{2^n}\right]^2 \\ &\le (T^{1+2^n}x,x)(Tx,x)^{2^n-1}\|y\|^{2^{n+1}}, \end{aligned}$$

and the induction process is completed.

(2.1)⇒(2.2). The inequality (2.1) gives

$$|(Tx,y)|^{2^n} \le \|T\| \left\|T^{2^{n-1}}\right\| \|x\|^2 (Tx,x)^{2^{n-1}-1} \|y\|^{2^n}.$$

Taking the 2^{n-1}-th root of both sides of the inequality above yields

$$|(Tx,y)|^2 \le \|T\|^{\frac{1}{2^{n-1}}} \left\|T^{2^{n-1}}\right\|^{\frac{1}{2^{n-1}}} \|x\|^{\frac{2}{2^{n-1}}} (Tx,x)^{1-\frac{1}{2^{n-1}}} \|y\|^2.$$

Passing to the limit as $n \to \infty$ we have the desired conclusion.

We mention before we continue that the methods of the proof of all others are similar to above.

(1.1)⇒(2.3). $|(Sx,Cy)|^2 \le (S^*Sx,x)(C^*Cy,y)$ for $n=1$.
For the inductive step, note first that

$$((S^*S)^{2^{n-1}}x,x)^2 \le ((S^*S)^{2^n}x,x)\|x\|^2.$$

So,

$$\begin{aligned} |(Sx,Cy)|^{2^{n+1}} &\le \left[\left((S^*S)^{2^{n-1}}x,x\right)\left((C^*C)^{2^{n-1}}y,y\right)\|x\|^{2^n-2}\|y\|^{2^n-2}\right]^2 \\ &= \left((S^*S)^{2^n}x,x\right)((C^*C)^{2^n}y,y)\|x\|^{2^{n+1}-2}\|y\|^{2^{n+1}-2}, \end{aligned}$$

and the process is completed.

(2.3)⇒(2.4). The inequality (2.3) gives

$$|(Sx,Cy)|^{2^n} \le \left\|(S^*S)^{2^{n-1}}\right\| \left\|(C^*C)^{2^{n-1}}\right\| \|x\|^{2^n}\|y\|^{2^n},$$

which yields

$$|(Sx,Cy)|^2 \le \left\|(S^*S)^{2^{n-1}}\right\|^{\frac{1}{2^{n-1}}} \left\|(C^*C)^{2^{n-1}}\right\|^{\frac{1}{2^{n-1}}} \|x\|^2\|y\|^2.$$

(2.4) follows immediately if we take the limit in above as $n \to \infty$.

(1.1)⇒(2.5). As T is positive and both TS and TC are selfadjoint, we see that $S^*TS = (TS)^*S = TS^2$. And by induction we get $(S^*)^iTS^i = TS^{2i}$ (for $i = 1,2,...$). Similary, $(C^*)^iTC^i = TC^{2i}$ (for $i=1,2,...$). It follows, for $n=1$, that

$$|(TSx,Cy)|^2 \le (TSx,Sx)(TCy,Cy) = (TS^2x,x)(TC^2y,y).$$

Since

$$(TS^{2^n}x,x)^2 \le ((S^*)^{2n}TS^{2^n}x,x)(Tx,x) = (TS^{2^{n+1}}x,x)(Tx,x),$$

we have

$$\begin{aligned}|(TSx,Cy)|^{2^{n+1}} &\le (TS^{2^n}x,x)^2(Tx,x)^{2^n-2}(TC^{2^n}y,y)^2(Ty,y)^{2^n-2}\\ &\le (TS^{2^{n+1}}x,x)(Tx,x)^{2^n-1}(TC^{2^{n+1}}y,y)(Ty,y)^{2^n-1}.\end{aligned}$$

This proves, by induction, the inequality (2.5).

(2.5)⇒(2.6). (2.5) yields

$$|(TSx,Cy)|^{2^n} \le \|T\|^2 \left\|S^{2^n}\right\| \left\|C^{2^n}\right\| \|x\|^2 \|y\|^2 (Tx,x)^{2^{n-1}-1}(Ty,y)^{2^{n-1}-1},$$

which implies, by taking the 2^n-th root,

$$\begin{aligned}|(TSx,Cy)| \le \|T\|^{\frac{1}{2^{n-1}}} \left\|S^{2^n}\right\|^{\frac{1}{2^n}} \left\|C^{2^n}\right\|^{\frac{1}{2^n}}&\\ \times \|x\|^{\frac{1}{2^{n-1}}} \|y\|^{\frac{1}{2^{n-1}}} (Tx,x)^{\frac{1}{2}-\frac{1}{2^n}}(Ty,y)^{\frac{1}{2}-\frac{1}{2^n}}&.\end{aligned}$$

Thus, we have the inequality (2.6) after passing to the limit as $n\to\infty$.

(1.1)⇒(2.7). Since $|(Ax,By)|^2 \le (A^2x,x)(B^2y,y)$,

$$\begin{aligned}|(Ax,By)|^4 &\le (A^2x,x)^2(B^2y,y)^2\\ &\le (A^2x,A^2x)(B^2y,B^2y)\|x\|^2\|y\|^2\\ &= (A^4x,x)(B^4y,y)\|x\|^2\|y\|^2\end{aligned}$$

for $n=2$. Note that $A^2\ge 0$, and

$$(A^{2^{n-1}+2}x,x)^2 = (A^2A^{2^{n-1}}x,x)^2 \le (A^{2^n+2}x,x)\|Ax\|^2,$$

and similarly for $B^2\ge 0$. Therefore,

$$|(Ax,By)|^{2^{n+1}} \le (A^{2^n+2}x,x)(B^{2^n+2}y,y)\|Ax\|^{2^n-2}\|By\|^{2^n-2}\|x\|^{2^n}\|y\|^{2^n},$$

and (2.7) holds by induction.

(2.7)⇒(2.8). The inequality (2.7) gives

$$\begin{aligned}|(Ax,By)|^{2^n} \le \|A\|^2 \left\|A^{2^{n-1}}\right\| \|Ax\|^{2^{n-1}-2}&\\ \times \|B\|^2 \left\|B^{2^{n-1}}\right\| \|By\|^{2^{n-1}-2}\|x\|^{2^{n-1}+2}\|y\|^{2^{n-1}+2}&.\end{aligned}$$

The next step is taking the 2^{n-1}-th root, and then passing to the limit as $n\to\infty$; the same as we did many times before. The proof of the theorem is now completed. ∎

By a well-known result that if E is a normal operator (selfadjoint operator, in particular) on a complex Hilbert space, then $r(E)=\|E\|$ [7, Theorem 6.2-E]. Thus, the proofs of (1.1)⇔(2.2), (1.1)⇔(2.4) and (1.1)⇔(2.8) in Theorem 1 are trivial. However, our proofs do not rely on this result. It should be pointed out that (2.5) and (2.6) in Theorem 1 are generalizations of Halmos' inequalities (1.2) and (1.3), respectively. The next result, a generalization of Reid's inequality, is obviously a consequence of (2.6) in Theorem 1 and the proof should be omitted.

Corollary 1. *Let $T \geq 0$, and S and C be arbitrary operators. If TS and TC are selfadjoint operators, then for all $x, y \in H$, $y \neq x$, the following inequality is equivalent to (1.1):*

$$|(TSx, Cy)| \leq \|S\| \|C\| (Tx, x)^{\frac{1}{2}} (Ty, y)^{\frac{1}{2}}. \tag{2.9}$$

The Cauchy-Schwarz inequality (1.1) can produce various kinds of inequalities which are not immediately apparent. The next results are consequences of Theorem 1 and Corollary 1. This also shows why the condition $y \neq x$ is imposed in both results.

Corollary 2. *Let $T \geq 0$, and S and C be arbitrary operators. Also let TS, TC, A and B be all selfadjoint operators. If n is a positive integer, then for every $x \in H$ the following hold:*

$$(Tx, x)^{2^{n-1}+1} \leq (T^{1+2^{n-1}}x, x) \|x\|^{2^n} \quad \textit{for} \;\; n \geq 1; \tag{2.10}$$

$$(Tx, x) \leq r(T) \|x\|^2 ; \tag{2.11}$$

$$|(Sx, Cx)|^{2^n} \leq ((S^*S)^{2^{n-1}}x, x)((C^*C)^{2^{n-1}}x, x) \|x\|^{2^{n+1}-4} \quad \textit{for} \;\; n \geq 1; \tag{2.12}$$

$$|(Sx, x)|^{2^n} \leq ((S^*S)^{2^{n-1}}x, x) \|x\|^{2^{n+1}-2} \quad \textit{for} \;\; n \geq 1; \tag{2.13}$$

$$|(Sx, Cx)|^2 \leq r(S^*S) r(C^*C) \|x\|^4 ; \tag{2.14}$$

$$|(Sx, x)|^2 \leq r(S^*S) \|x\|^4 ; \tag{2.15}$$

$$|(TSx, Cx)|^{2^n} \leq (TS^{2^n}x, x)(Tx, x)^{2^n-2}(TC^{2^n}x, x) \quad \textit{for} \;\; n \geq 1; \tag{2.16}$$

$$|(TSx, x)|^{2^n} \leq (TS^{2^n}x, x)(Tx, x)^{2^n-1} \quad \textit{for} \;\; n \geq 0 \tag{2.17}$$

(Halmos' inequality (1.2));

$$|(TSx, Cx)| \leq r(S) r(C) (Tx, x); \tag{2.18}$$

$$|(TSx, x)| \leq r(S) (Tx, x) \quad \textit{(Halmos' inequality (1.3));} \tag{2.19}$$

$$|(Ax, Bx)|^{2^n} \leq (A^{2^{n-1}+2}x, x)(B^{2^{n-1}+2}x, x) \times \|Ax\|^{2^{n-1}-2} \|Bx\|^{2^{n-1}-2} \|x\|^{2^{n+1}-2} \quad \textit{for} \;\; n \geq 2; \tag{2.20}$$

$$|(Ax, x)|^{2^n} \leq (A^{2^{n-1}+2}x, x) \|Ax\|^{2^{n-1}-2} \|x\|^{5(2^{n-1})-2} \quad \textit{for} \;\; n \geq 2; \tag{2.21}$$

$$|(Ax, Bx)|^2 \leq r(A) r(B) \|Ax\| \|Bx\| \|x\|^2 . \tag{2.22}$$

$$|(Ax, x)|^2 \leq r(A) \|Ax\| \|x\|^3 ; \tag{2.23}$$

$$|(TSx, Cx)| \leq \|S\| \|C\| (Tx, x); \tag{2.24}$$

$$|(TSx, x)| \leq \|S\| (Tx, x) \quad \textit{(Reid's inequality).} \tag{2.25}$$

Proof. The proof is simple. Let, in particular, $y = x$ in Theorem 1 and Corollary 1 above, so that the Cauchy-Schwarz inequality, (1.1) and (2.10) in Corollary 2, becomes the trivial case $(x, x) = \|x\|^2$. ∎

The classical Löwner-Heinz inequality was initiated in [4] and established in [5], which is a basic tool in theory of operator inequalities on H. More precisely, the inequality $P^\alpha \geq Q^\alpha$ holds if $P \geq Q \geq 0$, where $\alpha \in [0,1]$. There are known examples showing that the inequality does not hold in general if $\alpha > 1$. The proof of the inequality was neither elementary nor short. However, there is a classical characterization of the inequality, namely $P^{\frac{1}{2}} \geq Q^{\frac{1}{2}}$ holds if $P \geq Q \geq 0$, which is known as the Löwner inequality. We propose next a new proof that the Löwner-Heinz inequality may follow by way of the Cauchy-Schwarz inequality (Corollary 3 below). First of all, more generally we have

Theorem 2. *The Cauchy-Schwarz inequality implies a generalized Löwner inequality, i.e.,*

$$r(C)P^{\frac{1}{2}} \geq C^* Q^{\frac{1}{2}}$$

if $P \geq Q \geq 0$, *both* $P^{\frac{1}{2}}C$ *and* $C^*Q^{\frac{1}{2}}$ *are selfadjoint for some operator* C.

Proof. It suffices to show that a slightly generalized Reid's inequality (2.18) in Corollary 2 implies the required inequality. Now, we may assume without loss of generality that P is invertible, then $P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \leq I$ as $P \geq Q \geq 0$. Let $S = P^{-\frac{1}{2}}Q^{\frac{1}{2}}$. Then $SS^* = P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \leq I$, i.e., S is a contraction. Next, let $T = P^{\frac{1}{2}} \geq 0$, then $C^*TS = C^*Q^{\frac{1}{2}}$. As both $P^{\frac{1}{2}}C$ and $C^*Q^{\frac{1}{2}}$ are selfadjoint by assumption (thus, $T \geq 0$, and both TS and TC are selfadjoint), it follows from the inequality $|(TSx, Cx)| \leq r(S)r(C)(Tx, x)$ that

$$\left(C^*Q^{\frac{1}{2}}x, x\right) \leq r(S)\left(r(C)P^{\frac{1}{2}}x, x\right) \leq \left(r(C)P^{\frac{1}{2}}x, x\right)$$

for every $x \in H$. ∎

Corollary 3. *The Cauchy-Schwarz inequality implies the Löwner-Heinz inequality.*

Proof. It suffices to ahow that (2.19) (Halmos' inequality (1.3)) in Corollary 2 implies the Löwner inequality. This is precisely the inequality in Theorem 2, where we let $C = I$. ∎

As usual, let $|E|$ mean the positive square root of the positive operator E^*E.

Corollary 4. *Let* $T \geq 0$ *and* TS *be a selfadjoint operator. Then the following are equivalent.*

(1) $|(|TS|\, x, x)| \leq \|S\|\,(Tx, x)$ *for every* $x \in H$;
(2) $|(TSx, x)| \leq \|S\|\,(Tx, x)$ *for every* $x \in H$ *(Reid's inequality);*
(3) $P^{\frac{1}{2}} \geq Q^{\frac{1}{2}}$ *if* $P \geq Q \geq 0$ *(Löwner inequality).*

Proof.
(1)⇒(2). We use a familiar relation that $-|A| \leq A \leq |A|$ holds if A is selfadjoint. In other words, $|(Ax, x)| \leq (|A|\, x, x)$ for every $x \in H$.

(2)⇒(3). In the proof of Theorem 2 let $C = I$ and use (2.25) in Corollary 2 instead of (2.18).

(3)⇒(1). Since $S/\|S\|$ is a contraction, i.e., $SS^* \leq \|S\|^2 I$, we have

$$0 \leq (TS)^2 = TS(TS)^* = TSS^*T \leq \|S\|^2 T^2.$$

It follows from (2.12) that $|TS| \leq \|S\|\, T$. Therefore,

$$(|TS|\, x, x) \leq (\|S\|\, Tx, x) = \|S\|\,(Tx, x).$$

$\blacksquare$

Notice that the equivalence of the Reid's inequality and the Löwner-Heinz inequality has been pointed out in [8]. In conclusion, in view of Corollary 4, let us pose a question:

Problem: *Could we prove that the generalized Löwner inequality in Theorem 2 implies the inequality (2.18) in Corollary 2? In other words, are the two inequalities equivalent?*

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