



VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

An Ostrowski Type Inequality for Mappings Whose Second Derivatives are Bounded and Applications

This is the Published version of the following publication

Dragomir, Sever S and Barnett, Neil S (1998) An Ostrowski Type Inequality for Mappings Whose Second Derivatives are Bounded and Applications. RGMIA research report collection, 1 (2).

The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository <https://vuir.vu.edu.au/17116/>

AN OSTROWSKI TYPE INEQUALITY FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE BOUNDED AND APPLICATIONS

S.S. DRAGOMIR AND N.S. BARNETT

ABSTRACT. An integral inequality of Ostrowski's type for mappings whose second derivatives are bounded is proved. Applications in Numerical Integration and for special means are pointed out.

1 INTRODUCTION

In [1], S.S. Dragomir and S. Wang obtained the following Ostrowski type inequality [2, p. 468]:

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and a differentiable on (a, b) . If $f' \in L_1(a, b)$ and there exists the constants γ, Γ so that*

$$(1.1) \quad \gamma \leq f'(x) \leq \Gamma \quad \text{for all } x \in (a, b),$$

then we have the inequality:

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma)$$

for all $x \in [a, b]$.

The proof used essentially the identity

$$(1.3) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t) dt$$

for all $x \in [a, b]$, where f is as above and the kernel $p(\cdot, \cdot) : [a, b]^2 \rightarrow \mathbf{R}$ is given by

$$(1.4) \quad p(x, t) := \begin{cases} t - a & \text{if } t \in [a, x] \\ t - b & \text{if } t \in (x, b] \end{cases}$$

and Grüss' integral inequality which says (see for example [1]) that:

$$(1.5) \quad \left| \frac{1}{b-a} \int_a^b g(x) h(x) dx - \frac{1}{b-a} \int_a^b g(x) dx \cdot \frac{1}{b-a} \int_a^b h(x) dx \right| \leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma)$$

provided $g, h : [a, b] \rightarrow \mathbf{R}$ are integrable and

$$(1.6) \quad \varphi \leq g(x) \leq \Phi, \quad \gamma \leq h(x) \leq \Gamma$$

for all $x \in [a, b]$.

The main aim of this paper is to point out a new estimation of the left membership of (1.2) and to apply it for special means and in Numerical Integration.

Date. December, 1998

1991 Mathematics Subject Classification. Primary 26 D 15; Secondary 41 A 55.

Key words and phrases. Ostrowski's Inequality, Numerical Integration, Special Means.

2 A NEW INTEGRAL INEQUALITY

The following results holds:

Theorem 2.1. Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b) , whose second derivative $f'' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) . Then we have the inequality

$$(2.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \\ \leq \frac{1}{2} \left\{ \left[\frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \|f''\|_\infty \\ \leq \frac{\|f''\|_\infty}{6} (b-a)^2$$

for all $x \in [a, b]$.

Proof. For the sake of completeness, we give a short proof of the identity (1.3) which will be used in the sequel.

Integrating by parts, we have

$$\int_a^x (t-a) f'(t) dt = (x-a) f(x) - \int_a^x f(t) dt$$

and

$$\int_x^b (t-b) f'(t) dt = (b-x) f(x) - \int_x^b f(t) dt.$$

Adding these two equalities, we get

$$\int_a^x (t-a) f'(t) dt + \int_x^b (t-b) f'(t) dt = (b-a) f(x) - \int_a^b f(t) dt$$

which is equivalent to (1.3).

Applying the identity (1.3) for $f'(\cdot)$ we can state

$$f'(t) = \frac{1}{b-a} \int_a^b f'(s) ds + \frac{1}{b-a} \int_a^b p(t,s) f''(s) ds$$

which is equivalent to

$$f'(t) = \frac{f(b) - f(a)}{b-a} + \frac{1}{b-a} \int_a^b p(t,s) f''(s) ds.$$

Substituting $f'(t)$ in the right membership of (1.3) we get

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt \\ + \frac{1}{b-a} \int_a^b p(x,t) \left[\frac{f(b) - f(a)}{b-a} + \frac{1}{b-a} \int_a^b p(t,s) f''(s) ds \right] dt$$

$$\begin{aligned}
&= \frac{1}{b-a} \int_a^b f(t) dt + \frac{f(b) - f(a)}{(b-a)^2} \int_a^b p(x,t) dt \\
&\quad + \frac{1}{(b-a)^2} \int_a^b \int_a^b p(x,t) p(t,s) f''(s) ds dt
\end{aligned}$$

and as

$$\begin{aligned}
\int_a^b p(x,t) dt &= \int_a^x (t-a) dt + \int_x^b (t-b) dt \\
&= (b-a) \left(x - \frac{a+b}{2} \right)
\end{aligned}$$

we get the integral identity:

$$\begin{aligned}
(2.2) \quad f(x) &= \frac{1}{b-a} \int_a^b f(t) dt + \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \\
&\quad + \frac{1}{(b-a)^2} \int_a^b \int_a^b p(x,t) p(t,s) f''(s) ds dt
\end{aligned}$$

for all $x \in [a, b]$.

Now, using the identity (2.2) we get

$$\begin{aligned}
(2.3) \quad &\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \\
&\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b |p(x,t) p(t,s)| |f''(s)| ds dt \\
&\leq \frac{\|f''\|_\infty}{(b-a)^2} \int_a^b \int_a^b |p(x,t)| |p(t,s)| ds dt.
\end{aligned}$$

We have

$$\int_a^b |p(t,s)| ds = \frac{(t-a)^2 + (b-t)^2}{2}.$$

Also

$$\begin{aligned}
A &:= \int_a^b |p(x,t)| \left[\frac{(t-a)^2 + (b-t)^2}{2} \right] dt \\
&= \frac{1}{2} \left[\int_a^x (t-a) [(t-a)^2 + (b-t)^2] dt + \int_x^b (b-t) [(t-a)^2 + (b-t)^2] dt \right]
\end{aligned}$$

$$= \frac{1}{2} \left[\int_a^x [(t-a)^3 + (t-a)(b-t)^2] dt + \int_x^b [(t-a)^2(b-t) + (b-t)^3] dt \right].$$

Note that

$$\begin{aligned} \int_a^x (t-a)^3 dt &= \frac{(x-a)^4}{4}, \\ \int_a^x (t-a)(b-t)^2 dt \\ &= -\frac{1}{3}(b-x)^3(x-a) - \frac{1}{12}(b-x)^4 + \frac{1}{12}(b-a)^4; \\ \int_x^b (t-b)(t-a)^2 dt \\ &= \frac{1}{3}(x-a)^3(b-x) - \frac{1}{12}(b-a)^4 + \frac{1}{12}(x-a)^4; \\ \int_x^b (t-b)^3 dt &= \frac{(x-b)^4}{4}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} A &= \frac{1}{2} \left[\frac{(x-a)^4}{4} - \frac{1}{3}(b-x)^3(x-a) - \frac{1}{12}(b-x)^4 + \frac{1}{12}(b-a)^4 \right. \\ &\quad \left. - \frac{1}{3}(x-a)^3(b-x) + \frac{1}{12}(b-a)^4 - \frac{1}{12}(x-a)^4 + \frac{(x-b)^4}{4} \right] \\ &= \frac{1}{12} \left[(x-a)^4 - 2(b-x)^3(x-a) - 2(x-a)^3(b-x) \right. \\ &\quad \left. + (b-x)^4 + (b-a)^4 \right]. \end{aligned}$$

Now, observe that,

$$(x-a)^4 + (b-x)^4 = [(x-a)^2 + (b-x)^2]^2 - 2(x-a)^2(b-x)^2$$

and

$$\begin{aligned} &-2(b-x)^3(x-a) - 2(x-a)^3(b-x) \\ &= -2(x-a)(b-x) [(x-a)^2 + (b-x)^2] \end{aligned}$$

then

$$\begin{aligned} B := 12A &= \left[(x-a)^2 + (b-x)^2 \right]^2 - 2(x-a)(b-x) \left[(x-a)^2 + (b-x)^2 \right] \\ &\quad - 2(x-a)^2(b-x)^2 + (b-a)^4 \\ &= \left[(x-a)^2 + (b-x)^2 - (x-a)(b-x) \right]^2 - 3(x-a)^2(b-x)^2 + (b-a)^4. \end{aligned}$$

But a simple calculation shows that

$$(x-a)^2 + (b-x)^2 = \frac{1}{2}(b-a)^2 + 2\left(x - \frac{a+b}{2}\right)^2$$

and as

$$(x-a)^2 + (b-x)^2 + 2(x-a)(b-x) = (b-a)^2$$

we get

$$2(x-a)(b-x) = (b-a)^2 - \left[(x-a)^2 + (b-x)^2 \right]$$

i.e.,

$$\begin{aligned} (x-a)(b-x) &= \frac{1}{2}(b-a)^2 - \frac{1}{2} \left[(x-a)^2 + (b-x)^2 \right] \\ &= \frac{1}{4}(b-a)^2 - \left(x - \frac{a+b}{2} \right)^2. \end{aligned}$$

Consequently,

$$\begin{aligned} B &= \left[\frac{1}{2}(b-a)^2 + 2\left(x - \frac{a+b}{2}\right)^2 - \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right]^2 - \\ &\quad - 3 \left[\frac{1}{4}(b-a)^2 - \left(x - \frac{a+b}{2}\right)^2 \right]^2 + (b-a)^4 \\ &= 6\left(x - \frac{a+b}{2}\right)^2 + 3(b-a)^2\left(x - \frac{a+b}{2}\right)^2 + \frac{7}{8}(b-a)^4 \end{aligned}$$

and then

$$A = \frac{1}{12} \left[6\left(x - \frac{a+b}{2}\right)^2 + 3(b-a)^2\left(x - \frac{a+b}{2}\right)^2 + \frac{7}{8}(b-a)^4 \right].$$

Now, using the inequality (2.3) and simple algebraic manipulations, we get the first result in (2.1).

The second part is obvious by the fact that

$$\left| x - \frac{a+b}{2} \right| \leq \frac{b-a}{2}$$

for all $x \in [a, b]$. ■

3 APPLICATIONS IN NUMERICAL INTEGRATION

Let $I_h : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) a sequence of intermediate points and $h_i := x_{i+1} - x_i$ ($i = 0, 1, \dots, n-1$). As in [1], consider the perturbed Riemann's sum defined by

$$(3.1) \quad A_G(f, I_h, \xi) := \sum_{i=0}^{n-1} f(\xi_i) h_i - \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) (f(x_{i+1}) - f(x_i)).$$

In that paper Dragomir and Wang proved the following result:

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative $f' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) and assume that*

$$(3.2) \quad \gamma \leq f'(x) \leq \Gamma \quad \text{for all } x \in (a, b).$$

Then we have the quadrature formula:

$$(3.3) \quad \int_a^b f(x) dx = A_G(f, I_h, \xi) + R_G(f, I_h, \xi)$$

and the remainder $R_G(f, I_h, \xi)$ satisfies the estimation

$$(3.4) \quad |R_G(f, I_h, \xi)| \leq \frac{1}{4} (\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^2,$$

for all $\xi = (\xi_0, \dots, \xi_{n-1})$ as above.

Here, we prove another type of estimation for the remainder $R_G(f, I_h, \xi)$ in the case when f is twice differentiable.

Theorem 3.2. *Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b) , whose second derivative $f'' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) . Denote $\|f''\|_\infty := \sup_{t \in (a, b)} |f''(t)| <$*

∞ . Then we have the quadrature formula (3.3) and the remainder $R_G(f, I_h, \xi)$ satisfies the estimation:

$$(3.5) \quad |R_G(f, I_h, \xi)| \leq \frac{\|f''\|_\infty}{2} \sum_{i=0}^{n-1} \left\{ \left[\frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2}{h_i^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} h_i^3 \\ \leq \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} h_i^3$$

for all ξ_i as above.

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) to obtain

$$\left| f(\xi_i) h_i - \int_{x_i}^{x_{i+1}} f(t) dt - \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) (f(x_{i+1}) - f(x_i)) \right| \\ \leq \frac{\|f''\|_\infty}{2} \left[\left[\frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2}{h_i^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right] h_i^3 \leq \frac{\|f''\|_\infty}{6} h_i^3$$

for all $\xi_i \in [x_i, x_{i+1}]$ and $i \in \{0, \dots, n-1\}$.

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired inequality (3.5).

We omit the details. ■

4 APPLICATIONS FOR SPECIAL MEANS

Recall the following means :

(a) The arithmetic mean

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0;$$

(b) The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;$$

(c) The harmonic mean:

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b \geq 0;$$

(d) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \quad a, b > 0;$$

(e) The identric mean:

$$I := I(a, b) = \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} \quad a, b > 0;$$

(f) The p -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases}$$

where $p \in \mathbf{R} \setminus \{-1, 0\}$ and $a, b > 0$.

The following simple relationships are well known in the literature

$$(4.1) \quad H \leq G \leq L \leq I \leq A$$

and

$$(4.2) \quad L_p \text{ is monotonically increasing in } p \in \mathbf{R} \text{ with } L_0 := I \text{ and } L_{-1} := L.$$

1. Consider the mapping $f(x) = x^p$ ($p \geq 2$) on $[a, b] \subset (0, \infty)$.

Applying the inequality (2.1) for $f(x) = x^p$, we get:

$$(4.3) \quad \left| x^p - L_p^p - pL_{p-1}^{p-1}(x-A) \right| \\ \leq \frac{p(p-1)b^{p-2}}{2} \left\{ \left[\frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2$$

$$\leq \frac{p(p-1)b^{p-2}}{6}(b-a)^2$$

for all $x \in [a, b]$.

Choosing in (4.3), $x = A$, we get

$$(4.4) \quad 0 \leq L_p^p - A^p \leq \frac{7}{96}p(p-1)b^{p-2}(b-a)^2.$$

2. Consider the mapping $f(x) = \frac{1}{x}$ ($x \in [a, b] \subset (0, \infty)$).

Applying the inequality (2.1) for this mapping we get:

$$(4.5) \quad \left| \frac{1}{x} - \frac{1}{L} - \frac{x-A}{G^2} \right|$$

$$\leq \frac{1}{3a^3} \left\{ \left[\frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \leq \frac{1}{3a^3} (b-a)^2$$

for all $x \in [a, b]$.

Choosing in (4.5) $x = A$, we get

$$(4.6) \quad 0 \leq \frac{A-L}{AL} \leq \frac{7}{48a^3} (b-a)^2.$$

Also, choosing in (4.5) $x = L$, we get

$$(4.7) \quad 0 \leq \frac{A-L}{G^2} \leq \frac{1}{3a^3} \left\{ \left[\frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2$$

$$\leq \frac{1}{3a^3} (b-a)^2.$$

3. Finally, let us consider the mapping $f(x) = -\ln x$ ($x \in [a, b] \subset (0, \infty)$). Then, by (2.1), we get:

$$(4.8) \quad \left| \ln \left(\frac{I \left(\frac{b}{a} \right)^{\frac{x-A}{b-a}}}{x} \right) \right| \leq \frac{1}{2a^2} \left\{ \left[\frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \leq \frac{1}{6a^2} (b-a)^2$$

for all $x \in [a, b]$.

Putting $x = A$ in (4.8) we get

$$(4.9) \quad 1 \leq \frac{A}{I} \leq \exp \left[\frac{7}{96a^2} (b-a)^2 \right].$$

Acknowledgement. The authors would like to thank *Professor* Peter Cerone for his suggestions in improving this paper.

REFERENCES

- [1] S.S. DRAGOMIR, and S. WANG, An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.* **33**(1997), 15-20.
- [2] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publisher, Dordrecht, 1994.

SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO Box 14428,
MELBOURNE CITY MC, VICTORIA 8001, AUSTRALIA.

E-mail address: {sever, neil}@matilda.vut.edu.au

