



VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

Some Estimations of Kraft Numbers and Related Results

This is the Published version of the following publication

Dragomir, Nicoleta, Dragomir, Sever S and Pranesh, Kumar (1998) Some Estimations of Kraft Numbers and Related Results. RGMIA research report collection, 1 (2).

The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository <https://vuir.vu.edu.au/17120/>

SOME ESTIMATIONS OF KRAFT NUMBERS AND RELATED RESULTS

N.M. DRAGOMIR, S.S. DRAGOMIR AND K. PRANESH

ABSTRACT. Some inequalities for Kraft numbers which are important in coding theory [2, 3], for they lead to a simple criterion to determine whether or not there is an instantaneous code with given codeword lengths, are pointed out.

1 INTRODUCTION

The following remarkable theorem, published by L.G. Kraft in 1949 gives a simple criterion to determine whether or not there is an instantaneous code [1, p. 43] with given code word lengths [1, p. 44].

Theorem 1.1. (*Kraft's Theorem*) *We have*

1. *If C is an r -ary instantaneous code with code word lengths l_1, \dots, l_n , then these lengths must satisfy Kraft's inequality*

$$(1.1) \quad \sum_{k=1}^n \frac{1}{r^{l_k}} \leq 1.$$

2. *If the numbers l_1, l_2, \dots, l_n and r satisfy Kraft's inequality (1.1), then there is an instantaneous r -ary code with codeword lengths l_1, \dots, l_n .*

It is interesting to observe that Kraft's inequality is also necessary and sufficient for the existence of a uniquely decipherable code. Of course, Kraft's inequality is sufficient since any instantaneous code is also uniquely decipherable. The necessity of Kraft's inequality was proved by McMillan in 1956 [1, p. 47]:

Theorem 1.2. (*McMillan's Theorem*). *If $C = \{c_1, \dots, c_n\}$ is a uniquely decipherable r -ary code, then its code word lengths must satisfy Kraft's inequality (1.1).*

Define now for an r -ary code C having the code word lengths l_1, \dots, l_n the Kraft numbers

$$K_r(l_1, \dots, l_n) = \sum_{k=1}^n \frac{1}{r^{l_k}}.$$

In what follows we shall point out some new inequalities for Kraft numbers which are closely connected with the inequalities (1.1). Some related results with Kraft's theorem are also given.

2 THE RESULTS

We shall start with the following lemma which is of interest in itself.

Lemma 2.1. *Let r, l_i ($i = 1, \dots, n$) be real numbers with $r > 1$. Then we have the double inequality*

$$(2.1) \quad \ln r \sum_{i=1}^n \frac{\log_r(r^{l_i})}{r^{l_i}} \leq 1 - \sum_{i=1}^n \frac{1}{r^{l_i}} \leq \ln r \left[\frac{1}{n} \sum_{i=1}^n l_i - \log_r n \right].$$

The equality holds iff $l_i = \log_r n$ for all $i \in \{1, \dots, n\}$.

Proof. The exponential map $f : \mathbf{R} \rightarrow (0, \infty)$, $f(x) = r^x$ is strictly convex on \mathbf{R} .

Recall that for a convex mapping f which is differentiable on its domain, we have the double inequality:

$$(2.2) \quad f'(y)(x-y) \leq f(x) - f(y) \leq f'(x)(x-y)$$

for all x, y in the domain of f .

As $f'(x) = r^x \ln r$, then by (2.2) we get

$$(2.3) \quad r^y(x-y) \ln r \leq r^x - r^y \leq r^x(x-y) \ln r, \quad x, y \in \mathbf{R}.$$

Now if we choose into the inequality (2.3) $x = -l_i, y = \log_r\left(\frac{1}{n}\right)$ we deduce

$$r^{-l_i} \left[-l_i - \log_r\left(\frac{1}{n}\right) \right] \ln r \geq r^{-l_i} - \frac{1}{n} \geq \frac{1}{n} \left[-l_i - \log_r\left(\frac{1}{n}\right) \right] \ln r$$

for all $i \in \{1, \dots, n\}$, which is equivalent to:

$$(2.4) \quad (l_i - \log_r n) r^{-l_i} \ln r \leq \frac{1}{n} - \frac{1}{r^{l_i}} \leq \frac{1}{n} (l_i - \log_r n) \ln r$$

for all $i \in \{1, \dots, n\}$.

Summing in (2.4) over i from 1 to n , we deduce (2.1). The case of equality follows by the strict convexity of the mapping $f(x) = r^x$ ($r > 1, x \in \mathbf{R}$). We shall omit the details. ■

Theorem 2.2. *Let $C = (c_1, \dots, c_n)$ be an r -ary code having the codeword lengths l_1, \dots, l_n . Then we have the estimation for the Kraft's number:*

$$(2.5) \quad \frac{1}{n \ln r} \sum_{i=1}^n \left[\ln(nr) - l_i [\ln r]^2 \right] \leq K_r(l_1, \dots, l_n) \\ \leq \frac{1}{n \ln r} \sum_{i=1}^n \left[\frac{r^{l_i} \ln r + n \ln n - n l_i [\ln r]^2}{r^{l_i}} \right].$$

The equality holds iff $l_i = \log_r n$.

Proof. By Lemma 2.1, we have

$$K_r(l_1, \dots, l_n) \geq 1 - \ln r \left[\frac{1}{n} \sum_{i=1}^n l_i - \log_r n \right] \\ = \frac{1}{n \ln r} \sum_{i=1}^n \left[\ln(nr) - l_i (\ln r)^2 \right]$$

and

$$\begin{aligned} K_r(l_1, \dots, l_n) &\leq 1 - \ln r \sum_{i=1}^n \frac{\log_r \left(\frac{r^{l_i}}{n} \right)}{r^{l_i}} \\ &= \frac{1}{n \ln r} \sum_{i=1}^n \left[\frac{r^{l_i} \ln r + n \ln r - n l_i [\ln r]^2}{r^{l_i}} \right]. \end{aligned}$$

The case of equality is obvious by the same lemma. ■

Corollary 2.3. Let $C = (c_1, \dots, c_n)$ be an r -ary code having the codeword lengths l_1, \dots, l_n . If

$$(2.6) \quad \frac{1}{n} (l_1 + \dots + l_n) < \log_r n,$$

then C is not uniquely decipherable.

Corollary 2.4. If the real numbers $r, l_i (i = 1, \dots, n)$ satisfy the inequality:

$$(2.7) \quad \frac{\sum_{i=1}^n \frac{l_i}{r^{l_i}}}{\sum_{i=1}^n \frac{1}{r^{l_i}}} \geq \log_r n$$

then there is an instantaneous r -ary code with codeword lengths l_1, \dots, l_n .

Proof. Note that the inequality (2.7) is clearly equivalent to

$$\sum_{i=1}^n \frac{l_i - \log_r n}{r^{l_i}} \geq 0$$

but by the inequality (2.1) we have

$$0 \leq \ln r \sum_{i=1}^n \frac{l_i - \log_r n}{r^{l_i}} \leq 1 - K_r(l_1, \dots, l_n)$$

and, then

$$K_r(l_1, \dots, l_n) \leq 1.$$

Applying Kraft's theorem we deduce the desired conclusion. ■

Lemma 2.5. Let $r, l_i \geq 1 (i = 1, \dots, n)$ be real numbers. Then we have the inequality:

$$(2.8) \quad \frac{1}{n} \sum_{i=1}^n l_i \left(1 - \frac{n^{\frac{1}{l_i}}}{r} \right) \geq 1 - \sum_{i=1}^n \frac{1}{r^{l_i}} \geq r \sum_{i=1}^n \frac{l_i}{r^{l_i} n^{\frac{1}{l_i}}} \left(1 - \frac{n^{\frac{1}{l_i}}}{r} \right).$$

The equality holds iff $l_i = \log_r n, i = 1, \dots, n$.

Proof. The mapping $g(x) = x^p, p \geq 1$ is strictly convex on $(0, \infty)$ so by the inequality (2.2), we have the inequality

$$(2.9) \quad pb^{p-1}(a-b) \leq a^p - b^p \leq pa^{p-1}(a-b)$$

for all $a, b \in [0, \infty)$.

Let choose in (2.9)

$$p = l_i \geq 1, \quad a = \frac{1}{r}, \quad b = \left(\frac{1}{r}\right)^{\frac{1}{l_i}}$$

to get for all $i \in \{1, \dots, n\}$

$$l_i \left(\frac{1}{n}\right)^{\frac{l_i-1}{l_i}} \left(\frac{1}{r} - \left(\frac{1}{n}\right)^{\frac{1}{l_i}}\right) \leq r^{-l_i} - \frac{1}{n} \leq l_i \left(\frac{1}{r}\right)^{l_i-1} \left(\frac{1}{r} - \left(\frac{1}{n}\right)^{\frac{1}{l_i}}\right)$$

which is equivalent to

$$(2.10) \quad \frac{1}{rn} l_i n^{\frac{1}{l_i}} - \frac{l_i}{n} \leq r^{-l_i} - \frac{1}{n} \leq l_i \left(\frac{1}{r}\right)^{l_i} - l_i \left(\frac{1}{r}\right)^{l_i-1} \left(\frac{1}{n}\right)^{\frac{1}{l_i}}$$

for all $i \in \{1, \dots, n\}$.

Summing into the inequality (2.10) over i from 1 to n , we derive

$$\frac{1}{rn} \sum_{i=1}^n l_i n^{\frac{1}{l_i}} - \frac{1}{n} \sum_{i=1}^n l_i \leq \sum_{i=1}^n \frac{1}{r^{l_i}} - 1 \leq \sum_{i=1}^n l_i \left(\frac{1}{r}\right)^{l_i} - \sum_{i=1}^n \frac{l_i}{r^{l_i-1}} \frac{1}{n^{\frac{1}{l_i}}}$$

which is equivalent to (2.8).

The case of equality holds from the strict convexity of g and taking into account that $\frac{1}{r} = \left(\frac{1}{n}\right)^{\frac{1}{l_i}}$ iff $\frac{1}{l_i} \log_r \frac{1}{n} = -1$, i.e., $l_i = \log_r n, i = 1, \dots, n$. ■

In the following theorem we give an estimation of Kraft numbers $K_r(l_1, \dots, l_n)$ holds.

Theorem 2.6. *Let $C = (c_1, \dots, c_n)$ be an r -ary code with the codeword lengths l_1, \dots, l_n . Then we have the estimation*

$$(2.11) \quad \frac{1}{nr} \sum_{i=1}^n \left[n^{\frac{1}{l_i}+1} - r(l_i - 1) \right] \leq K_r(l_1, \dots, l_n) \\ \leq \frac{1}{nr} \sum_{i=1}^n \left[\frac{r^{l_i+1} n^{\frac{1}{l_i}} \left(n^{\frac{1}{l_i}+1} + 1 \right) - nr^2 l_i}{r^{l_i} n^{\frac{1}{l_i}}} \right]$$

The equality holds in (2.11) iff $l_i = \log_r n$.

Proof. By Lemma 2.5 we have

$$K_r(l_1, \dots, l_n) \geq 1 - \frac{1}{n} \sum_{i=1}^n \left(l_i - \frac{n^{\frac{1}{l_i}}}{r} \right) = \frac{1}{nr} \sum_{i=1}^n \left[r(1 - l_i) + n^{\frac{1}{l_i}+1} \right] \\ = \frac{1}{nr} \sum_{i=1}^n \left[n^{\frac{1}{l_i}+1} - r(l_i - 1) \right]$$

and

$$K_r(l_1, \dots, l_n) \leq 1 - \sum_{i=1}^n \left[\frac{r l_i}{r^{l_i} n^{\frac{1}{l_i}}} - n^{\frac{1}{l_i}} \right] \\ = \frac{1}{nr} \sum_{i=1}^n \left[\frac{r^{l_i+1} n^{\frac{1}{l_i}} \left(1 + n^{\frac{1}{l_i}+1} \right)}{r^{l_i} n^{\frac{1}{l_i}}} \right]$$

and the inequality (2.11) is proved. The case of equality follows by Lemma 2.5, too. ■

Proposition 2.7. Let $C = (c_1, \dots, c_n)$ be an r -ary code with the codeword lengths l_1, \dots, l_n . If

$$(2.12) \quad \frac{\sum_{i=1}^n l_i n^{\frac{1}{l_i}}}{\sum_{i=1}^n l_i} > r$$

then C is not uniquely decipherable.

Proof. If we would assume that C is uniquely decipherable, then by McMillan's theorem we have that $K_r(l_1, \dots, l_n) \geq 1$ which implies

$$0 \leq 1 - K_r(l_1, \dots, l_n) \leq \sum_{i=1}^n l_i \left(1 - \frac{n^{\frac{1}{l_i}}}{r}\right)$$

and then $\sum_{i=1}^n l_i \geq \frac{1}{r} \sum_{i=1}^n n^{\frac{1}{l_i}} l_i$ which contradicts (2.12). ■

Finally, we obtain the following sufficient condition for the existence of an instantaneous code having a given the non-negative integers r and the lengths l_1, \dots, l_n .

Theorem 2.8. If the non-negative integers r, l_i ($i = 1, \dots, n$) satisfy the inequality:

$$r \geq \frac{\sum_{i=1}^n \frac{l_i}{r^{l_i}}}{\sum_{i=1}^n \frac{l_i}{n^{\frac{1}{l_i}}}}$$

then there is one instantaneous r -ary code with codeword lengths l_1, \dots, l_n .

The proof follows by Lemma 2.5 and Kraft's theorem. We shall omit the details.

REFERENCES

- [1] S. ROMAN, *Coding and Information Theory*, Springer Verlag, Berlin, 1992.
- [2] T. M. COVER and J. A. THOMAS, *Elements of Information Theory*, Wiley, New York, 1991.
- [3] C. CLAUDE and C. GROZEA, Kraft-Chaitin Inequality Revisited, *J. UCS*, 5(1996), 306-310.

(N.M. DRAGOMIR AND S.S. DRAGOMIR) SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO BOX 14428, MELBOURNE CITY MC, VICTORIA 8001, AUSTRALIA.

(K. PRANESH) UNIVERSITY OF TRANSKEI, PRIVATE BAG X1, UNITRA UMTATA, 5117, SOUTH AFRICA

E-mail address: N.M. Dragomir nico@matilda.vut.edu.au

S.S. Dragomir sever@matilda.vut.edu.au

K. Pradesh kumar@getafix.utr.ac.za

⁰ The present paper is accepted for publication in *Soochow J. of Math.*, Taiwan.

