



**VICTORIA UNIVERSITY**  
MELBOURNE AUSTRALIA

*On Some Shift Invariant Multivariate, Integral Operators, Revisited.*

This is the Published version of the following publication

Gal, Sorin G and Anastassiou, George A (1999) On Some Shift Invariant Multivariate, Integral Operators, Revisited. RGMIA research report collection, 2 (1).

The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository <https://vuir.vu.edu.au/17125/>

## A NOTE ON BOUNDS FOR THE ESTIMATION ERROR VARIANCE OF A CONTINUOUS STREAM WITH STATIONARY VARIOGRAM

N.S. BARNETT AND S.S. DRAGOMIR

ABSTRACT. In this paper, by the use of an Ostrowski type integral inequality for double integrals, we establish an upper bound for the estimation error variance of a continuous stream with stationary variogram.

### 1 INTRODUCTION

In [1], the authors considered  $X(t)$  as defining the quality of a product at time  $t$  where  $X(t)$  is a continuous time stochastic process which may be non-stationary. Typically,  $X(t)$  represents a continuous stream industrial process such as is common in many areas of the chemical industry. The paper was concerned with issues related to sampling the stream with a view to estimating the mean quality characteristic of the flow,  $\bar{X}$ , over the interval  $[0, d]$ . Specifically, focus was on obtaining the sampling location, said to be optimal, which minimizes the estimation error variance,  $E\left[(\bar{X} - X(t))^2\right]$ ,  $0 < t < d$ .

Given that  $t$  is as specified, the problem is to find the value of  $t$  (the sampling location) that minimizes  $E\left[(\bar{X} - X(t))^2\right]$ . It is shown that for constant stream flows the optimal sampling point is the mid point of  $[0, d]$  for situations where the process variogram,

$$V(u) = \frac{1}{2}E\left[(X(t) - X(t+u))^2\right],$$

$$V(0) = 0, V(-u) = V(u),$$

is stationary (note that variogram stationarity is not equivalent to process stationarity).

The paper continues to consider optimal sampling locations for situations where the stream flow rate varies. The optimal sampling location is seen to depend on both the flow rate function and the form of the process variogram - some examples are given.

In this note, rather than focusing on the optimal sampling point, we focus on the actual value of the estimation error variance itself. In particular, we focus on obtaining an upper bound for its value. To do this, we use a result obtained in [2], an inequality of the Ostrowski type. For other results of this type see Chapter XV of [7] and [3, 4, 5, 6].

### 2 AN OSTROWSKI INEQUALITY FOR DOUBLE INTEGRALS

Let  $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$  be so that  $f(\cdot, \cdot)$  is continuous on  $[a, b] \times [c, d]$ ,  $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$  exists on  $(a, b) \times (c, d)$  and is bounded, i.e.,

$$\|f''_{s,t}\|_{\infty} := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty.$$

---

*Date.* December, 1998

*1991 Mathematics Subject Classification.* Primary 65Xxx; Secondary 26D15.

*Key words and phrases.* Error Variance, Stationary Variogram Ostrowski's Inequality

Then we have the inequality:

$$(2.1) \quad \left| \int_a^b \int_c^d f(s, t) ds dt - \left[ (b-a) \int_c^d f(x, t) dt + (d-c) \int_a^b f(s, y) ds \right. \right. \\ \left. \left. - (d-c)(b-a)f(x, y) \right] \right| \\ \leq \left[ \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \times \left[ \frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2 \right] \|f''_{s,t}\|_{\infty}$$

for all  $(x, y) \in [a, b] \times [c, d]$ .

For the sake of completeness we give here a short proof of this inequality.

Integrating by parts successively, we have the equality:

$$(2.2) \quad \int_a^x \int_c^y (s-a)(t-c) f''_{s,t}(s, t) dt ds \\ = (y-c)(x-a)f(x, y) - (y-c) \int_a^x f(s, y) ds \\ - (x-a) \int_c^y f(x, t) dt + \int_a^x \int_c^y f(s, t) ds dt.$$

Also, by similar computations we have

$$(2.3) \quad \int_a^x \int_y^d (s-a)(t-d) f''_{s,t}(s, t) ds dt \\ = (x-a)(d-y)f(x, y) - (d-y) \int_a^x f(s, y) ds \\ - (x-a) \int_y^d f(x, t) dt + \int_a^x \int_c^y f(s, t) ds dt.$$

Now,

$$(2.4) \quad \int_x^b \int_y^d (s-b)(t-d) f''_{s,t}(s, t) ds dt \\ = (d-y)(b-x)f(x, y) - (d-y) \int_x^b f(s, y) ds$$

$$-(b-x) \int_y^d f(x,t) dt + \int_x^b \int_y^d f(s,t) dsdt$$

and finally

$$(2.5) \quad \int_x^b \int_c^y (s-b)(t-c) f''_{s,t}(s,t) dsdt$$

$$= (y-c)(b-x) f(x,y) - (y-c) \int_x^b f(s,y) ds$$

$$- (b-x) \int_c^y f(x,t) dt + \int_x^b \int_c^y f(s,t) dsdt.$$

If we add the equalities (2.2) – (2.5) we get, in the right membership:

$$[(y-c)(x-a) + (x-a)(d-y)$$

$$+ (d-y)(b-x) + (y-c)(b-x)] f(x,y)$$

$$- (d-c) \int_a^x f(s,y) ds - (d-c) \int_x^b f(s,y) ds - (b-a) \int_c^y f(x,t) dt$$

$$- (b-a) \int_y^d f(x,t) dt + \int_a^x \int_c^y f(s,t) dsdt + \int_a^x \int_y^d f(s,t) dsdt$$

$$+ \int_x^b \int_y^d f(s,t) dsdt + \int_x^b \int_c^y f(s,t) dsdt$$

$$= (d-c)(b-a) f(x,y) - (d-c) \int_a^b f(s,y) ds$$

$$- (b-a) \int_c^b f(x,t) dt + \int_a^b \int_c^d f(s,t) dsdt.$$

For the first membership, let us define the kernels:  $p : [a, b]^2 \rightarrow \mathbf{R}$ ,  $q : [c, d]^2 \rightarrow \mathbf{R}$  given by:

$$p(x, s) := \begin{cases} s - a & \text{if } s \in [a, x] \\ s - b & \text{if } s \in (x, b] \end{cases}$$

and

$$q(y, t) := \begin{cases} t - c & \text{if } t \in [c, y] \\ t - d & \text{if } t \in (y, d] \end{cases}.$$

Now, using this notation, we deduce that the left membership can be represented as :

$$\int_a^b \int_c^d p(x, s) q(y, t) f''_{s,t}(s, t) ds dt.$$

Consequently, we get the identity

$$\begin{aligned} (2.6) \quad & \int_a^b \int_c^d p(x, s) q(y, t) f''_{s,t}(s, t) ds dt \\ &= (d - c)(b - a) f(x, y) - (d - c) \int_a^b f(s, y) ds \\ & \quad - (b - a) \int_c^d f(x, t) dt + \int_a^b \int_c^d f(s, t) ds dt \end{aligned}$$

for all  $(x, y) \in [a, b] \times [c, d]$ .

Now, using the identity (2.6) we get,

$$\begin{aligned} & \left| \int_a^b \int_c^d f(s, t) ds dt - [(b - a) \int_c^d f(x, t) dt + (d - c) \int_a^b f(s, y) ds \right. \\ & \quad \left. - (d - c)(b - a) f(x, y)] \right| \\ & \leq \int_a^b \int_c^d |p(x, s)| |q(y, t)| |f''_{s,t}(s, t)| ds dt \\ & \leq \|f''_{s,t}\|_{\infty} \int_a^b \int_c^d |p(x, s)| |q(y, t)| ds dt. \end{aligned}$$

Observe that

$$\int_a^b |p(x, s)| ds = \frac{1}{4} (b - a)^2 + \left(x - \frac{a + b}{2}\right)^2$$

and,

$$\int_c^d |q(y, t)| dt = \frac{1}{4} (d - c)^2 + \left(y - \frac{c + d}{2}\right)^2.$$

Finally, using (2.6), we get the desired inequality (2.1).

3 BOUND ON ESTIMATION ERROR VARIANCE. CONSTANT FLOW RATE

From [1], using an identity given in [8], it can be shown that

$$(3.1) \quad E \left[ (\bar{X} - X(t))^2 \right] \\ = -\frac{1}{d^2} \int_0^d \int_0^d V(v-u) dudv + \frac{2}{d} \left\{ \int_0^t V(u) du + \int_0^{d-t} V(u) du \right\}.$$

Assume that  $V$  is twice differentiable on  $(-d, d)$  and having the second derivative  $V$  bounded on that interval.

Applying inequality [1] for the mapping  $f(u, v) = V(v - u)$  we can state the inequality

$$\left| \int_0^d \int_0^d V(v-u) dudv - \left[ d \int_0^d V(v-x) dv + d \int_0^d V(y-u) du - d^2 V(y-x) \right] \right| \\ \leq \left[ \frac{1}{4} d^2 + \left( x - \frac{d}{2} \right)^2 \right] \left[ \frac{1}{4} d^2 + \left( y - \frac{d}{2} \right)^2 \right] \|V''\|_\infty$$

for all  $x, y \in [0, d]$ .

Let  $x = y = t$ . Then we get

$$(3.2) \quad \left| \int_0^d \int_0^d V(v-u) dudv - \left[ d \int_0^d V(v-t) dv + d \int_0^d V(t-u) du \right] \right| \\ \leq \left[ \frac{1}{4} d^2 + \left( t - \frac{d}{2} \right)^2 \right]^2 \|V''\|_\infty$$

as  $V(0) = 0$ .

Now, observe that

$$\int_0^d V(v-t) dv = \int_0^t V(u) du + \int_0^{d-t} V(u) du$$

and

$$\int_0^d V(t-u) du = \int_0^t V(u) du + \int_0^{d-t} V(u) du.$$

By the inequality (3.2) we get that

$$\left| \int_0^d \int_0^d V(v-u) dudv - 2d \left[ \int_0^t V(v) dv + \int_0^{d-t} V(v) dv \right] \right| \\ \leq \left[ \frac{1}{4} d^2 + \left( t - \frac{d}{2} \right)^2 \right]^2 \|V''\|_\infty$$

and dividing by  $d^2$

$$\left| \frac{1}{d^2} \int_0^d \int_0^d V(v-u) dudv - \frac{2}{d} \left[ \int_0^t V(v)dv + \int_0^{d-t} V(v)dv \right] \right|$$

$$\leq \left[ \frac{1}{4} + \frac{(t - \frac{d}{2})^2}{d^2} \right]^2 d^2 \|V''\|_\infty.$$

Using the equation (3.1) we conclude that the following inequality for the variance  $E [(\bar{X} - X(t))^2]$  holds

$$(3.3) \quad E [(\bar{X} - X(t))^2] \leq \left[ \frac{1}{4} + \frac{(t - \frac{d}{2})^2}{d^2} \right]^2 d^2 \|V''\|_\infty.$$

Note that the best inequality we can get from (3.3) is that one for which  $t = t_o = \frac{d}{2}$  giving the bound

$$E [(\bar{X} - X(t_o))^2] \leq \frac{d^2}{16} \|V''\|_\infty.$$

**Remark 3.1.** *It should be noted that this result requires double differentiability of  $V$  in  $(-d, d)$  and that this condition does not hold for the case of a linear variogram, i.e.,*

$$V(u) = a |u|, \quad u \in \mathbf{R}.$$

#### REFERENCES

- [1] N.S. BARNETT, I.S. GOMM and L. ARMOUR, Location of the optimal sampling point for the quality assessment of continuous streams, *Australian J. Statistics*, **37**(2), 1995, 145-152.
- [2] N.S. BARNETT and S.S. DRAGOMIR, An Ostrowski's type inequality for double integrals and applications to cubature formulae, *submitted*.
- [3] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in  $L_1$  norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28**(1997), 239-244.
- [4] S.S. DRAGOMIR and S. WANG, An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.*, **33**(1997), 15-20.
- [5] S.S. DRAGOMIR and S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11**(1998), 105-109.
- [6] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in  $L_p$  norm and applications to some special means and to some numerical quadrature rules, *submitted*.
- [7] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK: *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1994.

- [8] I.W. SAUNDERS, G.K. ROBINSON, T. LWIN and R.J. HOLMES, A simplified variogram method for the estimation error variance in sampling from continuous stream, *Internat. J. Mineral Processing*, **25**(1989), 175-198.

SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO Box 14428,  
MELBOURNE CITY MC, VICTORIA 8001, AUSTRALIA.  
*E-mail address:* {neil, sever}@matilda.vut.edu.au

---

<sup>0</sup> The present paper is accepted for publication in Journal KSIAM, Korea



