An Inequality of Ostrowski Type and its Applications for Simpson's Rule and Special Means

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AN INEQUALITY OF OSTROWSKI TYPE FOR Mappings Whose Second Derivatives Belong to $L_1(A, B)$ AND APPLICATIONS

P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS

Abstract. An inequality of Ostrowski type for twice differentiable mappings whose derivatives belong to $L_1(a, b)$ and applications in Numerical Integration and for special means (logarithmic mean, identric mean, p-logarithmic mean etc...) are given.

1 Introduction

In 1938, Ostrowski (see for example [3, p. 468]) proved the following integral inequality:

**Theorem 1.1.** Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^o$ ($I^o$ is the interior of $I$), and let $a, b \in I^o$ with $a < b$. If $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on $(a, b)$, i.e., $\|f'\|_{\infty} := \sup_{t \in (a, b)} |f'(t)| < \infty$, then we have the inequality:

$$
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \leq \frac{1}{4} + \frac{\left( \frac{x-a+b}{2} \right)^2}{(b-a)^2} (b-a) \|f'\|_{\infty}
$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer to the recent paper [1] by S.S. Dragomir and S. Wang.

In paper [2], the same authors considered another inequality of Ostrowski type for $\|\|_1$ norm as follows:

**Theorem 1.2.** Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping in $I^o$ and $a, b \in I^o$ with $a < b$. If $f' \in L_1[a, b]$, then we have the inequality:

$$
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \leq \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{(b-a)} \|f'\|_1
$$

for all $x \in [a, b]$.

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski’s inequality for $n$-time differentiable mappings (see for example [3, p. 468]) from which we would like to mention only the case of twice differentiable mappings [3, p. 470]:

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Theorem 1.3. Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable mapping such that \( f'' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\), i.e., \( \| f'' \|_{\infty} = \sup_{t \in (a, b)} |f''(t)| < \infty \). Then we have the inequality:

\[
\begin{align*}
12 & \left| \frac{1}{2} \left[ f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right] - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \\
& \leq \frac{\| f'' \|_{\infty}}{4} (b-a)^2 \left[ \frac{1}{12} + \frac{(x-a+b)^2}{(b-a)^2} \right]
\end{align*}
\]

for all \( x \in (a, b) \).

In this paper we point out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the \( \| \cdot \|_1 \) -norm of the second derivative \( f'' \) and apply it in numerical integration and for some special means such as: logarithmic mean, identric mean, p-logarithmic mean etc.

2 Some Integral Inequalities

The following inequality of Ostrowski's type for mappings which are twice differentiable, holds.

Theorem 2.1. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), twice differentiable on \((a, b)\) and \( f'' \in L_1(a, b) \). Then we have the inequality:

\[
\begin{align*}
& \left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt - \left( x - \frac{a+b}{2} \right) f'(x) \right| \\
& \leq \frac{1}{2(b-a)} \left( \left| x - \frac{a+b}{2} \right| + \frac{1}{2} (b-a) \right)^2 \| f'' \|_1 \leq \frac{b-a}{2} \| f'' \|_1
\end{align*}
\]

for all \( x \in [a, b] \).

Proof. Let us define the mapping \( K(\cdot, \cdot) : [a, b]^2 \to \mathbb{R} \) given by

\[
K(x, t) := \begin{cases} 
\frac{(t-a)^2}{2} & \text{if } t \in [a, x] \\
\frac{(t-b)^2}{2} & \text{if } t \in (x, b]
\end{cases}
\]

Integrating by parts, we have successively

\[
\begin{align*}
\int_{a}^{b} K(x, t) f''(t) \, dt \\
= \int_{a}^{x} \frac{(t-a)^2}{2} f''(t) \, dt + \int_{x}^{b} \frac{(t-b)^2}{2} f''(t) \, dt \\
= \left( \frac{t-a}{2} \right)^2 f'(t) \bigg|_{a}^{x} - \int_{a}^{x} \left( t-a \right) f'(t) \, dt + \frac{(t-b)^2}{2} f'(t) \bigg|_{x}^{b} - \int_{x}^{b} (t-b) f'(t) \, dt
\end{align*}
\]
\[(x - a)^2 \frac{f'(x)}{2} - \left[ (t - a) f(t) \right]_a^x - \int_a^x f(t) dt \]
\[- \frac{(b - x)^2}{2} \frac{f'(x)}{2} - \left[ (t - b) f(t) \right]_x^b - \int_x^b f(t) dt \]
\[= \frac{1}{2} \left[ (x - a)^2 - (b - x)^2 \right] f'(x) - (x - a) f(x) \]
\[+ \int_a^x f(t) dt + (x - b) f(x) + \int_x^b f(t) dt \]
\[= (b - a) \left( x - \frac{a + b}{2} \right) f'(x) - (b - a) f(x) + \int_a^b f(t) dt \]

from where we get the integral identity:

\[(2.2) \quad \int_a^b f(t) dt = (b - a) f(x) - (b - a) \left( x - \frac{a + b}{2} \right) f'(x) + \int_a^b K(x, t) f''(t) dt \]

for all \( x \in [a, b] \), which is interesting in itself, too.

Using the identity (2.2) we have

\[(2.3) \quad \left| f(x) - \frac{1}{b - a} \int_a^b f(t) dt - \left( x - \frac{a + b}{2} \right) f'(x) \right| \]
\[= \frac{1}{b - a} \left| \int_a^b K(x, t) f''(t) dt \right| \]
\[= \frac{1}{b - a} \left| \int_a^x \frac{(t - a)^2}{2} f''(t) dt + \int_x^b \frac{(t - b)^2}{2} f''(t) dt \right| \]
\[\leq \frac{1}{b - a} \left[ \int_a^x \frac{(t - a)^2}{2} \left| f''(t) \right| dt + \int_x^b \frac{(t - b)^2}{2} \left| f''(t) \right| dt \right] \]
\[\leq \frac{1}{b - a} \left[ \frac{(x - a)^2}{2} \int_a^x \left| f''(t) \right| dt + \frac{(b - x)^2}{2} \int_x^b \left| f''(t) \right| dt \right] \]
\[\leq \frac{1}{b - a} \max \left\{ \frac{(x - a)^2}{2}, \frac{(b - x)^2}{2} \right\} \left[ \int_a^x \left| f''(t) \right| dt + \int_x^b \left| f''(t) \right| dt \right] . \]

Now, let observe that

\[\max \left\{ \frac{(x - a)^2}{2}, \frac{(b - x)^2}{2} \right\} \]
\[
= \frac{1}{2} \left[ \frac{(x-a)^2 + (b-x)^2}{2} + (b-a) \left| x - \frac{a+b}{2} \right| \right] \\
= \frac{1}{2} \left[ \frac{(x-a)^2 + (b-x)^2}{2} + (b-a) \left| x - \frac{a+b}{2} \right| \right] \\
= \frac{1}{2} \left[ \frac{1}{4} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 + (b-a) \left| x - \frac{a+b}{2} \right| \right] \\
= \frac{1}{2} \left( \left| x - \frac{a+b}{2} \right| + \frac{1}{2} (b-a) \right)^2.
\]

Using (2.3) we deduce the desired inequality (2.1). \[\blacksquare\]

**Corollary 2.2.** Let \( f \) be as above. Then we have the midpoint inequality:

\[
(2.4) \quad \left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{8} (b-a) \| f'' \|_1.
\]

The following trapezoid inequality also holds:

**Corollary 2.3.** Under the above assumptions we have:

\[
(2.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{b-a}{4} \left( f'(b) - f'(a) \right) \right| \\
\leq \frac{1}{2} (b-a) \| f'' \|_1.
\]

**Proof.** Choose in (2.1) \( x = a \) and \( x = b \) to get:

\[
\left| f(a) - \frac{1}{b-a} \int_a^b f(t) \, dt + \frac{b-a}{2} f'(a) \right| \leq \frac{b-a}{2} \| f'' \|_1,
\]

and

\[
\left| f(b) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{b-a}{2} f'(b) \right| \leq \frac{b-a}{2} \| f'' \|_1.
\]

Adding the above two inequalities, using the triangle inequality and dividing by 2, we get the desired inequality (2.5). \[\blacksquare\]

### 3 Applications in Numerical Integration

Let \( I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) be a division of the interval \([a, b], \xi_i \in [x_i, x_{i+1}] \) \((i = 0, \ldots, n-1)\). We have the following quadrature formula:

**Theorem 3.1.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and twice differentiable on \((a, b)\), whose second derivative \( f'' : (a, b) \to \mathbb{R} \) belongs to \( L_1(a, b) \), i.e.,

\[
\| f'' \|_1 := \int_a^b |f''(t)| \, dt < \infty.
\]

Then the following perturbed Riemann’s type quadrature formula holds:

\[
(3.1) \quad \int_a^b f(x) \, dx = A(f, f', \xi, I_n) + R(f, f', \xi, I_n)
\]

\[\blacksquare\]
where
\[ A(f, f', \xi, I_n) = \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} f'(\xi_i) \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i \]

and the remainder satisfies the estimation:
\[
|R(f, f', \xi, I_n)| \leq \frac{1}{2} \left[ \frac{1}{2} \nu(h) + \sup_{i=0, \ldots, n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \| f'' \|_1
\]
\[
\leq \frac{\nu^2(h)}{2} \| f'' \|_1
\]
for all \( \xi_i \) as above, where \( \nu(h) = \max \{ x_{i+1} - x_i \mid i = 0, \ldots, n - 1 \} \).

Proof. Apply Theorem 2.1 on the interval \([x_i, x_{i+1}]\) \((i = 0, \ldots, n - 1)\) to get
\[
\left| \int_{x_i}^{x_{i+1}} f(t) \, dt - h_i f(\xi_i) + \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) \right|
\]
\[
\leq \frac{1}{2} \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) + \frac{1}{2} (x_{i+1} - x_i) \int_{x_i}^{x_{i+1}} |f''(t)| \, dt.
\]
Summing over \( i \) from 0 to \( n - 1 \) and using the generalized triangle inequality we deduce:
\[
|R(f, f', \xi, I_n)| \leq \frac{1}{2} \sum_{i=0}^{n-1} \left[ \frac{1}{2} (x_{i+1} - x_i) + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \int_{x_i}^{x_{i+1}} |f''(t)| \, dt
\]
\[
\leq \frac{1}{2} \sup_{i=0, \ldots, n-1} \left[ \frac{1}{2} (x_{i+1} - x_i) + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f''(t)| \, dt
\]
\[
\leq \frac{1}{2} \left[ \frac{1}{2} \nu(h) + \sup_{i=0, \ldots, n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \| f'' \|_1
\]
and the estimation (3.2) is obtained. \( \square \)

Remark 3.1. If we choose above \( \xi_i = \frac{x_i + x_{i+1}}{2} \), we recapture the midpoint quadrature formula
\[
\int_{a}^{b} f(x) \, dx = A_M(f, I_n) + R_M(f, I_n)
\]
where the remainder \( R_M(f, I_n) \) satisfies the estimation
\[
|R_M(f, I_n)| \leq \frac{1}{8} \nu^2(h) \| f'' \|_1.
\]
4 Applications for Special Means

Let us recall the following means:

(a) The arithmetic mean

\[ A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0; \]

(b) The geometric mean:

\[ G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0; \]

(c) The harmonic mean:

\[ H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b \geq 0; \]

(d) The logarithmic mean:

\[ L = L(a, b) := \begin{cases} 
    a & \text{if } a = b \\
    \frac{b - a}{\ln b - \ln a} & \text{if } a \neq b 
\end{cases}, \quad a, b > 0; \]

(e) The identric mean:

\[ I = I(a, b) := \begin{cases} 
    a & \text{if } a = b \\
    \frac{1}{e} \left( \frac{b^b \cdot a^a}{a^b \cdot b^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b 
\end{cases}, \quad a, b > 0; \]

(f) The \( p \)-logarithmic mean:

\[ L_p = L_p(a, b) := \begin{cases} 
    \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^\frac{1}{p} & \text{if } a \neq b; \\
    a & \text{if } a = b 
\end{cases}, \quad \text{where } p \in \mathbb{R} \setminus \{-1, 0\}, a, b > 0. \]

The following simple relationships are known in the literature

\[ H \leq G \leq L \leq I \leq A. \]

It is also known that \( L_p \) is monotonically increasing in \( p \in \mathbb{R} \) with \( L_0 = I \) and \( L_{-1} = L \).

Consider the mapping \( f : (0, \infty) \to \mathbb{R}, f(x) = x^r, r \in \mathbb{R} \setminus \{-1, 0\}. \)

Then we have for \( 0 < a < b : \)

\[ \frac{1}{b-a} \int_a^b f(x) \, dx = L_1^r(a, b) \]

and

\[ \|f''\|_1 = |r (r - 1)| (b - a) L_{r-1}^r(a, b). \]
Using the inequality (2.1) we get:

\[(4.1) \quad |x^r - L_r^r - r (x - A) x^{r-1}| \leq \frac{1}{2} \left[ |x - A| + \frac{1}{2} (b - a) \right]^2 |r (r-1)| L_{r-1}^{-1} \]

for all \( x \in [a, b] \).

If in (4.1) we choose \( x = A \), we get

\[(4.2) \quad |A^r - L_r^r| \leq \frac{|r (r-1)| (b - a)^2}{8} L_{r-1}^{-1}. \]

Consider the mapping \( f : (0, \infty) \rightarrow \mathbb{R} \), \( f(x) = \frac{1}{x} \).

Then we have for \( 0 < a < b \):

\[
\frac{1}{b-a} \int_a^b f(x) \, dx = L^{-1}(a, b)
\]

and

\[
\|f''\|_1 = 2(b-a) L_3^{-2}(a, b).
\]

Using the inequality (2.1), we get:

\[
\left| \frac{1}{x} - \frac{1}{L} + \frac{x - A}{x^2} \right| \leq L_{-3}^{-3} \left[ |x - A| + \frac{1}{2} (b - a) \right]^2
\]

which is equivalent to

\[(4.3) \quad |x (L - x) + L (x - A)| \leq x^2 L L_{-3}^{-3} \left[ |x - A| + \frac{1}{2} (b - a) \right]^2
\]

for all \( x \in [a, b] \).

Now, if we choose in (4.3), \( x = A \), we get

\[(4.4) \quad 0 \leq A - L \leq \frac{1}{4} A L L_{-3}^{-3} (b - a)^2.
\]

If in (4.3) we choose \( x = L \), we get

\[(4.5) \quad 0 \leq A - L \leq L^2 L_{-3}^{-3} \left[ L - A + \frac{1}{2} (b - a) \right]^2.
\]

Let us consider the mapping \( f(x) = \ln x \), \( x \in [a, b] \subset (0, \infty) \).

Then we have:

\[
\frac{1}{b-a} \int_a^b f(x) \, dx = \ln f(a, b),
\]

and

\[
\|f''\|_1 = (b-a) L_2^{-2}(a, b).
\]
Then the inequality (2.1) gives us

\[
(4.6) \quad \left| \ln x - \ln I - \frac{x - A}{x} \right| \leq \frac{1}{2} \left[ |x - A| + \frac{1}{2} (b - a) \right]^2 L_{-2}^2
\]

for all \( x \in [a, b] \).

Now, if in (4.6) we choose \( x = A \), we get

\[
(4.7) \quad 1 \leq \frac{A}{I} \leq \exp \left[ \frac{1}{8} (b - a)^2 L_{-2}^2 \right].
\]

If in (4.6) we choose \( x = I \), we get

\[
(4.8) \quad 0 \leq A - I \leq \frac{I}{2} \left[ A - I + \frac{1}{2} (b - a) \right]^2 L_{-2}^2.
\]

REFERENCES

