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ON THE OSTROWSKI’S INTEGRAL INEQUALITY FOR MAPPINGS WITH BOUNDED VARIATION AND APPLICATIONS

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Abstract. A generalization of Ostrowski’s inequality for mappings with bounded variation and applications in Numerical Analysis for Euler’s Beta function is given.

1 Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski’s inequality [2, p. 469].

Theorem 1.1. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) whose derivative is bounded on \((a, b)\) and denote \( \|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty \). Then for all \( x \in [a, b] \) we have the inequality

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{4} \left[ \frac{(x-a)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty.
\]

The constant \( \frac{1}{4} \) is sharp in the sense that it cannot be replaced by a smaller one.

In this paper we prove an Ostrowski’s type inequality for mappings with bounded variation and apply it in obtaining a Riemann’s type quadrature formula for this class of mappings. Applications for Euler’s Beta function are also given.

2 Ostrowski’s Inequality for Mappings With Bounded Variation

The following inequality for mappings with bounded variation holds:

Theorem 2.1. Let \( u : [a, b] \to \mathbb{R} \) be mapping with bounded variation on \([a, b]\). Then for all \( x \in [a, b] \), we have the inequality

\( (2.1) \quad \left| \int_a^b u(t) \, dt - u(x)(b-a) \right| \leq \left[ \frac{1}{2}(b-a) + \frac{|x-a+b|}{2} \right] \sqrt{V_u(u)}.
\]

where \( \sqrt{V_u(u)} \) denotes the total variation of \( u \).

The constant \( \frac{1}{2} \) is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

\[
\int_a^b (t-a) \, du(t) = u(x)(x-a) - \int_a^x u(t) \, dt
\]

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and

\[
\int_x^b (t - b) du(t) = u(x)(b - x) - \int_x^b u(t) dt.
\]

If we add the above two equalities, we get

\[
\begin{align*}
\frac{b}{x} u(x)(b - a) - \frac{b}{x} \int_a^x u(t) dt &= \frac{b}{x} \int_a^x p(x, t) du(t) \\
\end{align*}
\]

where

\[
p(x, t) := \begin{cases}
  t - a & \text{if } t \in [a, x] \\
  t - b & \text{if } x \in [x, b],
\end{cases}
\]

for all \( x, t \in [a, b] \).

Now, assume that \( \Delta_n : a = x^{(n)}_0 < x^{(n)}_1 < \ldots < x^{(n)}_{n-1} < x^{(n)}_n = b \) is a sequence of divisions with \( \nu(\Delta_n) \to 0 \) as \( n \to \infty \), where \( \nu(\Delta_n) := \max_{i \in [0, n-1]} (x^{(n)}_{i+1} - x^{(n)}_i) \) and \( \xi^{(n)}_i \in [x^{(n)}_i, x^{(n)}_{i+1}] \).

If \( p : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and \( v : [a, b] \to \mathbb{R} \) is with bounded variation on \([a, b]\), then

\[
\frac{b}{x} \int_a^x p(x) dv(x) = \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} p(\xi^{(n)}_i) \left[ v \left( x^{(n)}_{i+1} - v(x^{(n)}_i) \right) \right]
\]

\[
\leq \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} p(\xi^{(n)}_i) \left| v \left( x^{(n)}_{i+1} - v(x^{(n)}_i) \right) \right|
\]

\[
\leq \sup_{x \in [a, b]} |p(x)| \sum_{i=0}^{n-1} |v \left( x^{(n)}_{i+1} - v(x^{(n)}_i) \right)| = \sup_{x \in [a, b]} |p(x)| \int_a^x \sqrt{v}.
\]

Applying the inequality (2.3) for \( p(x, t) \) as above and \( v(x) = u(x), x \in [a, b] \), we get

\[
\frac{b}{x} \int_a^x p(x, t) du(t) \leq \sup_{t \in [a, b]} |p(x, t)| \sqrt{u}
\]

\[
= \max \left\{ x - a, b - x \right\} \int_a^x \sqrt{u} = \left| \frac{b - a}{2} + \frac{x - a + b}{2} \right| \sqrt{u}
\]

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant \( C > 0 \), i.e.,

\[
\int_a^x |u(t)| dt - u(x)(b - a) \leq \left[ C(b - a) + \frac{x - a + b}{2} \right] \sqrt{u}
\]

for all \( x \in [a, b] \).

Consider the mapping \( u : [a, b] \to \mathbb{R} \), given by

\[
u(x) = \begin{cases}
  0 & \text{if } x \in [a, b] \setminus \{ a + \frac{b}{2} \} \\
  1 & \text{if } x = a + \frac{b}{2}
\end{cases}
\]

in (2.5). Then \( u \) is with bounded variation on \([a, b]\), and

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\[
\forall (u) = 2, \quad \int_a^b u(t)dt = 0
\]

and for \(x = \frac{a+b}{2}\), we get in (2.5)
\[
1 \leq 2C
\]

which implies that \(C \geq \frac{1}{2}\) and the theorem is completely proved.

The following corollary holds:

**Corollary 2.2.** Let \(u : [a, b] \rightarrow \mathbb{R}\) be a monotonous mapping on \([a, b]\). Then we have the inequality
\[
\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leq \frac{1}{2} \left( b-a \right) \left( \left| f(b) - f(a) \right| \right).
\]

The case of lipschitzian mappings is embodied in the following corollary.

**Corollary 2.3.** Let \(u : [a, b] \rightarrow \mathbb{R}\) be an \(L\)-lipschitzian mapping on \([a, b]\), i.e., we recall
\[
|u(x) - u(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b].
\]

Then we have the inequality
\[
\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leq L \left[ \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] (b-a).
\]

The best inequality we can get from (2.1) is that one for which \(x = \frac{a+b}{2}\) obtaining

**Corollary 2.4.** Let \(u : [a, b] \rightarrow \mathbb{R}\) be as above. Then we have the inequality:
\[
\left( 2.6 \right)
\]
\[
\left| \int_a^b u(t)dx - u \left( \frac{a+b}{2} \right) (b-a) \right| \leq \frac{1}{2} (b-a) \int_a^b u(x)dx.
\]

Similar inequalities can be found if we assume that \(u\) is monotonous or lipschitzian on \([a, b]\). We shall omit the details.

**Remark 2.1.** If we assume that \(u\) is continuous differentiable on \((a, b)\) and \(u'\) is integrable on \((a, b)\), then by (2.1) we get
\[
\left| \int_a^b u(t)dx - u \left( \frac{a+b}{2} \right) (b-a) \right| \leq \frac{1}{2} (b-a) \left\| u' \right\|_1
\]

which is the inequality obtained by Dragomir and Wang in the recent paper [1].

**Remark 2.2.** It is well known that if \(f : [a, b] \rightarrow \mathbb{R}\) is a convex mapping on \([a, b]\), then Hermite-Hadamard’s inequality holds
\[
\left( 2.7 \right)
\]
\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

Now, if we assume that \(f : I \subset \mathbb{R} \rightarrow \mathbb{R}\) is convex on \(I\) and \(a, b \in \text{Int}(I), a < b\), then \(f'\) is monotonous nondecreasing on \([a, b]\) and by Corollary 2.4 we get
\[
\left( 2.8 \right)
\]
\[
0 \leq \frac{1}{b-a} \int_a^b f(x)dx - f \left( \frac{a+b}{2} \right) \leq \frac{1}{2} \left\| f' \right\|_1
\]

which gives a counterpart for the first membership of Hadamard’s inequality.

Similar results can be obtained if we assume that \(f\) is convex and monotonous or convex and lipschitzian on \([a, b]\).
3 A Quadrature Formula of Riemann Type

Let \( I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) be a division of the interval \([a, b]\) and \( \xi_i \in [x_i, x_{i+1}] \) \((i = 0, \ldots, n - 1)\) a sequence of intermediate points for \( I_n \). Construct the Riemann sums

\[
R_n (f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i) h_i
\]

where \( h_i := x_{i+1} - x_i \).

We have the following quadrature formula

**Theorem 3.1.** Let \( f : [a, b] \to \mathbb{R} \) be a mapping with bounded variation on \([a, b]\) and \( I_n, \xi_i \) \((i = 0, \ldots, n - 1)\) be as above. Then we have the Riemann quadrature formula

\[
\int_a^b f(x) dx = R_n (f, I_n, \xi) + W_n (f, I_n, \xi) \quad \text{(3.1)}
\]

where the remainder satisfies the estimation

\[
[W_n (f, I_n, \xi)] \leq \sup_{i=0, \ldots, n} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \| f \|_\alpha \quad \text{(3.2)}
\]

for all \( \xi_i \) \((i = 0, \ldots, n - 1)\) as above, where \( \nu(h) := \max_{i=0, \ldots, n} h_i \).

The constant \( \frac{1}{2} \) is sharp in (3.2).

**Proof.** Apply Theorem 2.1 on the interval \([x_i, x_{i+1}]\) to get

\[
\left| \int_{x_i}^{x_{i+1}} f(x) dx - f(\xi_i) h_i \right| \leq \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \| f \|_{x_i} \quad \text{(3.3)}
\]

Summing over \( i \) from 0 to \( n - 1 \) and using the generalized triangle inequality we get

\[
[W_n (f, I_n, \xi)] \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) dx - f(\xi_i) h_i \right|
\]

\[
\leq \sum_{i=0}^{n-1} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \| f \|_{x_i} \leq \sup_{i=0, \ldots, n} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \| f \|_{x_i}
\]

The second inequality follows by the properties of \( \sup(\cdot) \).

Now, as

\[
\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2} h_i
\]

for all \( \xi_i \in [x_i, x_{i+1}] \) \((i = 0, \ldots, n - 1)\) the last part of (3.2) is also proved.

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Corollary 3.2. Let \( u : [a, b] \to \mathbb{R} \) be a monotonic mapping on \( [a, b] \) and \( I_n, \xi_i (i = 0, \ldots, n - 1) \) be as above. Then we have the Riemann quadrature formula (3.1) and the remainder satisfies the estimation

\[
|W_n(f, I_n, \xi)| \leq \sup_{i=0,\ldots,n} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] |f(b) - f(a)|
\]

\[
\leq \frac{1}{2} \nu(h) + \sup_{i=0,\ldots,n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| |f(b) - f(a)| \leq \nu(h) |f(b) - f(a)|
\]

for all \( \xi_i (i = 0, \ldots, n - 1) \) as above.

The case of Lipschitz mappings is embodied into the following corollary.

Corollary 3.3. Let \( u : [a, b] \to \mathbb{R} \) be an \( L \)-Lipschitzian mapping on \( [a, b] \) and \( I_n, \xi_i (i = 0, \ldots, n - 1) \) be as above. Then we have the Riemann quadrature formula (3.1) and the remainder satisfies the estimation

\[
|W_n(f, I_n, \xi)| \leq L \sum_{i=0}^{n-1} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] h_i
\]

\[
\leq L \sum_{i=0}^{n-1} h_i^2
\]

The proof is obvious by Corollary 2.3 applied on the intervals \( [x_i, x_{i+1}] \) and summing the obtained inequalities.

We shall omit the details.

Note that the best estimation we can get from (3.2) is that one for which \( \xi_i = \frac{x_i + x_{i+1}}{2} \) obtaining the following midpoint formula:

Corollary 3.4. Let \( f, I_n \) be as Theorem 3.1. Then we have the midpoint rule

\[
\int_a^b f(x) \, dx = M_n(f, I_n) + S_n(f, I_n)
\]

where

\[
M_n(f, I_n) = \sum_{i=0}^{n-1} f \left( \frac{x_i + x_{i+1}}{2} \right) h_i
\]

and the remainder \( S_n(f, I_n) \) satisfies the estimation

\[
|S_n(f, I_n)| \leq \frac{1}{2} \nu(h) \int_a^b f(x) \, dx.
\]

Similar results can be obtained from Corollaries 3.2 and 3.3.

Remark 3.1. If we assume that \( f : [a, b] \to \mathbb{R} \) is differentiable on \( (a, b) \) and whose derivative \( f' \) is integrable on \( (a, b) \) we can put instead of \( \int_a^b f(x) \, dx \) the \( L_1 \)-norm \( \| f' \|_1 \) obtaining the estimation due to Dragomir-Wang from the paper [1].
4 Applications for Euler's Beta Mapping

Consider the mapping Beta for real numbers

$$B(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} \, dt, \quad p, q > 0$$

and the mapping $e_{p,q}(t) := t^{p-1} (1-t)^{q-1}, t \in [0, 1]$.

We have for $p, q > 1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t)[p - 1 - (p + q - 2)t]$$

and as

$$|p - 1 - (p + q - 2)t| \leq \max\{p - 1, q - 1\}$$

for all $t \in [0, 1]$, then

$$\|e'_{p,q}\|_1 \leq \max\{p - 1, q - 1\} \|e_{p-2,q-2}\|_1$$

$$= \max\{p - 1, q - 1\} B(p - 1, q - 1); \quad p, q > 1.$$ 

The following inequality for Beta mapping holds

**Proposition 4.1.** Let $p, q > 1$ and $x \in [0, 1]$. Then we have the inequality

$$|B(p, q) - x^{p-1}(1-x)^{q-1}|$$

$$\leq \max\{p - 1, q - 1\} \left( B(p - 1, q - 1) \left[ \left\| x - \frac{1}{2} \right\| \right] \right).$$

The proof follows by Theorem 2.1 applied for the mapping $e_{p,q}$ and taking into account that $\|e'_{p,q}\|_1$ satisfies the inequality (4.1).

**Corollary 4.2.** Let $p, q > 1$. Then we have the inequality

$$\left| B(p, q) - \frac{1}{2^{p+q-1}} \right| \leq \frac{1}{2} \max\{p - 1, q - 1\} B(p - 1, q - 1).$$

Now, if we apply Theorem 3.1 for the mapping $e_{p,q}$ we get the following approximation of Beta mapping in terms of Riemann sums.

**Proposition 4.3.** Let $I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}] (i = 0, \ldots, n - 1)$ a sequence of intermediate points for $I_n$ and $p, q > 1$. Then we have the formula

$$B(p, q) = \sum_{i=0}^{n-1} \xi_i^{p-1} (1-\xi_i)^{q-1} h_i + T_n(p, q)$$

where the remainder $T_n(p, q)$ satisfies the estimation

$$|T_n(p, q)|$$

$$\leq \max\{p - 1, q - 1\} \left[ \frac{1}{2} \nu(h) + \sup_{i=0, \ldots, n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] B(p - 1, q - 1)$$

$$\leq \max\{p - 1, q - 1\} \nu(h) B(p - 1, q - 1).$$

**Particularly, if we choose above $\xi_i = \frac{x_i + x_{i+1}}{2} (i = 0, \ldots, n - 1)$ then we get the approximation**

$$B(p, q) = \frac{1}{2^{p+q-1}} \sum_{i=0}^{n-1} (x_i + x_{i+1})^{p-1} (2 - x_i - x_{i+1})^{q-1} + V_n(p, q)$$

where

$$|V_n(p, q)| \leq \frac{1}{2} \max\{p - 1, q - 1\} \nu(h) B(p - 1, q - 1).$$

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