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ON THE OSTROWSKI INTEGRAL INEQUALITY FOR LIPSCHITZIAN MAPPINGS AND APPLICATIONS

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Abstract. A generalization of Ostrowski's inequality for lipschitzian mappings and applications in Numerical Analysis and for Euler's Beta function are given.

1 Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p. 469].

Theorem 1.1. *Let $f:[a,b]\to\mathbf{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then for all $x\in[a,b]$ we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty.$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In this paper we prove that Ostrowski's inequality also holds for lipschitzian mappings and apply it in obtaining a Riemann's type quadrature formula for this class of mappings. Applications for Euler's Beta function are also given.

2 Ostrowski's Inequality For Lipschitzian Mappings

The following inequality for lipschitzian mappings holds:

Theorem 2.1. *Let $u:[a,b]\to\mathbf{R}$ be an $L-$lipschitzian mapping on $[a,b]$, i.e.,*

$$|u(x)-u(y)|\le L|x-y|\quad \text{for all } x,y\in[a,b].$$

Then we have the inequality

$$\left|\int_a^b u(t)dt-u(x)(b-a)\right|\le L(b-a)^2\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]. \tag{2.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^x (t-a)du(t)=u(x)(x-a)-\int_a^x u(t)dt$$

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and

$$\int_x^b (t-b)\,du(t) = u(x)(b-x) - \int_x^b u(t)dt.$$

If we add the above two equalities, we get

$$u(x)(b-a) - \int_a^b u(t)dt = \int_a^b p(x,t)du(t) \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad t \in [a,x) \\ t-b & \text{if} \quad x \in [x,b] \end{cases},$$

for all $x, t \in [a,b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i \in \{0,\ldots,n-1\}} (x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$. If $p : [a,b] \to \mathbf{R}$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to \mathbf{R}$ is L-lipschitzian on $[a,b]$, then

$$\left| \int_a^b p(x)dv(x) \right| = \left| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)}) \left[v(x_{i+1}^{(n)}) - v(x_i^{(n)}) \right] \right|$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| p(\xi_i^{(n)}) \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right) \left| \frac{v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)}{x_{i+1}^{(n)} - x_i^{(n)}} \right|$$

$$\leq L \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| p(\xi_i^{(n)}) \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right) = L \int_a^b |p(x)|\, dx. \tag{2.3}$$

Applying the inequality (2.3) for $p(x,t)$ as above and $v(x) = u(x), x \in [a,b]$, we get

$$\left| \int_a^b p(x,t)du(t) \right| \leq L \int_a^b |p(x,t)|\, dt \tag{2.4}$$

$$= L \left[\int_a^x |t-a|\, dt + \int_x^b |t-b|\, dt \right] = \frac{L}{2} \left[(x-a)^2 + (b-x)^2 \right]$$

$$= L\,(b-a)^2 \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leq L\,(b-a)^2 \left[C + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{2.5}$$

for all $x \in [a,b]$.

Consider the mapping $f : [a,b] \to \mathbf{R}, f(x) = x$ in (2.5). Then

$$\left| x - \frac{a+b}{2} \right| \leq C + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}$$

for all $x \in [a,b]$; and then for $x = a$, we get

$$\frac{b-a}{2} \leq \left(C + \frac{1}{4} \right)(b-a)$$

which implies that $C \geq \frac{1}{4}$ and the theorem is completely proved. ∎

The following corollary holds:

Corollary 2.2. *Let $u : [a,b] \to \mathbf{R}$ be as above. Then we have the inequality:*

$$\left| \int_a^b u(t)dx - u\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{4}L(b-a)^2. \tag{2.6}$$

Remark 2.1. *It is well known that if $f : [a,b] \to \mathbf{R}$ is a convex mapping on $[a,b]$, then Hermite-Hadamard's inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f : I \subset \mathbf{R} \to \mathbf{R}$ is convex on I and $a,b \in Int(I), a < b$; then f'_+ is monotonous nondecreasing on $[a,b]$ and by Theorem 2.1 we get

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}f'_+(b)(b-a) \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

3 A Quadrature Formula of Riemann Type

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i)h_i$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be an $L-$lipschitzian mapping on $[a,b]$ and I_n, ξ_i $(i = 0, ..., n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi) \tag{3.1}$$

where the remainder satisfies the estimation

$$|W_n(f, I_n, \xi)| \leq \frac{1}{4}L\sum_{i=0}^{n-1} h_i^2 + L\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \tag{3.2}$$

$$\leq \frac{1}{2}L\sum_{i=0}^{n-1} h_i^2$$

for all ξ_i $(i = 0, ..., n-1)$ as above.
The constant $\frac{1}{4}$ is sharp in (3.2).

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i \right| \leq L\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]. \tag{3.3}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$|W_n(f, I_n, \xi)| \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i \right|$$

$$\leq L \sum_{i=0}^{n-1} \left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right].$$

Now, as

$$\left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \leq \frac{1}{4}h_i^2$$

for all $\xi_i \in [x_i, x_{i+1}] (i = 0, ..., n-1)$ the second part of (3.2) is also proved. ∎

Note that the best estimation we can get from (3.2) is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ obtaining the following midpoint formula:

Corollary 3.2. *Let f, I_n be as above. Then we have the midpoint rule*

$$\int_a^b f(x)dx = M_n(f, I_n) + S_n(f, I_n)$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2} \right) h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \leq \frac{1}{4} L \sum_{i=0}^{n-1} h_i^2.$$

Remark 3.1. *If we assume that $f : [a, b] \to \mathbf{R}$ is differentiable on (a, b) and whose derivative f' is bounded on (a, b) we can put instead of L the infinity norm $\|f'\|_\infty$ obtaining the estimation due to Dragomir-Wang from the paper [1].*

4 Applications for Euler's Beta Mapping

Consider the mapping Beta for real numbers

$$B(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad p, q > 0$$

and the mapping $e_{p,q}(t) := t^{p-1} (1-t)^{q-1}, t \in [0, 1]$.

We have for $p, q > 1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t) [p - 1 - (p + q - 2)t].$$

If $t \in \left[0, \frac{p-1}{p+q-2}\right)$ then $e'_{p,q}(t) > 0$ and if $t \in \left(\frac{p-1}{p+q-2}, 1\right]$ then $e'_{p,q}(t) < 0$ which shows that for $t_0 = \frac{p-1}{p+q-2}$ we have a maximum for $e_{p,q}$ and then:

$$\sup_{t \in [0,1]} e_{p,q}(t) = e_{p,q}(t_0) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}; \quad p, q > 1.$$

Consequently

$$\left|e'_{p,q}(t)\right| \le \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \max_{t\in[0,1]} |p-1-(p+q-2)t|$$

$$= \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}; \quad p, q > 2$$

for all $t \in [0,1]$ and then

$$\left\|e'_{p,q}\right\|_\infty \le \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \quad p, q > 2. \tag{4.1}$$

The following inequality for Beta mapping holds

Proposition 4.1. *Let $p, q > 2$ and $x \in [0,1]$. Then we have the inequality*

$$\left|B(p,q) - x^{p-1}(1-x)^{q-1}\right|$$

$$\le \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2\right]. \tag{4.2}$$

The proof follows by Theorem 2.1 applied for the mapping $e_{p,q}$ and taking into account that $\left\|e'_{p,q}\right\|_\infty$ satisfies the inequality (4.1).

Corollary 4.2. *Let $p, q > 2$. Then we have the inequality*

$$\left|B(p,q) - \frac{1}{2^{p+q-2}}\right| \le \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}.$$

Now, if we apply Theorem 3.1 for the mapping $e_{p,q}$ we get the following approximation of Beta mapping in terms of Riemann sums.

Proposition 4.3. *Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b], \xi_i \in [x_i, x_{i+1}], (i = 0, ..., n-1)$ a sequence of intermediate points for I_n and $p, q > 2$. Then we have the formula*

$$B(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1}(1-\xi_i)^{q-1} h_i + T_n(p,q)$$

where the remainder $T_n(p,q)$ satisfies the estimation

$$|T_n(p,q)| \le \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}$$

$$\times \left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]$$

$$\le \frac{1}{2} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

Particularly, if we choose above $\xi_i = \frac{x_i + x_{i+1}}{2}$ $(i = 0, ..., n-1)$ then we get the approximation

$$B(p,q) = \frac{1}{2^{p+q-2}} \sum_{i=0}^{n-1} (x_i + x_{i+1})^{p-1} (2 - x_i - x_{i+1})^{q-1} + V_n(p,q)$$

where

$$|V_n(p,q)| \le \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

References

[1] S.S. DRAGOMIR and S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules, *Appl. Math. Lett.*, **11**(1)(1998), 105-109.

[2] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.

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