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A NEW GENERALIZATION OF OSTROWSKI'S INTEGRAL INEQUALITY FOR MAPPINGS WHOSE DERIVATIVES ARE BOUNDED AND APPLICATIONS IN NUMERICAL INTEGRATION AND FOR SPECIAL MEANS

S.S. Dragomir, P. Cerone and J. Roumeliotis

Abstract. In this paper we establish a new inequality of Ostrowski type for functions with bounded deriviatives. This has immediate applications in Numerical Integration where new estimates are obtained for the remainder term of the trapezoid, mid-point and Simpson formulae. Application to special means are also investigated.

1 Introduction

In 1938, Ostrowski proved the following interesting inequality [1, p. 469]:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $[a, b]$ and its derivative $f' : (a, b) \to \mathbb{R}$ is bounded in (a, b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. Then for any $x \in [a, b]$, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty . \tag{1.1}$$

The inequality (1.1) *is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.*

For some extensions, generalizations and similar results, see Chapter XV of the book [1] by Mitrinović, Pečarić and Fink where further references are given.

In the recent paper [2], S. S. Dragomir and S. Wang have applied this inequality for special means: $p-$logarithmic, logarithmic and identric means. They have also applied it in Numerical Analysis to obtain some new adaptive quadrature formulae.

In this paper we point out some generalizations of (1.1) and apply them to some special means and in Numerical Integration to obtain amongst other things new estimations of the remainder term for trapezoid, midpoint and Simpson's formulae.

2 The Results

The following generalization of Ostrowski's inequality holds:

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b). Denote $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. Then*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \left[f(x)\cdot(1-h) + \frac{f(a)+f(b)}{2}\cdot h \right](b-a) \right| \\ \le\ &\left[\frac{1}{4}(b-a)^2\left[h^2 + (h-1)^2\right] + \left(x - \frac{a+b}{2}\right)^2 \right] \|f'\|_\infty \end{aligned} \tag{2.1}$$

for all $h \in [0, 1]$ and $a + h\cdot\frac{b-a}{2} \le x \le b - h\cdot\frac{b-a}{2}$.

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Proof. Let us define the mapping $p:[a,b]^2\to\mathbb{R}$ given by

$$p(x,t):=\begin{cases} t-\left[a+h\cdot\frac{b-a}{2}\right], t\in[a,x] \\ t-\left[b-h\cdot\frac{b-a}{2}\right], t\in(x,b] \end{cases}.$$

Integrating by parts, we have:

$$\begin{aligned} &\int_a^b p(x,t)\,f'(t)\,dt \\ =\;& \int_a^x\left(t-\left[a+h\cdot\frac{b-a}{2}\right]\right)f'(t)\,dt+\int_x^b\left(t-\left[b-h\cdot\frac{b-a}{2}\right]\right)f'(t)\,dt \\ =\;& (b-a)\cdot h\frac{(f(a)+f(b))}{2}+(b-a)\cdot(1-h)\cdot f(x)-\int_a^b f(t)\,dt. \end{aligned}\tag{2.2}$$

On the other hand,

$$\begin{aligned} &\left|\int_a^b p(x,t)\,f'(t)\,dt\right|\le\int_a^b|p(x,t)|\,|f'(t)|\,dt\le\|f'\|_\infty\int_a^b|p(x,t)|\,dt \\ =\;& \|f'\|_\infty\left[\int_a^x\left|t-\left(a+h\cdot\frac{b-a}{2}\right)\right|dt+\int_x^b\left|t-\left(b-h\cdot\frac{b-a}{2}\right)\right|dt\right] \\ =\;& \|f'\|_\infty L \end{aligned}$$

Now, let us observe that

$$\begin{aligned} \int_p^r|t-q|\,dt &= \int_p^q(q-t)\,dt+\int_q^r(t-q)\,dt \\ &= \frac{1}{2}\left[(q-p)^2+(r-q)^2\right]=\frac{1}{4}(p-r)^2+\left(q-\frac{r+p}{2}\right)^2 \end{aligned}$$

for all r,p,q such that $p\le q\le r$.
Using the previous identity, we have that

$$\begin{aligned} &\int_a^x\left|t-\left(a+h\cdot\frac{b-a}{2}\right)\right|dt \\ =\;& \frac{1}{4}(x-a)^2+\left[\left(a+h\cdot\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2 \end{aligned}$$

and

$$\begin{aligned} &\int_x^b\left|t-\left(b-h\cdot\frac{b-a}{2}\right)\right|dt \\ =\;& \frac{1}{4}(b-x)^2+\left[\left(b-h\cdot\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2. \end{aligned}$$

Then we get

$$\begin{aligned} L &= \frac{1}{2}\frac{(x-a)^2+(b-x)^2}{2}+\left(h\cdot\frac{b-a}{2}-\frac{x-a}{2}\right)^2+\left(\frac{b-x}{2}-h\cdot\frac{b-a}{2}\right)^2 \\ &= \frac{(b-a)^2}{4}\left[h^2+(h-1)^2\right]+\left(x-\frac{a+b}{2}\right)^2 \end{aligned}$$

and the theorem is thus proved. ∎

Remark 2.1. a. *If we choose in* (2.1), $h=0$, *we get Ostrowski's inequality* (1.1).

b. *If we choose in* (2.1), $h=1$ *and* $x=\frac{a+b}{2}$ *we get the trapezoid inequality:*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{4}(b-a)^2\|f'\|_\infty\tag{2.3}$$

Corollary 2.2. *Under the above assumptions, we have the inequality:*

$$\left|\int_a^b f(t)\,dt-\frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right](b-a)\right| \leq \left[\frac{1}{8}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty \tag{2.4}$$

for all $x\in\left[\frac{b+3a}{4},\frac{a+3b}{4}\right]$, *and, in particular, the following mixture of the trapezoid inequality and mid-point inequality:*

$$\left|\int_a^b f(t)\,dt-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right](b-a)\right|\leq\frac{1}{8}(b-a)^2\|f'\|_\infty . \tag{2.5}$$

Finally, we also have the following generalization of Simpson's inequality:

Corollary 2.3. *Under the above assumptions and with* $h=\frac{1}{3}$, *we obtain*

$$\left|\int_a^b f(t)\,dt-\frac{1}{6}\left[f(a)+4f(x)+f(b)\right](b-a)\right| \leq \left[\frac{5}{36}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty \tag{2.6}$$

for all $x\in\left[\frac{b+5a}{4},\frac{a+5b}{4}\right]$, *and, in particular when* x *is at the mid-point, the Simpson's inequality*

$$\left|\int_a^b f(t)\,dt-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right](b-a)\right| \leq \frac{5}{36}(b-a)^2\|f'\|_\infty . \tag{2.7}$$

It is interesting to note that the smallest bound for (2.1) is obtained at $x=\frac{a+b}{2}$ and $h=\frac{1}{2}$. Thus the quadrature rule (2.5) comprised of the linear combination of the mid-point and trapezoidal rule is optimal and has a lower bound than Simpson's rule (2.7).

3 Applications in Numerical Integration

The following approximation of the integral $\int_a^b f(x)\,dx$ holds.

Theorem 3.1. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *whose derivative is bounded on* (a,b). *If* $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ *is a partition of* $[a,b]$ *and* $h_i:=x_{i+1}-x_i$, $i=0,...,n-1$, *then we have:*

$$\int_a^b f(x)\,dx=A_T(I_n,\xi,\delta,f)+R_T(I_n,\xi,\delta,f) \tag{3.1}$$

where

$$A_T(I_n,\xi,\delta,f)=(1-\delta)\sum_{i=0}^{n-1}f(\xi_i)h_i+\delta\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}h_i, \tag{3.2}$$

$\delta\in[0,1]$, $x_i+\delta\cdot\frac{h_i}{2}\leq\xi_i\leq x_{i+1}-\delta\cdot\frac{h_i}{2}$, $i=0,...,n-1$; *and the remainder term satisfies the estimation:*

$$|R_T(I_n,\xi,\delta,f)| \leq \|f'\|_\infty\left[\frac{1}{4}\left[\delta^2+(\delta-1)^2\right]\sum_{i=0}^{n-1}h_i^2+\sum_{i=0}^{n-1}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right] \tag{3.3}$$

Proof. Applying Theorem 2.1 on the interval $[x_i, x_{i+1}]$, $i = 0, ..., n-1$ we get

$$\begin{aligned} & \left| h_i \left[(1-\delta) f(\xi_i) + \frac{f(x_i) + f(x_{i+1})}{2} \delta \right] - \int_{x_i}^{x_{i+1}} f(x)\, dx \right| \\ \leq & \left[\left[\delta^2 + (\delta - 1)^2 \right] \frac{h_i^2}{4} + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty \end{aligned}$$

for all $\delta \in [0,1]$ and ξ_i $(i = 0, ..., n-1)$ as above.
Summing over i from 0 to $n-1$ and using the triangle inequality we get the estimation (3.2). ∎

Remark 3.1. *a) If we choose $\delta = 0$, then we get the quadrature formula*

$$\int_a^b f(x)\, dx = A_T(I_n, \xi, f) + R_T(I_n, \xi, f) \tag{3.4}$$

where $A_T(I_n, \xi, f)$ is the Riemann's sum, i.e.,

$$A_T(I_n, \xi, f) := \sum_{i=0}^{n-1} f(\xi_i) h_i, \;\; \xi_i \in [x_i, x_{i+1}], i = 0, ..., n-1;$$

and the remainder term satisfies the estimate (see also [2]):

$$|R_T(I_n, \xi, f)| \leq \|f'\|_\infty \sum_{i=0}^{n-1} \left[\frac{h_i^2}{4} + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]. \tag{3.5}$$

b) If we choose $\delta = 1$, then we get the trapezoid formula

$$\int_a^b f(x)\, dx = A_T(I_n, f) + R_T(I_n, f) \tag{3.6}$$

where $A_T(I_n, f)$ is the trapezoidal rule

$$A_T(I_n, f) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} h_i$$

and the remainder terms satisfies the estimation

$$|R_T(I_n, f)| \leq \frac{\|f'\|_\infty}{4} \sum_{i=0}^{n-1} h_i^2. \tag{3.7}$$

Corollary 3.2. *Under the above assumptions we have*

$$\int_a^b f(x)\, dx = B_T(I_n, \xi, f) + Q_T(I_n, \xi, f) \tag{3.8}$$

where

$$\begin{aligned} B_T(I_n, \xi, f) &= \frac{1}{2} \left[\sum_{i=0}^{n-1} f(\xi_i) h_i + \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} h_i \right], \\ \xi_i &\in \left[\frac{x_{i+1} + 3x_i}{4}, \frac{x_i + 3x_{i+1}}{4} \right], \end{aligned}$$

and the remainder term satisfies the estimation

$$|Q_T(I_n, \xi, f)| \leq \|f'\|_\infty \left[\frac{1}{8} \sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]. \tag{3.9}$$

In particular, we have

$$\int_a^b f(x)\, dx = B_T(I_n, f) + Q_T(I_n, f) \tag{3.10}$$

where

$$B_T(I_n,f)=\frac{1}{2}\left[\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)h_i+\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}h_i\right]$$

and $Q_T(I_n,f)$ *satisfies the estimation:*

$$|Q_T(I_n,f)|\leq\frac{\|f'\|_\infty}{8}\sum_{i=0}^{n-1}h_i^2. \tag{3.11}$$

Finally, we have the following generalization of Simpson's inequality whose remainder term is estimated by the use of the first derivative only.

Corollary 3.3. *Under the above assumptions we have:*

$$\int_a^b f(x)\,dx=S_T(I_n,\xi,f)+W_T(I_n,\xi,f) \tag{3.12}$$

where

$$\begin{aligned} S_T(I_n,\xi,f) &= \frac{2}{3}\sum_{i=0}^{n-1}f(\xi_i)h_i+\frac{1}{6}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})]h_i,\\ \xi_i &\in \left[\frac{x_{i+1}+5x_i}{6},\frac{x_i+5x_{i+1}}{6}\right], \end{aligned}$$

and the remainder term $W_T(I_n,\xi,f)$ *satisfies the bound:*

$$|W_T(I_n,\xi,f)|\leq\|f'\|_\infty\left[\frac{5}{36}\sum_{i=0}^{n-1}h_i^2+\sum_{i=0}^{n-1}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right] \tag{3.13}$$

and, in particular, the Simpson's rule:

$$\int_a^b f(x)\,dx=S_T(I_n,f)+W_T(I_n,f) \tag{3.14}$$

where

$$S_T(I_n,f)=\frac{2}{3}\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)h_i+\frac{1}{6}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})]h_i$$

and the remainder term satisfies the estimation:

$$|W_T(I_n,f)|\leq\frac{5}{36}\|f'\|_\infty\sum_{i=0}^{n-1}h_i^2. \tag{3.15}$$

4 Applications for Special Means

Let us recall the following means:

(a) *The arithmetic mean*

$$A=A(a,b):=\frac{(a+b)}{2},a,b\geq0;$$

(b) *The geometric mean*

$$G=G(a,b):=\sqrt{ab};a,b\geq0;$$

(c) *The harmonic mean*

$$H=H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}};a,b>0;$$

(d) *The logarithmic mean*

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \quad a, b > 0;$$

(e) *The identric mean*

$$I = I(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases};$$

(f) *The p-logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases}$$

where $p \in \mathbb{R}\setminus\{0,-1\}$.

The following inequality is well-known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is well-known that L_p is monotonically increasing over p, assuming that $L_0 = I$ and $L_{-1} = L$.

Now, let us reconsider the inequality (2.1) in the following equivalent form:

$$\begin{aligned} &\left|(1-h) f(x) + \frac{f(a)+f(b)}{2} \cdot h - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq\; &\left[(b-a)\left(\frac{h^2+(h-1)^2}{4}\right) + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)}\right]\|f'\|_\infty \end{aligned} \tag{4.1}$$

for all $h \in [0,1]$ and $x \in [a,b]$ such that

$$a + h\cdot\left(\frac{b-a}{2}\right) \leq x \leq b - h\cdot\left(\frac{b-a}{2}\right)$$

(1) Consider the mapping $f : (0,\infty) \to (0,\infty)$, $f(x) = x^p, p \in \mathbb{R}\setminus\{-1,0\}$. Then, for $0 < a < b$, we have

$$\frac{f(a)+f(b)}{2} = A(a^p, b^p), \; \frac{1}{b-a}\int_a^b f(x)\,dx = L_p^p(a,b),$$

$$\|f'\|_\infty = \begin{cases} |p|\, b^{p-1} & \text{if } p > 1 \\ |p|\, a^{p-1} & \text{if } p \in (-\infty,1]\setminus\{-1,0\} \end{cases},$$

and then, by (4.1), we deduce that:

$$\begin{aligned} &\left|(1-h)x^p + hA(a^p,b^p) - L_p^p(a,b)\right| \\ \leq\; &\left\{(b-a)\left[\frac{h^2+(h-1)^2}{4}\right] + \frac{(x-A)^2}{(b-a)}\right\}\delta_p(a,b) \end{aligned} \tag{4.2}$$

where

$$\delta_p(a,b) := \begin{cases} |p|\, b^{p-1} & \text{if } p > 1 \\ |p|\, a^{p-1} & \text{if } p \in (-\infty,1]\setminus\{-1,0\} \end{cases}$$

and $h \in [0,1], x \in \left[a + h\cdot\frac{b-a}{2}, b - h\cdot\frac{b-a}{2}\right]$.

(2) Consider the mapping $f:(0,\infty)\rightarrow(0,\infty), f(x)=\frac{1}{x}$ and $0<a<b$. We have:

$$\frac{f(a)+f(b)}{2}=H^{-1}(a,b), \frac{1}{b-a}\int_a^b f(x)\,dx=L^{-1}(a,b),$$

$$\|f'\|_\infty=\frac{1}{a^2}$$

and then by (4.1), we deduce, for all $h\in[0,1]$, and $a+h\cdot\frac{b-a}{2}\leq x\leq b-h\cdot\frac{b-a}{2}$ that:

$$|(1-h)HL+Lxh-xH|\leq\frac{xHL}{a^2}\left[(b-a)\left[\frac{h^2+(h-1)^2}{4}\right]+\frac{(x-A)^2}{(b-a)}\right]. \tag{4.3}$$

(3) Consider the mapping $f:(0,\infty)\rightarrow\mathbb{R},\ f(x)=\ln x$ and $0<a<b$. We have:

$$\frac{f(a)+f(b)}{2}=\ln G, \frac{1}{b-a}\int_a^b f(x)\,dx=\ln I,$$

$$\|f'\|_\infty=\frac{1}{a}$$

and then, by (4.1), we deduce that

$$\left|\ln\left[\frac{x^{1-h}G^h}{I}\right]\right|\leq\frac{1}{a}\left[(b-a)\left[\frac{h^2+(h-1)^2}{4}\right]+\frac{(x-A)^2}{(b-a)}\right], \tag{4.4}$$

for all $h\in[0,1]$, and $x\in\left[a+h\cdot\frac{b-a}{2},b-h\cdot\frac{b-a}{2}\right]$.

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