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## *Logarithmic Convexity of the One-Parameter Mean Values*

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# LOGARITHMIC CONVEXITY OF THE ONE-PARAMETER MEAN VALUES

WING-SUM CHEUNG AND FENG QI

ABSTRACT. In this article, the logarithmic convexity of the one-parameter mean values  $J(r)$  and the monotonicity of the product  $J(r)J(-r)$  with  $r \in \mathbb{R}$  are presented. Some more general results are established. Three open problems are posed.

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## 1. INTRODUCTION

Define a function  $g(r; x, y)$  for  $x \neq y$  by

$$g(t) \triangleq g(t; x, y) = \begin{cases} \frac{y^t - x^t}{t}, & t \neq 0; \\ \ln y - \ln x, & t = 0. \end{cases} \quad (1)$$

The following integral form of  $g$  is presented and applied in [11, 13, 16, 17]:

$$g(t) = \int_x^y u^{t-1} du, \quad t \in \mathbb{R}, \quad (2)$$

$$g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1} du, \quad t \in \mathbb{R}. \quad (3)$$

Straightforward computation results in

$$\left( \frac{g'(t)}{g(t)} \right)' = \frac{g''(t)g(t) - [g'(t)]^2}{g^2(t)}, \quad (4)$$

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$$\left(\frac{g'(t)}{g(t)}\right)'' = \frac{g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3}{g^3(t)}. \quad (5)$$

In [11], Corollary 3 states that, for  $y > x > 0$ , if  $t > 0$ , then

$$g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3 < 0; \quad (6)$$

if  $t < 0$ , inequality (6) reverses.

The function  $g(t; x, y)$  and its integral expressions (2) and (3) are very important in the proofs of the logarithmic convexity [11, 13] and Schur-convexity [12, 13, 15] of the extended mean values  $E(r, s; x, y)$ , which is a generalization of the one-parameter mean values  $J(r)$  with  $E(r, r + 1; x, y) = J(r; x, y)$ . The monotonicity and comparison of  $E(r, s; x, y)$  were studied in [6, 7, 8, 13]. The concepts of mean values are generalized in [9, 10, 13, 19]. For more information about the extended mean values  $E(r, s; x, y)$ , please refer to the expository article [13] and the references therein.

The one-parameter mean values  $J(r; x, y)$  for  $x \neq y$  are defined in [1, 20] and introduced in [5, p. 44] by

$$J(r) \triangleq J(r; x, y) = \begin{cases} \frac{r(x^{r+1} - y^{r+1})}{(r+1)(x^r - y^r)}, & r \neq 0, -1; \\ \frac{x - y}{\ln x - \ln y}, & r = 0; \\ \frac{xy(\ln x - \ln y)}{x - y}, & r = -1. \end{cases} \quad (7)$$

In [4, p. 49], the following results in [2, 3] by Alzer are mentioned:

(1) When  $r \neq 0$ , we have

$$G(x, y) < \sqrt{J(r; x, y)J(-r; x, y)} < L < \frac{J(r; x, y) + J(-r; x, y)}{2} < A(x, y). \quad (8)$$

(2) For  $x_1 > 0$ ,  $x_2 > 0$ ,  $y_1 > 0$  and  $y_2 > 0$ , if  $r \geq 1$ , then

$$J(r; x_1 + y_1, x_2 + y_2) \leq J(r; x_1, x_2) + J(r; y_1, y_2); \quad (9)$$

if  $r \leq 1$ , inequality (9) is reversed.

(3) If  $(x_1, x_2)$  and  $(y_1, y_2)$  are similarly or oppositely ordered, then, if  $r < -\frac{1}{2}$ , we have

$$J(r; x_1y_1 + x_2y_2) \geq J(r; x_1, x_2)J(r; y_1, y_2); \quad (10)$$

if  $r \geq -\frac{1}{2}$ , then inequality (10) is reversed.

(4) For  $x > 0$  and  $y > 0$ , if  $r < s < t \leq -\frac{1}{2}$ , then

$$[J(s; x, y)]^{t-r} \leq [J(r; x, y)]^{t-s}[J(t; x, y)]^{s-r}; \quad (11)$$

if  $-\frac{1}{2} \leq r < s < t$ , inequality (11) is reversed.

Moreover, H. Alzer in [3] raised a question about the convexity of  $r \ln J(r; x, y)$  and proved that  $(r+1)J(r; x, y)$  is convex.

In April of 2004, Witkowski looked for the reference to the inequality

$$J(r; x, y)J(-r; x, y) \leq [J(0; x, y)]^2 = L^2(x, y), \quad (12)$$

which is contained in (8), through S. S. Dragomir by an e-mail which was forwarded to all members of the Research Group in Mathematical Inequalities and Applications at <http://rgmia.vu.edu.au>.

The main purpose of this paper is to prove the logarithmic convexity of the one-parameter mean values  $J(r; x, y)$  and the monotonicity of  $J(-r)J(r)$  for  $r \in \mathbb{R}$ . Our main results are as follows.

**Theorem 1.** *For fixed positive numbers  $x$  and  $y$  with  $x \neq y$ , we have*

- (i) *The one-parameter mean values  $J(r)$  defined by (7) are strictly increasing in  $r \in \mathbb{R}$ ;*
- (ii) *The one-parameter mean values  $J(r)$  defined by (7) are strictly logarithmically convex in  $(-\infty, -\frac{1}{2})$  and strictly logarithmically concave in  $(-\frac{1}{2}, \infty)$ .*

*Remark 1.* Though the monotonicity property of  $J(r; x, y)$  with  $r \in \mathbb{R}$  is well known, as a by-product of Theorem 1 and for completeness, we give it other two proofs below. However, we cannot affirm whether they are new proofs or not.

**Theorem 2.** *Let  $\mathcal{J}(r) = J(r)J(-r)$  with  $r \in \mathbb{R}$  for fixed positive numbers  $x$  and  $y$  with  $x \neq y$ . Then the function  $\mathcal{J}(r)$  is strictly increasing in  $(-\infty, 0)$  and strictly decreasing in  $(0, \infty)$ .*

*Remark 2.* Inequality (12) is clearly a direct consequence of Theorem 2.

## 2. PROOFS OF THEOREMS

*Proof of Theorem 1.* (i) Formula (6) implies that, for  $y > x > 0$ ,

$$\left(\frac{g'(t)}{g(t)}\right)'' \begin{cases} > 0, & t < 0, \\ = 0, & t = 0, \\ < 0, & t > 0. \end{cases} \quad (13)$$

From this, we obtain that the function  $\left(\frac{g'(t)}{g(t)}\right)'$  is strictly increasing in  $(-\infty, 0)$  and strictly decreasing in  $(0, \infty)$ .

In [14, 18], by using the Cauchy-Schwartz integral inequality or the Tchebycheff integral inequality, it is obtained that

$$\left(\frac{g'(t)}{g(t)}\right)' > 0 \quad (14)$$

for  $t \in \mathbb{R}$ . Then the function  $\frac{g'(t)}{g(t)}$  is strictly increasing in  $(-\infty, \infty)$ .

The one-parameter mean values  $J(r)$  can be rewritten in terms of  $g$  as

$$J(r) = \frac{g(r+1)}{g(r)} \quad (15)$$

with  $r \in \mathbb{R}$  for  $y > x > 0$ . Taking the logarithm of  $J(r)$  yields

$$\ln J(r) = \ln g(r+1) - \ln g(r) = \int_r^{r+1} \frac{g'(u)}{g(u)} du = \int_0^1 \frac{g'(u+r)}{g(u+r)} du \quad (16)$$

and

$$[\ln J(r)]' = \frac{g'(r+1)}{g(r+1)} - \frac{g'(r)}{g(r)} > 0. \quad (17)$$

Hence the functions  $\ln J(r)$  and  $J(r)$  are strictly increasing in  $r \in (-\infty, \infty)$ . This proves (i).

(ii) If  $r < -1$ , then  $r < r + 1 < 0$  and

$$[\ln J(r)]'' = \left( \frac{g'(r+1)}{g(r+1)} \right)' - \left( \frac{g'(r)}{g(r)} \right)' > 0 \quad (18)$$

which follows from the strictly increasing property of  $\left( \frac{g'(r)}{g(r)} \right)'$  in  $(-\infty, 0)$ .

If  $r > 0$ , then from the strictly decreasing property of  $\left( \frac{g'(r)}{g(r)} \right)'$  in  $(0, \infty)$ , we have  $[\ln J(r)]'' < 0$ .

If  $-1 < r < 0$ , then  $r < 0 < r + 1$ , and we have

$$\begin{aligned} [\ln J(r)]'' &= \left( \frac{g'(r+1)}{g(r+1)} \right)' - \left( \frac{g'(r)}{g(r)} \right)' \\ &= \frac{g''(r+1)g(r+1) - [g'(r+1)]^2}{g^2(r+1)} - \frac{g''(r)g(r) - [g'(r)]^2}{g^2(r)} \\ &= \frac{g''(u)g(u) - [g'(u)]^2}{g^2(u)} - \frac{g''(-r)g(-r) - [g'(-r)]^2}{g^2(-r)} \\ &= \frac{g''(u)g(u) - [g'(u)]^2}{g^2(u)} - \frac{g''(v)g(v) - [g'(v)]^2}{g^2(v)} \\ &= \left( \frac{g'(u)}{g(u)} \right)' - \left( \frac{g'(v)}{g(v)} \right)', \end{aligned} \quad (19)$$

where  $u = r + 1 > 0$  and  $v = -r > 0$ . Thus,  $[\ln J(r)]'' < 0$  for  $-1 < r < 0$  and  $r + 1 > -r$ . This means that  $[\ln J(r)]'' < 0$  for  $r \in (-\frac{1}{2}, 0)$ .

Similarly as above,  $[\ln J(r)]'' > 0$  for  $-1 < r < 0$  and  $-r > r + 1$ . This means that  $[\ln J(r)]'' > 0$  for  $r \in (-1, -\frac{1}{2})$ . This proves (ii).

The proof of Theorem 1 is completed.  $\square$

*Remark 3.* From (16), (13) and by direct calculation, we have

$$[\ln J(r)]'' = \int_0^1 \frac{d^2}{dr^2} \left( \frac{g'(u+r)}{g(u+r)} \right) du < 0 \quad (20)$$

for  $r \in (0, \infty)$ . This means that  $J(r; x, y)$  is strictly logarithmically concave in  $r \in (0, \infty)$ , whether  $x > y$  or  $x < y$ , since  $J(r; x, y) = J(r; y, x)$  holds.

By straightforward computation, we have

$$J(r) = \frac{xy}{J(-r-1)} \quad (21)$$

for  $r \in \mathbb{R}$ . Hence, if  $r \in (-\infty, -1)$ , from (6), (20) and (13), it follows that

$$[\ln J(r)]'' = -[\ln J(-r-1)]'' = - \int_0^1 \frac{d^2}{dr^2} \left( \frac{g'(u-r-1)}{g(u-r-1)} \right) du > 0. \quad (22)$$

This tells us that the one-parameter mean values  $J(r; x, y)$  are strictly logarithmically convex in  $r \in (-\infty, -1)$ , whether  $x > y$  or  $x < y$ , since  $J(r; x, y) = J(r; y, x)$ .

*Proof of Theorem 2.* By standard argument, we obtain

$$\mathcal{J}(r) = \frac{xyJ(r)}{J(r-1)} \quad (23)$$

for  $r \in \mathbb{R}$ . Then

$$\ln \mathcal{J}(r) = \ln(xy) + \ln J(r) - \ln J(r-1), \quad (24)$$

$$[\ln \mathcal{J}(r)]' = \frac{J'(r)}{J(r)} - \frac{J'(r-1)}{J(r-1)}. \quad (25)$$

Theorem 1 states that the function  $J(r)$  is strictly logarithmically convex in  $(-\infty, -\frac{1}{2})$ . Thus, being the derivative of  $\ln J(r)$ ,  $\frac{J'(r)}{J(r)}$  is strictly increasing in  $(-\infty, -\frac{1}{2})$ , that is

$$\frac{J'(r)}{J(r)} > \frac{J'(r-1)}{J(r-1)}, \quad (26)$$

or, equivalently,  $[\ln \mathcal{J}(r)]' > 0$  for  $r \in (-\infty, -\frac{1}{2})$ , thus  $\ln \mathcal{J}(r)$  and  $\mathcal{J}(r)$  are strictly increasing in  $(-\infty, -\frac{1}{2})$ .

From (21), it follows that

$$\ln J(r) = \ln(xy) - \ln J(-r-1), \quad (27)$$

$$\frac{J'(r)}{J(r)} = \frac{J'(-r-1)}{J(-r-1)}. \quad (28)$$

Then, from (25), we have

$$[\ln \mathcal{J}(r)]' = \frac{J'(-r-1)}{J(-r-1)} - \frac{J'(r-1)}{J(r-1)}. \quad (29)$$

For  $r \in (-\frac{1}{2}, 0)$ , we have  $-\frac{3}{2} < r-1 < -1$  and  $-1 < -r-1 < -\frac{1}{2}$ . Since  $\frac{J'(r)}{J(r)}$  is strictly increasing in  $(-\infty, -\frac{1}{2})$ ,  $[\ln \mathcal{J}(r)]' > 0$  for  $r \in (-\frac{1}{2}, 0)$ , therefore  $\ln \mathcal{J}(r)$  and  $\mathcal{J}(r)$  are also strictly increasing in  $(-\frac{1}{2}, 0)$ .

It is clear that the function  $\mathcal{J}(r)$  is even in  $(-\infty, \infty)$ . So, it is easy to see that  $\mathcal{J}(r)$  is strictly decreasing in  $(0, \infty)$ . The proof of Theorem 2 is completed.  $\square$

### 3. SOME RELATED RESULTS

For  $x \neq y$  and  $\alpha > 0$ , define

$$J_\alpha(r) \triangleq J_\alpha(r; x, y) = \begin{cases} \frac{r(x^{r+\alpha} - y^{r+\alpha})}{(r+\alpha)(x^r - y^r)}, & r \neq 0, -\alpha; \\ \frac{x^\alpha - y^\alpha}{\alpha(\ln x - \ln y)}, & r = 0; \\ \frac{\alpha x^\alpha y^\alpha (\ln x - \ln y)}{x^\alpha - y^\alpha}, & r = -\alpha. \end{cases} \quad (30)$$

We call  $J_\alpha(r; x, y)$  the generalized one-parameter mean values for two positive numbers  $x$  and  $y$  in the interval  $(-\infty, \infty)$ .

It is clear that  $J_1(r; x, y) = J(r; x, y)$  and  $J_\alpha(r; x, y) = \frac{g(r+\alpha)}{g(r)}$ .

By the same arguments as in the proofs of Theorems 1 and 2, we can obtain the following

**Theorem 3.** *For positive numbers  $x$  and  $y$  with  $x \neq y$ , we have*

- (1) *The generalized one-parameter mean values  $J_\alpha(r)$  defined by (30) are strictly increasing in  $r \in \mathbb{R}$ ;*

- (2) The generalized one-parameter mean values  $J_\alpha(r)$  defined by (30) are strictly logarithmically convex in  $(-\infty, -\frac{\alpha}{2})$  and strictly logarithmically concave in  $(-\frac{\alpha}{2}, \infty)$ .
- (3) Let  $\mathcal{J}_\alpha(r) = J_\alpha(r)J_\alpha(-r)$  with  $r \in \mathbb{R}$  for positive numbers  $x$  and  $y$  with  $x \neq y$ . Then the function  $\mathcal{J}_\alpha(r)$  is strictly increasing in  $(-\infty, 0)$  and strictly decreasing in  $(0, \infty)$ .

*Proof.* These follow from combining the identity

$$J_\alpha(r; x, y) = J\left(\frac{r}{\alpha}; x^\alpha, y^\alpha\right) \quad (31)$$

with Theorems 1 and 2.  $\square$

**Theorem 4.** The function  $(r + \alpha)J_\alpha(r)$  is strictly increasing and strictly convex in  $(-\infty, \infty)$ , and is strictly logarithmically concave for  $r > -\frac{\alpha}{2}$ .

*Proof.* Direct computation gives

$$(r + \alpha)J_\alpha(r; x, y) = \alpha\left(\frac{r}{\alpha} + 1\right)J\left(\frac{r}{\alpha}; x^\alpha, y^\alpha\right), \quad (32)$$

$$\{\ln[(r + \alpha)J_\alpha(r)]\}'' = -\frac{1}{(r + \alpha)^2} + [\ln J_\alpha(r)]''. \quad (33)$$

From the result by Alzer in [3] that the function  $(r+1)J(r; x, y)$  is strictly convex, it is not difficult to obtain that the function  $(r + \alpha)J_\alpha(r; x, y)$  is also strictly convex in  $(-\infty, \infty)$  by using (32).

By standard argument, we have

$$\lim_{r \rightarrow -\infty} [(r + \alpha)J'_\alpha(r)] = \lim_{r \rightarrow -\infty} \frac{\alpha(z^{r+\alpha} - 1)}{(r + \alpha)(z^r - 1)} - \lim_{r \rightarrow -\infty} \frac{rz^r(z^\alpha - 1)\ln z}{(z^r - 1)^2} = 0 \quad (34)$$

and

$$\lim_{r \rightarrow -\infty} J_\alpha(r) = \min\{x^\alpha, y^\alpha\}, \quad (35)$$

where  $z = \frac{y}{x} \neq 1$ . This leads to

$$\lim_{r \rightarrow -\infty} [(r + \alpha)J_\alpha(r)]' = \lim_{r \rightarrow -\infty} J_\alpha(r) + \lim_{r \rightarrow -\infty} [(r + \alpha)J'_\alpha(r)] = \min\{x^\alpha, y^\alpha\} > 0. \quad (36)$$

The convexity of  $(r + \alpha)J_\alpha(r)$  means that  $[(r + \alpha)J_\alpha(r)]'$  is strictly increasing, in view of (36),  $[(r + \alpha)J_\alpha(r)]' > 0$ , and so  $(r + \alpha)J_\alpha(r)$  is strictly increasing in  $(-\infty, \infty)$ .

Since  $J_\alpha(r)$  is strictly logarithmically concave in  $(-\frac{\alpha}{2}, \infty)$ , we have  $[\ln J_\alpha(r)]'' < 0$ , then  $\{\ln[(r + \alpha)J_\alpha(r)]\}'' < 0$  by (33). This means that the function  $(r + \alpha)J_\alpha(r)$  is strictly logarithmically concave in  $(-\frac{\alpha}{2}, \infty)$ .  $\square$

**Corollary 1.** If  $r < -\alpha$ , then

$$0 < \frac{J'_\alpha(r)}{J_\alpha(r)} = \frac{J'_\alpha(-r - \alpha)}{J_\alpha(-r - \alpha)} < -\frac{1}{r + \alpha}, \quad (37)$$

$$0 < \frac{J''_\alpha(r)}{J'_\alpha(r)} < -\frac{2}{r + \alpha}. \quad (38)$$

*Proof.* From the monotonicity and convexity of  $(r + \alpha)J_\alpha(r)$ , we have

$$[(r + \alpha)J_\alpha(r)]' = J_\alpha(r) + (r + \alpha)J'_\alpha(r) > 0, \quad (39)$$

$$[(r + \alpha)J_\alpha(r)]'' = 2J'_\alpha(r) + (r + \alpha)J''_\alpha(r) > 0. \quad (40)$$

Inequality (37) follows from the combination of (39) and

$$J_\alpha(r) = \frac{xy}{J_\alpha(-r - \alpha)}. \quad (41)$$

Inequality (38) is a direct consequence of (40).  $\square$

**Theorem 5.** *The function  $r \ln J_\alpha(r)$  is strictly convex in  $(-\frac{\alpha}{2}, 0)$ .*

*Proof.* Direct calculation yields

$$[r \ln J_\alpha(r)]'' = 2[\ln J_\alpha(r)]' + r[\ln J_\alpha(r)]''. \quad (42)$$

Since  $J_\alpha(r)$  is strictly increasing in  $(-\infty, \infty)$  and strictly logarithmically concave in  $(-\frac{\alpha}{2}, \infty)$ , it follows that  $[\ln J_\alpha(r)]' > 0$  and  $[\ln J_\alpha(r)]'' < 0$  in  $(-\frac{\alpha}{2}, \infty)$ . Therefore,  $[r \ln J_\alpha(r)]'' > 0$  and  $r \ln J_\alpha(r)$  is strictly convex in  $(-\frac{\alpha}{2}, 0)$ .  $\square$

*Remark 4.* If  $\alpha = 1$  and  $\beta = 0$ , then  $r \ln J(r)$  is strictly convex in  $(-\frac{1}{2}, 0)$ . This partially answers the question raised by Alzer in [3].

#### 4. OPEN PROBLEMS

Finally, we pose the following

**Open Problem 1.** *The generalized one-parameter mean values  $J_\alpha(r)$  defined by (30) are strictly concave in  $(-\frac{\alpha}{2}, \infty)$ .*

**Open Problem 2.** *The function  $\mathcal{J}_\alpha(t) = J_\alpha(t)J_\alpha(-t)$  is strictly logarithmically convex for  $t \notin [-\frac{\alpha}{2}, \frac{\alpha}{2}]$  and strictly concave and strictly logarithmically concave for  $t \in (-\frac{\alpha}{2}, \frac{\alpha}{2})$ .*

**Open Problem 3.** *The function  $J_\alpha(r) + J_\alpha(-r)$  is strictly decreasing in  $(-\infty, 0)$ , strictly increasing in  $(0, \infty)$ , strictly convex in  $(-r_\alpha, r_\alpha)$ , and strictly concave for  $r \notin [-r_\alpha, r_\alpha]$ , where  $r_\alpha > 0$  is a constant dependent on  $\alpha$ .*

*Remark 5.* The following conclusions are well known.

- (1) Although a logarithmically convex function is also convex, a convex function may be not logarithmically convex.
- (2) A logarithmically concave function may be not concave.
- (3) A concave function may be not logarithmically concave.

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#### REFERENCES

- [1] H. Alzer, *On Stolarsky's mean value family*, Internat. J. Math. Ed. Sci. Tech. **20** (1987), no. 1, 186–189.
- [2] H. Alzer, *Über eine einparametrische familie von Mitlewerten*, Bayer. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. **1987** (1988), 23–29. (German)
- [3] H. Alzer, *Über eine einparametrische familie von Mitlewerten, II*, Bayer. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. **1988** (1989), 23–29. (German)
- [4] J.-Ch. Kuang, *Applied Inequalities*, 2nd ed., Hunan Education Press, Changsha City, Hunan Province, China, 1993. (Chinese)
- [5] J.-Ch. Kuang, *Applied Inequalities*, 3rd ed., Shangdong Science and Technology Press, Jinan City, Shangdong Province, China, 2004. (Chinese)



- [6] E. Leach and M. Sholander, *Extended mean values*, Amer. Math. Monthly **85** (1978), 84–90.
- [7] E. Leach and M. Sholander, *Extended mean values II*, J. Math. Anal. Appl. **92** (1983), 207–223.
- [8] Z. Páles, *Inequalities for differences of powers*, J. Math. Anal. Appl. **131** (1988), 271–281.
- [9] F. Qi, *Generalized abstracted mean values*, J. Inequal. Pure Appl. Math. **1** (2000), no. 1, Art. 4. <http://jipam.vu.edu.au/article.php?sid=97>. RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 4, 633–642. <http://rgmia.vu.edu.au/v2n5.html>.
- [10] F. Qi, *Generalized weighted mean values with two parameters*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **454** (1998), no. 1978, 2723–2732.
- [11] F. Qi, *Logarithmic convexity of extended mean values*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1787–1796. RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 5, 643–652. Available online at <http://rgmia.vu.edu.au/v2n5.html>.
- [12] F. Qi, *Schur-convexity of the extended mean values*, Rocky Mountain J. Math. (2004), in press. RGMIA Res. Rep. Coll. **4** (2001), no. 4, Art. 4, 529–533. Available online at <http://rgmia.vu.edu.au/v4n4.html>.
- [13] F. Qi, *The extended mean values: definition, properties, monotonicities, comparison, convexities, generalizations, and applications*, Cubo Matemática Educacional **5** (2003), no. 3, 63–90. RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 5, 57–80. Available online at <http://rgmia.vu.edu.au/v5n1.html>.
- [14] F. Qi and Q.-M. Luo, *A simple proof of monotonicity for extended mean values*, J. Math. Anal. Appl. **224** (1998), no. 2, 356–359.
- [15] F. Qi, J. Sándor, S. S. Dragomir, and A. Sofo, *Notes on the Schur-convexity of the extended mean values*, Taiwanese J. Math. **9** (2005), in press. RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 3, 19–27. Available online at <http://rgmia.vu.edu.au/v5n1.html>.
- [16] F. Qi and S.-L. Xu, *Refinements and extensions of an inequality, II*, J. Math. Anal. Appl. **211** (1997), no. 2, 616–620.
- [17] F. Qi and S.-L. Xu, *The function  $(b^x - a^x)/x$ : Inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359.
- [18] F. Qi, S.-L. Xu, and L. Debnath, *A new proof of monotonicity for extended mean values*, Internat. J. Math. Math. Sci. **22** (1999), no. 2, 415–420.
- [19] F. Qi and Sh.-Q. Zhang, *Note on monotonicity of generalized weighted mean values*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455** (1999), no. 1989, 3259–3260.
- [20] R. Yang and D. Cao, *Generalizations of the logarithmic mean*, J. Ningbo Univ. **2** (1989), no. 2, 105–108.

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