A Further Generalization of Hardy-Hilbert's Integral Inequality with Parameter and Applications

A FURTHER GENERALIZATION OF HARDY-HILBERT'S INTEGRAL INEQUALITY WITH PARAMETER AND APPLICATIONS

HE LEPING, SEVER S. DRAGOMIR, AND YANG QIAOSHUN

ABSTRACT. In this paper, by introducing some parameters and by employing a sharpening of Hölder's inequality, a new generalization of Hardy-Hilbert integral inequality involving the Beta function is established. At the same time, an extension of Widder's theorem is given.

1. Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f, g : (0, \infty) \to (0, \infty)$ are so that

$$0 < \int_0^\infty f^p(t)dt < \infty, \qquad 0 < \int_0^\infty g^q(t)dt < \infty.$$

Then we may state the following integral inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)dt \right)^{\frac{1}{q}}, \tag{1.1}$$

in which the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

The inequality (1.1) is well known in the litarature as the Hardy-Hilbert's integral inequality.

Recently, some improvements and generalizations of Hardy-Hilbert's integral inequality have been given. For instance, we refer the reader to the papers [2]–[7] and the bibliography therein.

The main purpose of this paper is to establish a new extended Hardy-Hilbert's type inequality, which includes improvements and generalisations of the corresponding results from [2]–[3].

2. Lemmas and their Proofs

For convenience, we firstly introduce some notations:

$$(f^r, g^s) = \int_\alpha^\infty f^r(x)g^s(x)dx, \quad \|f\|_p = \left(\int_\alpha^\infty f^p(x)dx \right)^{\frac{1}{p}},$$

$$\|f\|_2 = \|f\|, \quad S_r(H, x) = \left(H^{r/2}, x \right) \|H\|_r^{-r/2},$$

where x is a parametric variable unit vector. Clearly, $S_r(H, x) = 0$ when the vector x selected is orthogonal to $H^{\frac{p}{2}}$.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hardy-Hilbert's integral inequality, Hölder's inequality, Weight coefficient, Beta function, Widder's theorem.

Throughout this paper, m is taken to be $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

In order to state our results, we need to point out the following lemmas.

Lemma 1. *Let $f(x), g(x) > 0, x \in (0, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < ||f||_p < \infty, 0 < ||g||_q < \infty$, then*

$$(f, g) < ||f||_p ||g||_q (1 - R)^m, \tag{2.1}$$

where $R = (S_p(f, h) - S_q(g, h))^2$, $||h|| = 1$, $f^{p/2}(x), g^{q/2}(x)$ and $h(x)$ are linearly independent.

The lemma is proved in [4], and we omit the details.

In the following, we define

$$k_\lambda = B\left(\frac{p + \lambda - 2}{p}, \frac{q + \lambda - 2}{q}\right),$$

$$\theta_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du \quad (r = p, q),$$

where

$$B(u, v) = \int_0^1 \frac{t^{(-1+u)}}{(1+t)^{u+v}} dt \quad (u, v > 0)$$

is the *Beta function.*

The following lemma also holds.

Lemma 2. *Let $b < 1, \lambda > 0$. Define the function*

$$\varphi(b, y) = y^{-1+b} \int_0^y \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^b du, \qquad y \in (0, 1].$$

Then we have

$$\varphi(b, y) > \varphi(b, 1), \qquad (0 < y < 1). \tag{2.2}$$

A proof of Lemma 2 is given in paper [5], and we omit it here.

Another technical result that will be required in the following is:

Lemma 3. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $\alpha \geq -\beta$. Define the weight function ω_λ by*

$$\omega_\lambda(\alpha, \beta, r, x) = \int_\alpha^\infty \frac{1}{(x + y + 2\beta)^\lambda}\left(\frac{x+\beta}{y+\beta}\right)^{\frac{(2-\lambda)}{r}} dy \qquad x \in (\alpha, \infty). \tag{2.3}$$

(i) *For $\alpha = -\beta$,*

$$\omega_\lambda(-\beta, \beta, r, x) = k_\lambda (x + \beta)^{1-\lambda} \qquad x \in (-\beta, \infty). \tag{2.4}$$

(ii) *For $\alpha > -\beta$,*

$$\omega_\lambda(\alpha, \beta, r, x) < \left[k_\lambda - \theta_\lambda(r)\left(\frac{\alpha+\beta}{x+\beta}\right)^{1+\frac{(\lambda-2)}{r}}\right](x+\beta)^{1-\lambda} \quad x \in (\alpha, \infty). \tag{2.5}$$

Proof. Setting $u = (y + \beta)/(x + \beta)$, we have

$$\omega_\lambda(\alpha, \beta, r, x) = (x + \beta)^{1-\lambda} \int_{\frac{\alpha+\beta}{x+\beta}}^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du.$$

(i) For $\alpha = -\beta$, (2.4) is valid.
(ii) For $\alpha > -\beta$, we have

$$\omega_\lambda(\alpha, \beta, r, x)$$
$$= (x+\beta)^{1-\lambda} \left\{ \int_0^\infty \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du - \int_0^{\frac{\alpha+\beta}{x+\beta}} \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du \right\} \tag{2.6}$$

$$= (x+\beta)^{1-\lambda} \left\{ k_\lambda - \left(\frac{\alpha+\beta}{x+\beta}\right)^{1+\frac{(\lambda-2)}{r}} \varphi\left(\frac{2-\lambda}{r}, \frac{\alpha+\beta}{x+\beta}\right) \right\}. \tag{2.7}$$

Putting $b = \frac{2-\lambda}{r}$, and since $\lambda > 2 - \min\{p, q\}$, $b < 1$ is valid, then by Lemma 2 we get

$$\varphi\left(\frac{2-\lambda}{r}, \frac{\alpha+\beta}{x+\beta}\right) > \varphi\left(\frac{2-\lambda}{r}, 1\right) = \theta_\lambda(r) \qquad (x \in (0, \infty)). \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain (2.5). The proof is completed.

∎

Finally, the following result is needed as well.

Lemma 4. *Let $a_n (n = 0, 1, 2, 3, \dots)$ be complex numbers. If*

$$A(z) := \sum_{n=0}^\infty a_n z^n$$

is analytic on unit disk $|z| \le 1$, and

$$A^*(z) := \sum_{n=0}^\infty \frac{a_n z^n}{n!}$$

is analytic on $|z| < \infty$, then

$$\int_0^1 |A(x)|^2 dx = \int_0^1 \left| \int_0^\infty e^{-s/x} A^*(s) ds \right|^2 \frac{1}{x^2} dx, \tag{2.9}$$

where $s \in (0, \infty)$, $\quad x \in (0, 1]$.

Proof. Since $A^*(z)$ is analytic on the complex plane, the series

$$\sum_{n=0}^\infty \frac{e^{-t} a_n (xt)^n}{n!}$$

is uniformly convergent in $(0,\infty)$, and we obtain

$$\begin{aligned}\int_0^\infty e^{-t}A^*(tx)dt &= \int_0^\infty e^{-t}\sum_{n=0}^\infty \frac{a_n(xt)^n}{n!}dt\\ &= \sum_{n=0}^\infty \frac{a_nx^n}{n!}\int_0^\infty t^ne^{-t}dt\\ &= \sum_{n=0}^\infty a_nx^n = A(x).\end{aligned}$$

Setting $tx=s$, then

$$A(x)=\frac{1}{x}\int_0^\infty e^{-s/x}A^*(s)ds$$

whence

$$\int_0^1 |A(x)|^2\,dx=\int_0^1\left|\int_0^\infty e^{-s/x}A^*(s)ds\right|^2\frac{1}{x^2}dx.$$

The lemma is thus proved. ∎

3. Main Results

For the sake of convenience, we need the following notations:

$$F(x,y)=\frac{f(x)}{(x+y+2\beta)^{\lambda/p}}\left(\frac{x+\beta}{y+\beta}\right)^{\frac{(2-\lambda)}{pq}},$$

$$G(x,y)=\frac{g(y)}{(x+y+2\beta)^{\lambda/q}}\left(\frac{y+\beta}{x+\beta}\right)^{\frac{(2-\lambda)}{pq}},$$

$$\phi(r,x)=\int_0^{\frac{\alpha+\beta}{x+\beta}}\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r}du, \tag{3.1}$$

$$S_p(F,h)=\left\{\int_\alpha^\infty\int_\alpha^\infty F^{p/2}hdxdy\right\}\left\{\int_\alpha^\infty[k_\lambda-\phi(q,x)](x+\beta)^{1-\lambda}f^p(x)dx\right\}^{-\frac{1}{2}},$$

and

$$S_q(G,h)=\left\{\int_\alpha^\infty\int_\alpha^\infty G^{q/2}hdxdy\right\}\left\{\int_\alpha^\infty[k_\lambda-\phi(p,x)](x+\beta)^{1-\lambda}g^q(x)dx\right\}^{-\frac{1}{2}},$$

where $h=h(x,y)$ is a unit vector satisfying the property

$$||h||=\left\{\int_\alpha^\infty\int_\alpha^\infty h^2(x,y)dxdy\right\}^{\frac{1}{2}}=1$$

and $F^{p/2},G^{q/2},h$ are linearly independent.

The first main result is incorporated in the following theorem.

Theorem 1. *Let* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda>2-\min\{p,q\},\ \alpha\geq-\beta,$ *and* $f,g>0.$ *Assume also that*

$$0<\int_\alpha^\infty(t+\beta)^{1-\lambda}f^p(t)dt<\infty,$$

and

$$0<\int_\alpha^\infty(t+\beta)^{1-\lambda}g^q(t)dt<\infty.$$

(i) *If* $\alpha > -\beta$, *then we have*

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)g(y)}{(x+y+2\beta)^{\lambda}}dxdy \tag{3.2}$$
$$< \left\{\int_{\alpha}^{\infty}\left(k_{\lambda}-\theta_{\lambda}(q)\left(\frac{\alpha+\beta}{t+\beta}\right)^{1+(\lambda-2)/q}\right)(t+\beta)^{1-\lambda}f^{p}(t)dt\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\alpha}^{\infty}\left(k_{\lambda}-\theta_{\lambda}(p)\left(\frac{\alpha+\beta}{t+\beta}\right)^{1+(\lambda-2)/p}\right)(t+\beta)^{1-\lambda}g^{q}(t)dt\right\}^{\frac{1}{q}}(1-R_{\lambda})^{m}.$$

(ii) *If* $\alpha = -\beta$, *then we have*

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)g(y)}{(x+y+2\beta)^{\lambda}}dxdy \tag{3.3}$$
$$< k_{\lambda}\left(\int_{-\beta}^{\infty}(t+\beta)^{1-\lambda}f^{p}(t)dt\right)^{\frac{1}{p}}\left(\int_{-\beta}^{\infty}(t+\beta)^{1-\lambda}g^{q}(t)dt\right)^{\frac{1}{q}}(1-\overline{R}_{\lambda})^{m},$$

where

$$R_{\lambda} = \left(S_{p}(F,h)-S_{q}(G,h)\right)^{2},$$

while the function h *is defined by*

$$h(x,y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{e^{\alpha-x}}{(x+y-2\alpha)^{\frac{1}{2}}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{1}{4}}. \tag{3.4}$$

Proof. By Lemma 1 and the equality (2.3), we have

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)g(y)}{(x+y+2\beta)^{\lambda}}dxdy \tag{3.5}$$
$$= \int_{\alpha}^{\infty}\int_{\alpha}^{\infty}FGdxdy$$
$$\le \left\{\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}F^{p}dxdy\right\}^{\frac{1}{p}}\left\{\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}G^{q}dxdy\right\}^{\frac{1}{q}}(1-R_{\lambda})^{m}$$
$$= \left(\int_{\alpha}^{\infty}\omega_{\lambda}(\alpha,\beta,q,t)f^{p}(t)dt\right)^{\frac{1}{p}}\left(\int_{\alpha}^{\infty}\omega_{\lambda}(\alpha,\beta,p,t)g^{q}(t)dt\right)^{\frac{1}{q}}(1-R_{\lambda})^{m}.$$

Substituting (2.5) and (2.4) into the inequality (3.5) respectively, the inequalities (3.2) and (3.3) follow.

Next, let us discuss the expression R_{λ}.

We can choose the function h indicated by (3.4). Setting $s = x-\alpha$ and $t = y-\alpha$, we get

$$\|h\|^{2} = \int_{\alpha}^{\infty}\int_{\alpha}^{\infty}h^{2}(x,y)dxdy = \frac{2}{\pi}\int_{0}^{\infty}e^{-2s}ds\int_{0}^{\infty}\frac{1}{s+t}\left(\frac{s}{t}\right)^{\frac{1}{2}}dt = 1.$$

Hence, $\|h\| = 1$.

By Lemma 1 and the given h, we have

$$R_\lambda = \left\{ \left(\int_\alpha^\infty \int_\alpha^\infty F^{p/2} h \, dxdy \right) \left(\int_\alpha^\infty \int_\alpha^\infty F^p \, dxdy \right)^{-\frac{1}{2}} - \left(\int_\alpha^\infty \int_\alpha^\infty G^{q/2} h \, dxdy \right) \left(\int_\alpha^\infty \int_\alpha^\infty G^q \, dxdy \right)^{-\frac{1}{2}} \right\}^2 . \tag{3.6}$$

Substituting (2.3), (2.6) and (3.1) into (3.6), we get

$$R_\lambda = (S_p(F,h) - S_q(G,h))^2.$$

It is obvious that $F^{p/2}, G^{q/2}$ and h are linearly independent, so it is impossible for equality to hold in (3.5).

The proof is thus completed. ∎

Owing to $p, q > 1$, when $\lambda = 1, 2$; the condition $\lambda > 2 - \min\{p, q\}$ is satisfied. We have

$$\theta_1(r) = \int_0^1 \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{r}} du > \int_0^1 \frac{1}{1+u} du = \ln 2, \quad k_1 = B\left(\frac{1}{p}, \frac{1}{q}\right) = \frac{\pi}{\sin(\pi/p)},$$

$$\theta_2(r) = \int_0^1 \frac{1}{(1+u)^2} du = \frac{1}{2}, \quad k_2 = B\left(\frac{p+2-2}{p}, \frac{q+2-2}{q}\right) = B(1,1) = 1.$$

The following results are natural consequences of Theorem 1.

Corollary 1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq \beta$, $f, g > 0$,*

$$0 < \int_\alpha^\infty f^p(t) dt < \infty,$$

and

$$0 < \int_\alpha^\infty g^q(t) dt < \infty,$$

then

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{x+y+2\beta} dxdy < \left\{ \int_\alpha^\infty \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\alpha+\beta}{t+\beta}\right)^{\frac{1}{p}} \cdot ln2 \right) f^p(t) dt \right\}^{\frac{1}{p}} \times \left\{ \int_\alpha^\infty \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\alpha+\beta}{t+\beta}\right)^{\frac{1}{q}} \cdot ln2 \right) g^q(t) dt \right\}^{\frac{1}{q}} (1 - R_1)^m, \tag{3.7}$$

$$\int_{-\beta}^\infty \int_{-\beta}^\infty \frac{f(x)g(y)}{x+y+2\beta} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_{-\beta}^\infty f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{-\beta}^\infty g^q(t) dt \right)^{\frac{1}{q}} (1 - \overline{R}_1)^m, \tag{3.8}$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)dt\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)dt\right)^{\frac{1}{q}} (1-r_1)^m. \tag{3.9}$$

Remark 1. *When $p = q = 2$, the inequality (3.9) reduces, after some simple computation, to an inequality obtained in* [2].

Corollary 2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq \beta$, $f, g > 0$,*

$$0 < \int_\alpha^\infty (t+\beta)^{-1} f^p(t)dt < \infty$$

and

$$0 < \int_\alpha^\infty (t+\beta)^{-1} g^q(t)dt < \infty,$$

then

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y+2\beta)^2}dxdy < \left\{\int_\alpha^\infty \left(1 - \frac{\alpha+\beta}{2(t+\beta)}\right) \frac{1}{t+\beta} f^p(t)dt\right\}^{\frac{1}{p}} \times \left\{\int_\alpha^\infty \left(1 - \frac{\alpha+\beta}{2(t+\beta)}\right) \frac{1}{t+\beta} g^q(t)dt\right\}^{\frac{1}{q}} (1-R_2)^m \tag{3.10}$$

and

$$\int_{-\beta}^\infty \int_{-\beta}^\infty \frac{f(x)g(y)}{(x+y+2\beta)^2}dxdy < \left(\int_{-\beta}^\infty \frac{1}{t+\beta} f^p(t)dt\right)^{\frac{1}{p}} \left(\int_{-\beta}^\infty \frac{1}{t+\beta} g^q(t)dt\right)^{\frac{1}{q}} (1-\overline{R}_2)^m. \tag{3.11}$$

Remark 2. *The inequalities (3.2), (3.3) and (3.7) – (3.9) are generalizations of (1.1).*

Remark 3. *We can also define $h(x,y)$ as*

$$h(x,y) = \begin{cases} 1 & (x,y) \in [0,1] \times [0,1] \\ 0 & (x,y) \in (0,\infty) \times (0,\infty) \diagdown [0,1] \times [0,1]. \end{cases}$$

In this case, the expression of R_λ will be much simpler. The details are omitted.

4. Applications

We start with the following result:

Theorem 2. *Suppose that $a_n (n = 0, 1, 2, 3, \ldots)$ are complex numbers. Also, define $A(x) = \sum_{n=0}^\infty a_n x^n$, $A^*(x) = \sum_{n=0}^\infty \frac{a_n x^n}{n!}$, and the function f as:*

$$f(x) = e^{-x} A^*(x), \qquad x \in (0, \infty). \tag{4.1}$$

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_0^1 |A(x)|^2 \, dx < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_0^\infty |f(x)|^q \, dx \right)^{\frac{1}{q}} (1 - \overline{R})^m, \tag{4.2}$$

where

$$\overline{R} := \left(S_p \left(\overline{F}, h \right) - S_q \left(\overline{G}, h \right) \right)^2 > 0,$$

with $\|h\| = 1$, *and*

$$\overline{F} := \frac{|f(s)|}{(s+t)^{\frac{1}{p}}} \left(\frac{s}{t} \right)^{\frac{1}{pq}},$$

$$\overline{G} := \frac{|f(t)|}{(s+t)^{\frac{1}{q}}} \left(\frac{t}{s} \right)^{\frac{1}{pq}}; \quad \psi(t) := \int_0^\infty \frac{e^{-s}}{(s+t)} \left(\frac{t}{s} \right)^{\frac{(q-p)}{2pq}} ds,$$

$$S_p(\overline{F}, h) := \sqrt{2} \left\{ \int_0^\infty e^{-s} |f(s)|^{p/2} \, ds \right\} \cdot \left\{ \int_0^\infty |f(s)|^p \, ds \right\}^{-\frac{1}{2}},$$

$$S_q(\overline{G}, h) := \frac{\sqrt{2} \sin(\pi/p)}{\pi} \left\{ \int_0^\infty \psi(t) |f(t)|^{q/2} \, dt \right\} \cdot \left\{ \int_0^\infty |f(s)|^q \, ds \right\}^{-\frac{1}{2}}$$

and

$$(\overline{F})^{p/2}, (\overline{G})^{q/2}, h$$

are linearly independent.

Proof. Setting $y = \frac{1}{x}$ on the right-hand side of the equality (2.9), we have

$$\int_0^1 |A(x)|^2 \, dx = \int_1^\infty \left| \int_0^\infty e^{-sy} A^*(s) ds \right|^2 dy. \tag{4.3}$$

Next, put $u = y - 1$. According to the equalities (4.1) and (4.3), we get

$$\int_0^1 |A(x)|^2 \, dx = \int_0^\infty du \left| \int_0^\infty e^{-su} f(s) ds \right|^2.$$

Using Hardy's technique, we may state that

$$\begin{aligned}
\int_0^1 |A(x)|^2 \, dx &= \int_0^\infty du \left| \int_0^\infty e^{-su} f(s) ds \right|^2 \\
&= \int_0^\infty du \int_0^\infty e^{-su} f(s) ds \overline{\int_0^\infty e^{-tu} f(t) dt} \\
&\le \int_0^\infty \int_0^\infty \left(\int_0^\infty e^{-(s+t)u} du \right) |f(s)| \, |f(t)| \, ds dt \\
&= \int_0^\infty \int_0^\infty \frac{|f(s)| \, |f(t)|}{s+t} ds dt \\
&< \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty |f(x)|^p \right)^{\frac{1}{p}} \left(\int_0^\infty |f(x)|^q \right)^{\frac{1}{q}} (1 - \overline{R})^m.
\end{aligned}$$

Let us choose the function $h(s,t)$ to be defined by

$$h(s,t) := \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}} \left(\frac{s}{t} \right)^{\frac{1}{4}},$$

then

$$\|h\| = \left\{ \int_0^\infty \int_0^\infty h^2(s,t) ds dt \right\}^{\frac{1}{2}} = 1.$$

Notice that $k_1(p) = B(\frac{1}{q}, \frac{1}{p}) = \pi / \sin(\frac{\pi}{p})$, and, in a similar way to the one in Theorem 1, the expression of $\overline{R}$ is easily given. We omit the details. ∎

Remark 4. *In particular, when $p = q = 2$, it follows from (4.2) that*

$$\int_0^1 A^2(x) dx = \pi(1-r)^{\frac{1}{2}} \int_0^\infty f^2(x) dx. \tag{4.4}$$

If r in (4.4) is replaced by zero, then Widder's theorem (see [8]*) can be recaptured.*

Remark 5. *After simple computation, the inequality (4.4) is equivalent to the inequality (3.4) in* [2]*. Consequently, inequality (4.2) is an extension of (3.4) in* [2]*.*

References

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, UK, 1952.

[2] Gao Mingzhe, Tan Li and Lokenath Debnath, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.,* Vol. **229**(1999), 682-689.

[3] Gao Mingzhe, On the Hilbert inequality, *Zeitschrift für Analysis und ihre Anwendungen,* Vol. **18**, No. 4 (1999), 1117-1122.

[4] Leping He, Mingzhe Gao and Weijian Jia, On the improvement of the Hardy-Hilbert's integral inequality with parameters, *Journal of Inequalities in Pure and Applied Mathematics*, Vol. **4**(2003), No. 5, Art. 94, [ONLINE: http://jipam.vu.edu.au/v4n5/108_03.html].

[5] Yang Bicheng, On Hardy-Hilbert's integral inequality, *J. Math. Anal.Appl.*, Vol. **261**(2001), 295–306.

[6] He Leping, Gao Mingzhe and Wei Shangrong, A note on Hilbert's inequality, *Mathematical Inequalities & Applications*, Vol. **6**(2003), No. 2, 283-288.

[7] Gao Mingzhe, Wei Shangrong and He Leping, On the Hilbert inequality with weights, *Zeitschrift für Analysis und ihre Anwendungen*, Vol. **21**(2002), No. 1, 257-263.

[8] D.V. Widder, An inequality related to one of Hilbert's, *J. London Math. Soc.* Vol. **4**(1924), 194-198.

(He Leping) Department of Mathematics and Computer Science, Jishou University, Jishou Hunan 416000, People's Republic of China

E-mail address: lepinghe65729@163.com

(S.S. Dragomir) School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, Melbourne City M.C., Victoria 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

(Yang Qiaoshun) Department of Mathematics and Computer Science, Normal College, Jishou University, Jishou, Hunan, 416000, People's Republic of China

E-mail address: yqs244@163.com