A Further Generalization of Hardy-Hilbert's Integral Inequality with Parameter and Applications

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A FURTHER GENERALIZATION OF HARDY-HILBERT’S INTEGRAL INEQUALITY WITH PARAMETER AND APPLICATIONS

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Abstract. In this paper, by introducing some parameters and by employing a sharpening of Hölder’s inequality, a new generalization of Hardy-Hilbert integral inequality involving the Beta function is established. At the same time, an extension of Widder’s theorem is given.

1. Introduction

Suppose that \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), and \( f, g : (0, \infty) \to (0, \infty) \) are so that
\[
0 < \int_0^\infty f^p(t)dt < \infty, \quad 0 < \int_0^\infty g^q(t)dt < \infty.
\]
Then we may state the following integral inequality
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(t)dt \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(t)dt \right)^{\frac{1}{q}},
\]
in which the constant factor \( \frac{\pi}{\sin(\pi/p)} \) is the best possible.

The inequality (1.1) is well known in the literature as the Hardy-Hilbert’s integral inequality.

Recently, some improvements and generalizations of Hardy-Hilbert’s integral inequality have been given. For instance, we refer the reader to the papers [2]–[7] and the bibliography therein.

The main purpose of this paper is to establish a new extended Hardy-Hilbert’s type inequality, which includes improvements and generalisations of the corresponding results from [2]–[3].

2. Lemmas and their Proofs

For convenience, we firstly introduce some notations:
\[
(f^r, g^s) = \int_0^\infty f^r(x)g^s(x)dx, \quad \|f\|_p = \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}},
\]
\[
\|f\|_2 = \|f\|, \quad S_r(H, x) = \left( H^{r/2}, x \right) \|H\|^{r/2},
\]
where \( x \) is a parametric variable unit vector. Clearly, \( S_r(H, x) = 0 \) when the vector \( x \) selected is orthogonal to \( H \).

\[2000 \text{ Mathematics Subject Classification.} \quad 26D15.\]
\[\text{Key words and phrases.} \quad \text{Hardy-Hilbert’s integral inequality, Hölder’s inequality, Weight coefficient, Beta function, Widder’s theorem.}\]
Throughout this paper, \( m \) is taken to be \( m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\} \).

In order to state our results, we need to point out the following lemmas.

**Lemma 1.** Let \( f(x), g(x) > 0, x \in (0, \infty), \frac{1}{p} + \frac{1}{q} = 1 \) and \( p > 1 \). If \( 0 < \|f\|_p < \infty, 0 < \|g\|_q < \infty \), then

\[
(f, g) < \|f\|_p \|g\|_q (1 - R)^m,
\]

where \( R = (S_p(f, h) - S_q(g, h))^2, \|h\| = 1 \), \( f^{p/2}(x), g^{q/2}(x) \) and \( h(x) \) are linearly independent.

The lemma is proved in [4], and we omit the details.

In the following, we define

\[
k_\lambda = B \left( \frac{p + \lambda - 2}{p}, \frac{q + \lambda - 2}{q} \right),
\]

\[
\theta_\lambda(r) = \int_0^1 \frac{1}{(1 + u)^\lambda} \left( \frac{1}{u} \right) \left( \frac{u^{2-\lambda}}{r} \right) du \quad (r = p, q),
\]

where

\[
B(u, v) = \int_0^1 t^{(-1+u)} (1 + t)^{u+v} dt \quad (u, v > 0)
\]

is the Beta function.

The following lemma also holds.

**Lemma 2.** Let \( b < 1, \lambda > 0 \). Define the function

\[
\varphi(b, y) = y^{1+b} \int_0^y \frac{1}{(1+u)^\lambda} \left( \frac{1}{u} \right) du, \quad y \in (0, 1].
\]

Then we have

\[
\varphi(b, y) > \varphi(b, 1), \quad (0 < y < 1).
\]

A proof of Lemma 2 is given in paper [5], and we omit it here.

Another technical result that will be required in the following is:

**Lemma 3.** Let \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 2 - \min\{p, q\}, \alpha \geq -\beta \). Define the weight function \( \omega_\lambda \) by

\[
\omega_\lambda(\alpha, \beta, r, x) = \int_\alpha^{\infty} \frac{1}{(x + y + 2\beta)^\lambda} \left( \frac{x + \beta}{y + \beta} \right)^{\frac{2-\lambda}{r}} dy \quad x \in (\alpha, \infty).
\]

(i) For \( \alpha = -\beta \),

\[
\omega_\lambda(-\beta, \beta, r, x) = k_\lambda (x + \beta)^{1-\lambda} \quad x \in (-\beta, \infty).
\]

(ii) For \( \alpha > -\beta \),

\[
\omega_\lambda(\alpha, \beta, r, x) < \left[ k_\lambda - \theta_\lambda(r) \left( \frac{\alpha + \beta}{x + \beta} \right)^{\frac{(1-\lambda)\beta}{r}} \right] (x + \beta)^{1-\lambda} \quad x \in (\alpha, \infty).
\]

**Proof.** Setting \( u = (y + \beta)/(x + \beta) \), we have

\[
\omega_\lambda(\alpha, \beta, r, x) = (x + \beta)^{1-\lambda} \int_{\frac{\alpha + \beta}{x + \beta}}^{\infty} \frac{1}{(1 + u)^\lambda} \left( \frac{1}{u} \right)^{\frac{2-\lambda}{r}} du.
\]
(i) For $\alpha = -\beta$, (2.4) is valid.
(ii) For $\alpha > -\beta$, we have

$$\omega_\lambda(\alpha, \beta, r, x) = (x + \beta)^{1-\lambda} \left\{ \int_0^{\infty} \frac{1}{(1+u)^\lambda} \left( \frac{1}{u} \right)^{(2-\lambda)} du \right\}$$

$$- \int_0^{\frac{\alpha + \beta}{r + \beta}} \frac{1}{(1+u)^\lambda} \left( \frac{1}{u} \right)^{(2-\lambda)} du \right\}$$

$$= (x + \beta)^{1-\lambda} \left\{ k_\lambda = \left( \frac{\alpha + \beta}{x + \beta} \right)^{1+\frac{(\lambda - 2)}{r}} \varphi \left( \frac{2 - \lambda}{r}, \frac{\alpha + \beta}{x + \beta} \right) \right\}.$$  

Putting $b = \frac{2-\lambda}{r}$, and since $\lambda > 2 - \min\{p, q\}, b < 1$ is valid, then by Lemma 2 we get

$$\varphi \left( \frac{2 - \lambda}{r}, \frac{\alpha + \beta}{x + \beta} \right) > \varphi \left( \frac{2 - \lambda}{r}, 1 \right) = \theta_\lambda(r) \quad (x \in (0, \infty)).$$

Substituting (2.8) into (2.7), we obtain (2.5). The proof is completed.

Finally, the following result is needed as well.

**Lemma 4.** Let $a_n(n = 0, 1, 2, 3, \ldots)$ be complex numbers. If

$$A(z) := \sum_{n=0}^{\infty} a_n z^n$$

is analytic on unit disk $|z| \leq 1$, and

$$A^*(z) := \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

is analytic on $|z| < \infty$, then

$$\int_0^1 |A(x)|^2 dx = \int_0^1 \int_0^{\infty} e^{-s/x} A^*(s) ds \, dx^2 \, dx,$$

where $s \in (0, \infty), \quad x \in (0, 1]$.

**Proof.** Since $A^*(z)$ is analytic on the complex plane, the series

$$\sum_{n=0}^{\infty} \frac{e^{-t a_n(x t)^n}}{n!}$$
is uniformly convergent in \((0, \infty)\), and we obtain
\[
\int_0^\infty e^{-t} A^*(tx) dt = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{a_n(tx)^n}{n!} dt
\]
\[
= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \int_0^\infty t^n e^{-t} dt
\]
\[
= \sum_{n=0}^{\infty} a_n x^n = A(x).
\]

Setting \(tx = s\), then
\[
A(x) = \frac{1}{x} \int_0^\infty e^{-s/x} A^*(s) ds.
\]

whence
\[
\int_0^1 \left| A(x) \right|^2 dx = \int_0^1 \left| \int_0^\infty e^{-s/x} A^*(s) ds \right|^2 \frac{1}{x^2} dx.
\]

The lemma is thus proved.

3. Main Results

For the sake of convenience, we need the following notations:
\[
F(x, y) = \frac{f(x)}{(x + y + 2\beta)^{\lambda/p}},
\]
\[
G(x, y) = \frac{g(y)}{(x + y + 2\beta)^{\lambda/q}},
\]
\[
\phi(r, x) = \int_0^{2(-\lambda)/r} \frac{1}{(1 + u)^r} \left( \frac{1}{u} \right)^{(2-\lambda)/r} du,
\]
\[
S_p(F, h) = \left\{ \int_0^\infty \int_0^\infty F^{p/2} h dx dy \right\} \left\{ \int_0^\infty \left[ k_{\lambda} - \phi(q, x) \right] (x + \beta)^{1-\lambda} f^p(x) dx \right\}^{-\frac{1}{2}},
\]
and
\[
S_q(G, h) = \left\{ \int_0^\infty \int_0^\infty G^{q/2} h dx dy \right\} \left\{ \int_0^\infty \left[ k_{\lambda} - \phi(p, x) \right] (x + \beta)^{1-\lambda} g^q(x) dx \right\}^{-\frac{1}{2}},
\]
where \(h = h(x, y)\) is a unit vector satisfying the property
\[
\| h \| = \left\{ \int_0^\infty \int_0^\infty h^2(x, y) dx dy \right\}^{\frac{1}{2}} = 1
\]
and \(F^{p/2}, G^{q/2}, h\) are linearly independent.

The first main result is incorporated in the following theorem.

**Theorem 1.** Let \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 2 - \min\{p, q\}, \alpha \geq -\beta, \text{ and } f, g > 0.\)

Assume also that
\[
0 < \int_\alpha^\infty (t + \beta)^{1-\lambda} f^p(t) dt < \infty,
\]
and
\[
0 < \int_\alpha^\infty (t + \beta)^{1-\lambda} g^q(t) dt < \infty.
\]
(i) If $\alpha > -\beta$, then we have

\[
\int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x + y + 2\beta)^\lambda} \, dx \, dy < \left\{ \int_{\alpha}^{\infty} \left[ k_\lambda - \theta_\lambda(q) \left( \frac{\alpha + \beta}{t + \beta} \right)^{1+(\lambda-2)/q} \right] (t + \beta)^{1-\lambda} f^p(t) \, dt \right\}^{\frac{1}{p}} \times \left\{ \int_{\alpha}^{\infty} \left[ k_\lambda - \theta_\lambda(p) \left( \frac{\alpha + \beta}{t + \beta} \right)^{1+(\lambda-2)/p} \right] (t + \beta)^{1-\lambda} g^q(t) \, dt \right\}^{\frac{1}{q}} (1 - R_\lambda)^m.
\]

(ii) If $\alpha = -\beta$, then we have

\[
\int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x + y + 2\beta)^\lambda} \, dx \, dy < k_\lambda \left( \int_{-\beta}^{\infty} (t + \beta)^{1-\lambda} f^p(t) \, dt \right)^{\frac{1}{p}} \left( \int_{-\beta}^{\infty} (t + \beta)^{1-\lambda} g^q(t) \, dt \right)^{\frac{1}{q}} (1 - \overline{R}_\lambda)^m,
\]

where

\[
R_\lambda = (S_p(F,h) - S_q(G,h))^2,
\]

while the function $h$ is defined by

\[
h(x,y) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} e^{\alpha-x} \left( \frac{x - \alpha}{y - \alpha} \right) \left( \frac{x - \alpha}{y - \alpha} \right) \frac{1}{2}.
\]

**Proof.** By Lemma 1 and the equality (2.3), we have

\[
\int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x + y + 2\beta)^\lambda} \, dx \, dy = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} FG \, dx \, dy \\
\leq \left\{ \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} F^p \, dx \, dy \right\}^{\frac{1}{p}} \left\{ \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} G^q \, dx \, dy \right\}^{\frac{1}{q}} (1 - R_\lambda)^m \\
= \left( \int_{\alpha}^{\infty} \omega_\lambda(\alpha,\beta,q,t) f^p(t) \, dt \right)^{\frac{1}{p}} \left( \int_{\alpha}^{\infty} \omega_\lambda(\alpha,\beta,p,t) g^q(t) \, dt \right)^{\frac{1}{q}} (1 - R_\lambda)^m.
\]

Substituting (2.5) and (2.4) into the inequality (3.5) respectively, the inequalities (3.2) and (3.3) follow.

Next, let us discuss the expression $R_\lambda$.

We can choose the function $h$ indicated by (3.4). Setting $s = x - \alpha$ and $t = y - \alpha$, we get

\[
\|h\|^2 = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} h^2(x,y) \, dx \, dy = \frac{2}{\pi} \int_{0}^{\infty} e^{-2s} ds \int_{0}^{\infty} \frac{1}{s + t} \left( \frac{s}{t} \right)^{\frac{1}{2}} dt = 1.
\]

Hence, $\|h\| = 1$. 

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**HARDY-HILBERT’S INTEGRAL INEQUALITY**
By Lemma 1 and the given \( h \), we have
\[
R_\lambda = \left\{ \left( \int_\alpha^\infty \int_\alpha^\infty F^{p/2} h \, dx \, dy \right) \left( \int_\alpha^\infty \int_\alpha^\infty F^p \, dx \, dy \right)^{-\frac{1}{2}} - \left( \int_\alpha^\infty \int_\alpha^\infty G^{q/2} h \, dx \, dy \right) \left( \int_\alpha^\infty \int_\alpha^\infty G^q \, dx \, dy \right)^{-\frac{1}{2}} \right\}^2.
\]
Substituting (2.3), (2.6) and (3.1) into (3.6), we get
\[
R_\lambda = \left( S_p(F, h) - S_q(G, h) \right)^2.
\]
It is obvious that \( F^{p/2}, G^{q/2} \) and \( h \) are linearly independent, so it is impossible for equality to hold in (3.5).

The proof is thus completed.

Owing to \( p, q > 1 \), when \( \lambda = 1, 2; \) the condition \( \lambda > 2 - \min\{p, q\} \) is satisfied. We have
\[
\theta_1(r) = \int_0^1 \frac{1}{1 + u} \left( \frac{1}{u} \right)^\frac{1}{p} \, du > \int_0^1 \frac{1}{1 + u} \, du = \ln 2, \quad k_1 = B \left( \frac{1}{p}, \frac{1}{q} \right) = \frac{\pi}{\sin(\pi/p)}.
\]
\[
\theta_2(r) = \int_0^1 \frac{1}{(1 + u)^2} \, du = \frac{1}{2}, \quad k_2 = B \left( \frac{p + 2 - 2}{p}, \frac{q + 2 - 2}{q} \right) = B(1, 1) = 1.
\]
The following results are natural consequences of Theorem 1.

**Corollary 1.** If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha \geq \beta, f, g > 0, \)
\[
0 < \int_\alpha^\infty f^p(t) \, dt < \infty,
\]
and
\[
0 < \int_\alpha^\infty g^q(t) \, dt < \infty,
\]
then
\[
\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{x + y + 2\beta} \, dx \, dy < \left\{ \int_\alpha^\infty \left( \frac{\pi}{\sin(\pi/p)} - \left( \frac{\alpha + \beta}{\ell + \beta} \right)^\frac{1}{p} \cdot \ln 2 \right) f^p(t) \, dt \right\}^{\frac{1}{p}} \times \left\{ \int_\alpha^\infty \left( \frac{\pi}{\sin(\pi/p)} - \left( \frac{\alpha + \beta}{\ell + \beta} \right)^\frac{1}{q} \cdot \ln 2 \right) g^q(t) \, dt \right\}^{\frac{1}{q}} (1 - R_1)^m,
\]
\[
\int_{-\beta}^{\infty} \int_{-\beta}^{\infty} \frac{f(x)g(y)}{x + y + 2\beta} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} \left( \int_{-\beta}^{\infty} f^p(t) \, dt \right)^{\frac{1}{p}} \left( \int_{-\beta}^{\infty} g^q(t) \, dt \right)^{\frac{1}{q}} (1 - R_1)^m.
\]
and

\[
(3.9) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(t) \, dt \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(t) \, dt \right)^{\frac{1}{q}} (1 - r_1)^m.
\]

**Remark 1.** When \( p = q = 2 \), the inequality (3.9) reduces, after some simple computation, to an inequality obtained in [2].

**Corollary 2.** If \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \alpha \geq \beta \), \( f, g > 0 \),

\[
0 < \int_0^\infty (t + \beta)^{-1} f^p(t) \, dt < \infty
\]

and

\[
0 < \int_0^\infty (t + \beta)^{-1} g^q(t) \, dt < \infty,
\]

then

\[
(3.10) \quad \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y+2\beta)^2} \, dx \, dy < \left\{ \int_\alpha^\infty \left( 1 - \frac{\alpha + \beta}{2(t + \beta)} \right) \frac{1}{t + \beta} f^p(t) \, dt \right\}^{\frac{1}{p}} \times \left\{ \int_\alpha^\infty \left( 1 - \frac{\alpha + \beta}{2(t + \beta)} \right) \frac{1}{t + \beta} g^q(t) \, dt \right\}^{\frac{1}{q}} (1 - R_2)^m
\]

and

\[
(3.11) \quad \int_{-\beta}^\infty \int_{-\beta}^\infty \frac{f(x)g(y)}{(x+y+2\beta)^2} \, dx \, dy < \left( \int_{-\beta}^\infty \frac{1}{t + \beta} f^p(t) \, dt \right)^{\frac{1}{p}} \left( \int_{-\beta}^\infty \frac{1}{t + \beta} g^q(t) \, dt \right)^{\frac{1}{q}} (1 - \overline{R}_2)^m
\]

**Remark 2.** The inequalities (3.2), (3.3) and (3.7) – (3.9) are generalizations of (1.1).

**Remark 3.** We can also define \( h(x, y) \) as

\[
h(x, y) = \begin{cases} 
1 & (x, y) \in [0, 1] \times [0, 1] \\
0 & (x, y) \in (0, \infty) \times (0, \infty) \setminus [0, 1] \times [0, 1].
\end{cases}
\]

In this case, the expression of \( R_\lambda \) will be much simpler. The details are omitted.

4. Applications

We start with the following result:

**Theorem 2.** Suppose that \( a_n (n = 0, 1, 2, 3, \ldots) \) are complex numbers. Also, define\n\[ A(x) = \sum_{n=0}^\infty a_n x^n, \quad A^*(x) = \sum_{n=0}^\infty \frac{a_n x^n}{n!}, \]
and the function \( f \) as:

\[
f(x) = e^{-x} A^*(x), \quad x \in (0, \infty).
\]
If $p > 1$, then

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$(4.2) \int_0^1 |A(x)|^2 \, dx < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_0^\infty |f(x)|^q \, dx \right)^{\frac{1}{q}} (1 - R)^m,$$

where

$$R := (S_p(\mathcal{F}, h) - S_q(\mathcal{G}, h))^2 > 0,$$

with $\|h\| = 1$, and

$$\mathcal{F} := \frac{|f(s)|}{(s + t)^{\frac{1}{p}}} \left( \frac{t}{s} \right)^{\frac{1}{q}},$$

$$\mathcal{G} := \frac{|f(t)|}{(s + t)^{\frac{1}{q}}} \left( \frac{t}{s} \right)^{\frac{1}{p}}; \quad \psi(t) := \int_0^\infty \frac{e^{-t}}{(s + t)} \left( \frac{t}{s} \right)^{\frac{(n-p)}{2np}} \, ds,$$

$$S_p(\mathcal{F}, h) := \sqrt{2} \left\{ \int_0^\infty e^{-t} |f(s)|^{p/2} \, ds \right\} \left\{ \int_0^\infty |f(s)|^{q/2} \, ds \right\}^{-\frac{1}{2}},$$

$$S_q(\mathcal{G}, h) := \sqrt{2} \frac{\sin(\pi/p)}{\pi} \left\{ \int_0^\infty \psi(t) |f(t)|^{q/2} \, dt \right\} \left\{ \int_0^\infty |f(s)|^{q/2} \, ds \right\}^{-\frac{1}{2}}$$

are linearly independent.

**Proof.** Setting $y = \frac{1}{x}$ on the right-hand side of the equality (2.9), we have

$$(4.3) \int_1^0 |A(x)|^2 \, dx = \int_1^\infty \left| \int_0^\infty e^{-sy} A^*(s) \, ds \right|^2 \, dy.$$

Next, put $u = y - 1$. According to the equalities (4.1) and (4.3), we get

$$\int_0^1 |A(x)|^2 \, dx = \int_0^\infty du \left| \int_0^\infty e^{-su} f(s) \, ds \right|^2.$$

Using Hardy’s technique, we may state that

$$\int_0^1 |A(x)|^2 \, dx = \int_0^\infty du \left| \int_0^\infty e^{-su} f(s) \, ds \right|^2$$

$$= \int_0^\infty du \int_0^\infty e^{-su} f(s) \, ds \int_0^\infty e^{-tu} f(t) \, dt$$

$$\leq \int_0^\infty \int_0^\infty \left( \int_0^\infty e^{-(s+t)u} \, ds \right) |f(s)| |f(t)| \, ds \, dt$$

$$= \int_0^\infty \int_0^\infty |f(s)| |f(t)| \frac{ds}{s + t} \, dt$$

$$< \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty |f(x)|^p \right)^{\frac{1}{p}} \left( \int_0^\infty |f(x)|^q \right)^{\frac{1}{q}} (1 - R)^m.$$

Let us choose the function $h(s, t)$ to be defined by

$$h(s, t) := \left( \frac{2}{\pi} \right)^{\frac{1}{2}} e^{-s} \left( \frac{s}{t} \right)^{\frac{1}{p}} \left( \frac{t}{s} \right)^{\frac{1}{q}},$$

then
then
\[ \|h\| = \left\{ \int_0^\infty \int_0^\infty h^2(s,t)dsdt \right\}^{\frac{1}{2}} = 1. \]

Notice that \( k_1(p) = B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{\pi}{\sin(\frac{\pi}{p})} \), and, in a similar way to the one in Theorem 1, the expression of \( R \) is easily given. We omit the details.

**Remark 4.** In particular, when \( p = q = 2 \), it follows from (4.2) that

\[ (4.4) \quad \int_0^1 A^2(x)dx = \pi (1-\gamma) \int_0^\infty f^2(x) dx. \]

If \( r \) in (4.4) is replaced by zero, then Widder’s theorem (see [8]) can be recaptured.

**Remark 5.** After simple computation, the inequality (4.4) is equivalent to the inequality (3.4) in [2]. Consequently, inequality (4.2) is an extension of (3.4) in [2].

**References**


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