On Integral Representation of First Kind Bessel Function

ON INTEGRAL REPRESENTATION OF FIRST KIND BESSEL FUNCTION

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Abstract. A first kind Fredholm integral equation with nondegenerate kernel is given, which particular solution is the Bessel function of first kind. This equation is solved by means of Mellin transform pair.

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1. Introduction

Concerning the sources of special functions, the most exhaustive collection of 396 formulæ involving first kind Bessel functions the authors find on the widely known website [3]. The main purpose of this short note is to give a new, definite integral definition of Bessel function of the first kind (such that is not contained on the above mentioned website). This goal we realize by getting a Fredholm type integral equation of first kind with degenerate kernel. Particular solution of this equation is the desired Bessel J. To solve the integral equation we use the Mellin integral transform technique.

The most frequently used notations we need are $J_\nu(x)$ - the Bessel function of the first kind;

$$\mathcal{L}_x f := \int_0^\infty e^{-xv} f(v)dv, \qquad x > 0$$

denotes the real parameter ordinary Laplace transform of certain f, while

$$\mathcal{M}_p(g) := \int_0^\infty x^{p-1} g(x)dx, \qquad \mathcal{M}_x^{-1}(g) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\mathcal{M}_p(g)dp,$$

is the Mellin transform pair of g. Here the real c belongs to the so-called *fundamental strip* $\langle\cdot,\cdot\rangle$ of the inverse Mellin transform $\mathcal{M}^{-1}$, see [8].

We will say that functions f, g are *orthogonal a.e.* with respect to the ordinary Lebesgue measure on the positive halfline when $\int_0^\infty f(x)g(x)dx$ vanishes, writing this by $f \perp g$.

Let us introduce the series

$$S_\mu(r,\alpha) = \sum_{n=0}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha + r^2)^{\mu+1}}, \qquad \alpha, \mu, r > 0, \tag{1}$$

which has fundamental (but passive!) role in this note and such that is the subject of interest of many authors in the last time. This series we call *generalized Mathieu series* in the sequel following Qi, see [6],[7].

Finally, $[t]$, $\{t\}$ stay for the integer and the fractional part of real t; *if and only if* is abbreviated into **iff** and □ will denote the end of proofs.

2. Integral equation for $J_\nu(x)$

The generalized Mathieu series $S_\mu(r,\alpha)$ possesses several closed form representations involving definite integrals. The most recent ones are given by Cerone & Lenard and by Pogány, consult [1], [5]. (We have to remark at this point that Tomovski deduced a similar result. Namely, instead of Euler-McLaurin summation formula used by Pogány in [5] he apply the trapesoidal rule to derive the integral expression for $S_\mu(r,\alpha)$, [9]). So, the heart of the matter are the mentioned integral expressions of the same subject, of the generalized Mathieu series $S_\mu(r,\alpha)$.

Theorem 1. *The first kind Fredholm type convolutional integral equation*

$$\int_0^\infty x^{\nu+1}\psi(rx)\mathcal{L}_x[v^{2/\alpha}]dx = g_\nu(r,\alpha) \tag{2}$$

possesses particular solution $\psi(x) = J_\nu(x) + h(x),\ h \perp x^{\nu+1}\mathcal{L}_x\mathbb{I}_{[1,\infty)}(v)[v^{2/\alpha}]$ **iff**

$$g_\nu(r,\alpha) = \frac{(2r)^\nu\Gamma(\nu+5/2)}{\sqrt{\pi}(\alpha+2)}\int_1^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2)\int_1^{[t^{1/\alpha}]} u^{\alpha/2-1}\{u\}du}{(r^2+t)^{\nu+5/2}}\, dt. \tag{3}$$

Proof. In [1] Cerone and Lenard show that

$$S_\mu(r,\alpha) = \frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)}\int_0^\infty x^{\mu+1/2}J_{\mu-1/2}(rx)\left(\sum_{n=1}^\infty e^{-n^{\alpha/2}x}\right)dx. \tag{4}$$

On the other hand the standard Dirichlet-series

$$\mathcal{D}_\alpha(x) = \sum_{n=1}^\infty e^{-n^{\alpha/2}x}$$

possesses well-known integral representation of the form

$$\mathcal{D}_\alpha(x) = x\int_0^\infty e^{-xv}A(v)dv, \tag{5}$$

with the counting function

$$A(v) = \sum_{n:\ n^{\alpha/2}\le v} 1 = [v^{2/\alpha}],$$

compare [4, 5]. So, by reducing (4), we clearly get

$$S_\mu(r,\alpha) = \frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)} \int_0^\infty x^{\mu+1/2} J_{\mu-1/2}(rx) \int_1^\infty e^{-xv}[v^{2/\alpha}]dvdx. \tag{6}$$

By obvious reasons, taking the Laplace transform notation for $[v^{2/\alpha}]$ the relation (6) becomes

$$S_\mu(r,\alpha) = \frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)} \int_0^\infty x^{\mu+1/2} J_{\mu-1/2}(rx)\mathcal{L}_x[v^{2/\alpha}]\, dx. \tag{7}$$

Having in mind the second authors integral expression result [5, Eq.(13)] reads as follows

$$S_\mu(r,\alpha) = \frac{\mu+1}{\alpha+2} \int_1^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2)\int_1^{[t^{1/\alpha}]} u^{\alpha/2-1}\{u\}du}{(r^2+t)^{\mu+2}}\, dt \tag{8}$$

by comparing (7) and (8) and taking the basic identity $z\Gamma(z) = \Gamma(z+1)$ with $z = \mu+1$, we deduce

$$\begin{aligned}
&\int_0^\infty x^{\mu+1/2} J_{\mu-1/2}(rx)\mathcal{L}_x[v^{2/\alpha}]\, dx \\
&\quad = \frac{(2r)^{\mu-1/2}\Gamma(\mu+2)}{\sqrt{\pi}(\alpha+2)} \int_0^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2)\int_0^{[t^{1/\alpha}]} u^{\alpha/2-1}\{u\}du}{(r^2+t)^{\mu+2}}\, dt \\
&\quad = \frac{(2r)^{\mu-1/2}\Gamma(\mu+2)}{\sqrt{\pi}(\alpha+2)} \int_1^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2)\int_1^{[t^{1/\alpha}]} u^{\alpha/2-1}\{u\}du}{(r^2+t)^{\mu+2}}\, dt.
\end{aligned} \tag{9}$$

where the second relation holds because the integration domain reduces from $\mathbb{R}_+$ to $[1,\infty)$ by $[u]=0,\ 0 \le u < 1$.

Endly, writing $\nu = \mu - 1/2$ in the last equation the right hand expression in (9) becomes $g_\nu(r,\alpha)$ defined by (3). The derived first kind Fredholm convolution has a solution given in Theorem 1. □

Example 1. We give a simple construction of function h mentioned in the Theorem 1. In this goal consider the nonnegative random variable $\xi_{\nu,\alpha}$ defined on fixed probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ having probability density function

$$f_{\nu,\alpha}(x) = \begin{cases} \dfrac{x^{\nu+1}\mathcal{L}_x[v^{2/\alpha}]}{\Gamma(\nu+2)\mathcal{M}_{-\nu-1}([v^{2/\alpha}])} & x > 0, \\ 0 & x \le 0. \end{cases} \tag{10}$$

It is interesting that $f_{\nu,2}(x)$ reduces to

$$f_{\nu,2}(x) = \begin{cases} \dfrac{(\nu+1)x^\nu}{\Gamma(\nu+2)\zeta(\nu+1)(e^x-1)} & x > 0, \\ 0 & x \le 0, \end{cases}$$

where $\zeta(\cdot)$ denotes the Riemann zeta function, and by obvious reasons $\nu \ge 1$.

As the probability is monotonous increasing with respect to the increasing sequences of random events, there exists the unique median $x_{0.5}$, i.e. the solution of the equation

$$\mathsf{P}\{\xi_{\nu,\alpha} \le x_{0.5}\} = \int_0^{x_{0.5}} f_{\nu,\alpha}(x)dx = \frac{1}{2}.$$

Now the desired function

$$h(x) = \begin{cases} 1 & x \in [0, x_{0.5}), \\ -1 & x \ge x_{0.5}. \end{cases}$$

is the solution of the homogeneous variant of the equation (2). □

3. $J_\nu(x)$ given via Mellin transform

It is obvious that the equation (2) is invariant with respect to α, so without any loss of generality we can put $\alpha = 2$. Now (9) from the proof of the Theorem becomes

$$\int_0^\infty \frac{x^{\nu+1}}{e^x - 1} J_\nu(rx)dx = \frac{\Gamma(\nu+\frac{5}{2})(2r)^\nu}{\sqrt{\pi}} \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{(r^2+t)^{\nu+5/2}}\, dt, \tag{11}$$

see [1],[5, Eq.(16)]. By applying Mellin transform for the convolution as given in [2], from (11) we get the following equation

$$\Phi(s) = \frac{2^\nu \Gamma(\nu+\frac{5}{2})}{\sqrt{\pi}\Gamma(\nu-s+2)\zeta(\nu-s+2)} \mathcal{M}_s\left(r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{(r^2+t)^{\nu+5/2}} dt\right) \tag{12}$$

where $\Phi(s)$ is the Mellin transform of the Bessel function $J_\nu(x)$. To express the Bessel function explicitly we now apply the inverse Mellin transform on (12), and we get the following identity:

$$J_\nu(x) = \frac{2^{\nu-1}\Gamma(\nu+\frac{5}{2})}{\pi^{3/2} i} \int_{c-i\infty}^{c+i\infty} \frac{\mathcal{M}_s\left(r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{(r^2+t)^{\nu+5/2}} dt\right)}{\Gamma(\nu-s+2)\zeta(\nu-s+2)x^s}\, ds, \tag{13}$$

where c in the integral bounds is from the fundamental strip [8] of the function

$$\phi(r) = r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{(r^2+t)^{\nu+5/2}}\, dt. \tag{14}$$

Now, it can be easily shown that the fundamental strip for $\phi(r)$ contains $\langle -\nu, \nu+1\rangle$, so we can put $c = \frac{1}{2}$, say. Indeed, after some short calculation we have

$$\phi(r) \le \begin{cases} \dfrac{(4\nu+3)r^\nu}{(\nu+1)(2\nu+1)} & r \to 0, \\ \dfrac{(4\nu+3)r^\nu}{(\nu+1)(2\nu+1)(1+r^2)^{\nu+1/2}} & r \to \infty. \end{cases}$$

¿From this we deduce that the fundamental strip does contain $\langle -\nu, \nu+1\rangle$ since

$$\phi(r) = \begin{cases} \mathcal{O}(r^\nu) & r \to 0, \\ \mathcal{O}(r^{-\nu-1/2}) & r \to \infty. \end{cases}$$

Therefore we can clearly choose the value $c = \frac{1}{2}$ for the Bromwich - Wagner type integration contour in deriving the inverse Mellin transform. Therefore (13) becomes

$$J_\nu(x) = \frac{2^{\nu-1}\Gamma(\nu+\frac{5}{2})}{\pi^{3/2} i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathcal{M}_s\left(r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{(r^2+t)^{\nu+5/2}} dt\right)}{\Gamma(\nu-s+2)\zeta(\nu-s+2)x^s}\, ds \tag{15}$$

which is the desired particular solution of (11) and the integral representation of the Bessel function of the first kind.

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