Some Inequalities of Aczél Type for Gramians in Inner Product Spaces

SOME INEQUALITIES OF ACZÉL TYPE FOR GRAMIANS IN INNER PRODUCT SPACES

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Abstract. Some inequalities of Aczél type for Gramians which generalize Pečarićs result are given Applications connected to Schwartz's inequality are also noted.

1. Introduction

In 1956, J. Aczél established the following interesting inequality (see [9, p. 117]):

Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, ..., b_n)$ be two sequences of real numbers such that either

$$b_1^2 - b_2^2 - ... - b_n^2 > 0 \text{ or } a_1^2 - a_2^2 - ... - a_n^2 > 0.$$

Then

$$\left(a_1^2 - a_2^2 - ... - a_n^2\right)\left(b_1^2 - b_2^2 - ... - b_n^2\right) \le \left(a_1 b_1 - a_2 b_2 - ... - a_n b_n\right)^2 \tag{1.1}$$

with equality if and only if the sequences a and b are proportional.

In [7], S. Kurepa pointed out the following inequality of Aczél type which holds in Hilbert spaces (see [9, p. 602]):

Let X be a real Hilbert space and c a unit vector in X. Suppose that $a, b \in X$ are given vectors such that

$$\left(u^2 - \|a_0\|^2\right) \times \left(v^2 - \|b_0\|^2\right) \ge 0 \tag{1.2}$$

where $u = (a, c)\,,\ v = (b, c)\,,\ a_0 = a - uc$, and $b_0 = b - vc$. Then

$$\left(u^2 - \|a_0\|^2\right)\left(v^2 - \|b_0\|^2\right) \le \left(uv - (a_0, b_0)\right)^2. \tag{1.3}$$

If a and b are independent and strict inequality holds in (1.2), then strict inequality also holds in (1.3).

In [10], see also [9, p. 603], J.E. Pečarić proved an interesting converse of a known inequality of Kurepa [9, p. 599] which asserts that

$$\left|\det\begin{bmatrix} (x_1, y_1) & \cdots & (x_1, y_m) \\ \vdots & & \vdots \\ (x_m, y_1) & \cdots & (x_m, y_m) \end{bmatrix}\right|^2 \le \Gamma(x_1, ..., x_m)\,\Gamma(y_1, ..., y_m) \tag{1.4}$$

where $x_i, y_i \in X,\ \left(i = \overline{1, m}\right),\ X$ is an inner product space over the real or complex number field $\mathbb{K}$ and Γ is the Gramian of the vectors involved above.

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Pečarić's result is

$$\begin{aligned}(1.5)\qquad & \left(u^2-\Gamma\left(x_1,...,x_m\right)\right)\left(v^2-\Gamma\left(y_1,...,y_m\right)\right)\\ \leq & \left(uv-\det\left[\begin{array}{ccc}(x_1,y_1) & \cdots & (x_1,y_m)\\ \vdots & & \vdots\\ (x_m,y_1) & \cdots & (x_m,y_m)\end{array}\right]\right)^2\end{aligned}$$

provided that

$$u^2-\Gamma\left(x_1,...,x_m\right)>0 \text{ or } v^2-\Gamma\left(y_1,...,y_m\right)>0,$$

where x_i,y_i $\left(i=\overline{1,m}\right)$ are vectors in a real inner product space X.

Note that this result is a generalization for Gramians of the Aczél inequality (1.1).

The main aim of this paper is to point out some new inequalities of Aczél type for Gramians which also generalize and extend the result of Pečarić (1.5) and complement, in a sense, Chapter XX of the book [9]. Some applications to real or complex numbers which are closely connected with those embodied in Chapter IV of [9] are also given.

2. Some Inequalities of Aczél Type for Gramians

Let us denote by $\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)$ the determinant given by

$$\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right):=\det\left[\begin{array}{cccc}(x_1,y_1) & (x_1,y_2) & \cdots & (x_1,y_m)\\ \vdots & \vdots & & \vdots\\ (x_m,y_1) & (x_m,y_2) & \cdots & (x_m,y_m)\end{array}\right],$$

where $x_1,y_1,...,x_m,y_m$ are vectors in inner product space $(H;(\cdot,\cdot))$. In addition, we observe that if $y_1=x_1,...,y_m=x_m$, then

$$\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)=\Gamma\left(x_1,...,x_m\right).$$

By the use of this notation, Kurepa's inequality (1.4) may be written as:

$$\Gamma\left(x_1,...,x_m\right)\Gamma\left(y_1,...,y_m\right)\geq\left|\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|^2 \tag{2.1}$$

for all $x_i,y_i\in H$ $\left(i=\overline{1,m}\right)$.

The first result which gives a converse of (2.1) is embodied in the following theorem:

Theorem 1. *Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$ and a,b,c three real numbers satisfying the following condition:*

$$a,c\geq 0 \text{ and } b^2\geq ac.$$

Then for all $x_i,y_i\in H$ $\left(i=\overline{1,m}\right)$ such that either

$$a\geq\Gamma\left(x_1,...,x_m\right) \text{ or } c\geq\Gamma\left(y_1,...,y_m\right),$$

we have the inequality

$$
\begin{aligned}
(2.2)\qquad & [a-\Gamma(x_1,...,x_m)]\,[c-\Gamma(y_1,...,y_m)] \\
\leq\ & \min\left\{\left(b\pm\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right)^2;\left(b\pm\left|\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\right)^2\right. \\
& \left(b\pm\operatorname{Im}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right)^2;\left(b\pm\left|\operatorname{Im}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\right)^2; \\
& \left. b\pm\left|\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|^2\right\}.
\end{aligned}
$$

Proof. Suppose that $a>\Gamma(x_1,...,x_m)$ and consider the polynomial

$$P(t):=at^2-2bt+c,\ t\in\mathbb{R}.$$

Since $a>0$ and $b^2\geq ac$, it follows that there exists a $t_0\in\mathbb{R}$ such that $P(t_0)=0$. Now put

$$
\begin{aligned}
Q_1(t)\quad :\quad & =P(t) \\
& -\left(\Gamma(x_1,...,x_m)\,t^2\mp 2\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\,t+\Gamma(y_1,...,y_m)\right), \\
\text{for } t\quad \in\quad & \mathbb{R}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{Q}_1(t)\quad :\quad & =P(t) \\
& -\left(\Gamma(x_1,...,x_m)\,t^2\mp 2\left|\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|t+\Gamma(y_1,...,y_m)\right), \\
\text{for } t\quad \in\quad & \mathbb{R}.
\end{aligned}
$$

A simple calculation gives

$$
\begin{aligned}
Q_1(t)\quad =\quad & (a-\Gamma(x_1,...,x_m))\,t^2 \\
& -2\left(b\pm\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right)t+(c-\Gamma(y_1,...,y_m)), \\
\text{for } t\quad \in\quad & \mathbb{R}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{Q}_1(t)\quad =\quad & (a-\Gamma(x_1,...,x_m))\,t^2 \\
& -2\left(b\pm\left|\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\right)t+(c-\Gamma(y_1,...,y_m)), \\
\text{for } t\quad \in\quad & \mathbb{R}.
\end{aligned}
$$

Since

$$Q_1(t_0):=-\left[\Gamma(x_1,...,x_m)\,t_0^2\mp 2\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\,t_0+\Gamma(y_1,...,y_m)\right]\leq 0$$

as, by Kurepa's inequality, one has

$$\left|\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|^2\leq\Gamma(x_1,...,x_m)\,\Gamma(y_1,...,y_m)$$

which gives

$$\Gamma(x_1,...,x_m)\,t^2\mp 2\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\,t+\Gamma(y_1,...,y_m)\geq 0$$

for all $t \in \mathbb{R}$. Hence we conclude that Q_1 has at least one solution in $\mathbb{R}$, i.e.,

$$\begin{aligned} 0 &\leq \frac{1}{4}\Delta_1 = \left(b \pm \operatorname{Re}\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right)^2 \\ &= \left(a - \Gamma\left(x_1, ..., x_m\right)\right)\left(c - \Gamma\left(y_1, ..., y_m\right)\right). \end{aligned}$$

Similarly, $\bar{Q}_1$ has at least one solution in $\mathbb{R}$ which is equivalent to

$$\begin{aligned} 0 &\leq \frac{1}{4}\bar{\Delta}_1 = \left(b \pm \left|\operatorname{Re}\Gamma\left(x_1, y_1, ..., x_m, y_m\right)\right|\right)^2 \\ &= \left(a - \Gamma\left(x_1, ..., x_m\right)\right)\left(c - \Gamma\left(y_1, ..., y_m\right)\right) \end{aligned}$$

and the first part of (2.2) is proved.
The last part can be proved similarly by considering the polynomials:

$$\begin{aligned} & Q_2\left(t\right) \\ : &= P\left(t\right) - \left(\Gamma\left(x_1, ..., x_m\right) t^2 \mp 2 \operatorname{Im}\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right) t + \Gamma\left(y_1, ..., y_m\right)\right), \\ & \bar{Q}_2\left(t\right) \\ : &= P\left(t\right) - \left(\Gamma\left(x_1, ..., x_m\right) t^2 \mp 2 \left|\operatorname{Im}\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right| t + \Gamma\left(y_1, ..., y_m\right)\right) \end{aligned}$$

and

$$Q_2\left(t\right) := P\left(t\right) - \left(\Gamma\left(x_1, ..., x_m\right) t^2 \mp 2 \left|\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right| t + \Gamma\left(y_1, ..., y_m\right)\right)$$

respectively.
This completes the proof. ∎

Remark 1. *Let* $(H; (\cdot, \cdot))$ *be an inner product space and* $M_1, M_2 \in \mathbb{R}$. *Then for all* $x_i, y_i \in H$ $\left(i = \overline{1, m}\right)$ *with*

$$\Gamma\left(x_1, ..., x_m\right) \leq \left|M_1\right| \ \text{ or } \Gamma\left(y_1, ..., y_m\right) \leq \left|M_2\right|,$$

one has the inequality

$$\begin{aligned} & \left(M_1^2 - \Gamma\left(x_1, ..., x_m\right)\right)\left(M_2^2 - \ \Gamma\left(y_1, ..., y_m\right)\right) \\ \leq \ & \min\left\{\left(M_1 M_2 - \operatorname{Re}\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right)^2; \right. \\ & \left(M_1 M_2 - \left|\operatorname{Re}\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right|\right)^2; \left(M_1 M_2 - \operatorname{Im}\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right)^2; \\ & \left.\left(M_1 M_2 - \left|\operatorname{Im}\tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right|\right)^2; \left(M_1 M_2 - \tilde{\Gamma}\left(x_1, y_1, ..., x_m, y_m\right)\right)^2\right\} \end{aligned}$$

which improves Pečarić's result (1.5).

Now, using the above theorem, we can give the following inverse of Kurepa's inequality in inner product spaces.

Corollary 1. *Suppose that a, b, c, x_i, y_i $\left(i=\overline{1,m}\right)$ are as in Theorem 1. Then we have the inequalities:*

$$\begin{aligned}
0 &\leq \Gamma\left(x_1,...,x_m\right)\Gamma\left(y_1,...,y_m\right)-\left[\operatorname{Re}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right]^2\\
&\leq b^2-ac+a\Gamma\left(y_1,...,y_m\right)+c\Gamma\left(x_1,...,x_m\right)\\
&\quad +2\min\left\{\pm b\operatorname{Re}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right);\pm b\left|\operatorname{Re}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|\right\};\\
0 &\leq \Gamma\left(x_1,...,x_m\right)\Gamma\left(y_1,...,y_m\right)-\left[\operatorname{Im}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right]^2\\
&\leq b^2-ac+a\Gamma\left(y_1,...,y_m\right)+c\Gamma\left(x_1,...,x_m\right)\\
&\quad +2\min\left\{\pm b\operatorname{Im}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right);\pm b\left|\operatorname{Im}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|\right\};
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \Gamma\left(x_1,...,x_m\right)\Gamma\left(y_1,...,y_m\right)-\left|\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|^2\\
&\leq b^2-ac+a\Gamma\left(y_1,...,y_m\right)+c\Gamma\left(x_1,...,x_m\right)\pm 2b\left|\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|.
\end{aligned}$$

The proof follows by a simple calculation from (2.2). We omit the details.

The following result also holds:

Corollary 2. *Let H be as above and $x_i, y_i \in H$ $\left(i=\overline{1,m}\right)$ with*

$$\left[\Gamma\left(x_1,...,x_m\right)\right]^{\frac{1}{2}} \leq M \text{ or } \left[\Gamma\left(y_1,...,y_m\right)\right]^{\frac{1}{2}} \leq M.$$

Then we have the inequality

$$\begin{aligned}
0 &\leq \Gamma\left(x_1,...,x_m\right)\Gamma\left(y_1,...,y_m\right)-\left[\operatorname{Re}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right]^2\\
&\leq M^2\left[\Gamma\left(x_1,...,x_m\right)-2\operatorname{Re}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)+\Gamma\left(y_1,...,y_m\right)\right].
\end{aligned}$$

It is important to note that a similar theorem can also be stated:

Theorem 2. *Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$ and α, β, γ real numbers with the property that*

$$\alpha, \gamma > 0 \text{ and } \beta^2 \geq \alpha\gamma.$$

Then, for all $x_i, y_i \in H$ $\left(i=\overline{1,m}\right)$ such that

$$\left[\Gamma\left(x_1,...,x_m\right)\right]^{\frac{1}{2}} \leq \alpha \text{ or } \left[\Gamma\left(y_1,...,y_m\right)\right]^{\frac{1}{2}} \leq \gamma$$

we have the inequality

$$\begin{aligned}
&\left(\alpha-\left[\Gamma\left(x_1,...,x_m\right)\right]^{\frac{1}{2}}\right)\left(\gamma-\left[\Gamma\left(y_1,...,y_m\right)\right]^{\frac{1}{2}}\right)\\
\leq\ &\min\left\{\left(\beta\pm\left|\operatorname{Re}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|^{\frac{1}{2}}\right)^2;\right.\\
&\left.\left(\beta\pm\left|\operatorname{Re}\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|^{\frac{1}{2}}\right)^2;\left(\beta\pm\left|\tilde{\Gamma}\left(x_1,y_1,...,x_m,y_m\right)\right|^{\frac{1}{2}}\right)^2\right\}.
\end{aligned}$$

Proof. The argument is similar to that in the proof of the previous theorem. Choosing the polynomials

$$\tilde{Q}_i(t) := P(t) - \left([\Gamma(x_1,...,x_m)]^{\frac{1}{2}} t^2 \mp 2\phi_i t + [\Gamma(y_1,...,y_m)]^{\frac{1}{2}}\right),\ t\in\mathbb{R},\ i=1,2,3,$$

where

$$\begin{aligned}\phi_1 &= \left|\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|,\ \phi_2 = \left|\operatorname{Im}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|,\\ \phi_3 &= \left|\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\end{aligned}$$

and

$$P(t) = \alpha t^2 - 2\beta t + \gamma,\ t\in\mathbb{R}$$

respectively.
We omit the details. ∎

The following converse of Kurepa's inequality also holds:

Corollary 3. *Let $H,\alpha,\beta,\gamma,x_i,y_i\in H\ \left(i=\overline{1,m}\right)$ be as above. Then we have*

$$\begin{aligned}0 &\le [\Gamma(x_1,...,x_m)]^{\frac{1}{2}}[\Gamma(y_1,...,y_m)]^{\frac{1}{2}} - \left|\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\\ &\le \beta^2-\alpha\gamma+\alpha[\Gamma(x_1,...,x_m)]^{\frac{1}{2}}+\gamma[\Gamma(y_1,...,y_m)]^{\frac{1}{2}}\\ &\quad \pm 2\beta\left|\operatorname{Re}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|;\\ 0 &\le [\Gamma(x_1,...,x_m)]^{\frac{1}{2}}[\Gamma(y_1,...,y_m)]^{\frac{1}{2}} - \left|\operatorname{Im}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\\ &\le \beta^2-\alpha\gamma+\alpha[\Gamma(x_1,...,x_m)]^{\frac{1}{2}}+\gamma[\Gamma(y_1,...,y_m)]^{\frac{1}{2}}\\ &\quad \pm 2\beta\left|\operatorname{Im}\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\end{aligned}$$

and

$$\begin{aligned}0 &\le [\Gamma(x_1,...,x_m)]^{\frac{1}{2}}[\Gamma(y_1,...,y_m)]^{\frac{1}{2}} - \left|\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\\ &\le \beta^2-\alpha\gamma+\alpha[\Gamma(x_1,...,x_m)]^{\frac{1}{2}}+\gamma[\Gamma(y_1,...,y_m)]^{\frac{1}{2}}\\ &\quad \pm 2\beta\left|\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|.\end{aligned}$$

Corollary 4. *Let H be as above and $M>0$. Suppose $[\Gamma(x_1,...,x_m)]^{\frac{1}{2}}\le M$ or $[\Gamma(y_1,...,y_m)]^{\frac{1}{2}}\le M$. Then one has the inequality:*

$$\begin{aligned}0 &\le [\Gamma(x_1,...,x_m)]^{\frac{1}{2}}[\Gamma(y_1,...,y_m)]^{\frac{1}{2}} - \left|\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|\\ &\le M\left([\Gamma(x_1,...,x_m)]^{\frac{1}{2}}+[\Gamma(y_1,...,y_m)]^{\frac{1}{2}}-2\left|\tilde{\Gamma}(x_1,y_1,...,x_m,y_m)\right|^{\frac{1}{2}}\right).\end{aligned}$$

The following inequality is well-known in the literature as Hadamard's inequality for the Gram determinant:

$$\Gamma(x_1,...,x_m) \le \prod_{i=1}^{m}\|x_i\|^2 \tag{2.3}$$

for all $x_i\in H\ \left(i=\overline{1,m}\right)$ (see [9, p. 597].)

Equality holds in (2.3) iff $(x_i,y_i)=\delta_{ij}\|x_i\|\|x_j\|$ for all $i,j\in\{1,...,m\}$.

In the next theorem, we point out a converse inequality for (2.3).

Theorem 3. *Let $(H,(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$ and a,b,c real numbers satisfying the following condition:*

$$a,c>0 \text{ and } b^2\geq ac.$$

Then for all $x_i\in H$ $\left(i=\overline{1,m}\right)$ $(m\geq 2)$ such that

$$a\geq \prod_{i=1}^{k}\|x_i\|^4 \text{ or } c\geq \prod_{i=k+1}^{m}\|x_i\|^4,$$

where $1\leq k\leq m$, we have the inequality

$$\left(a-\prod_{i=1}^{k}\|x_i\|^4\right)\left(c-\prod_{i=k+1}^{m}\|x_i\|^4\right)\leq \left(b\pm\Gamma\left(x_1,...,x_m\right)\right)^2. \tag{2.4}$$

Proof. Fix $k\in\{1,...,m\}$ and suppose that $a\geq\prod_{i=1}^{k}\|x_i\|^4$. Consider the polynomial

$$P(t)=at^2-2bt+c,\ t\in\mathbb{R}.$$

Since $a>0$ and $b^2\geq ac$, it follows that there exists a $t_0\in\mathbb{R}$ such that $P(t_0)=0$. Now, put

$$\phi(t):=P(t)-\left(\left(\prod_{i=1}^{k}\|x_i\|^4\right)t^2\mp 2b\Gamma\left(x_1,...,x_m\right)t+\prod_{i=k+1}^{m}\|x_i\|^4\right),\ \alpha\in\mathbb{R}.$$

A simple calculation gives

$$\phi(t)=\left(a-\prod_{i=1}^{k}\|x_i\|^4\right)t^2-2\left(b\pm\Gamma\left(x_1,...,x_m\right)\right)t+\left(c-\prod_{i=k+1}^{m}\|x_i\|^4\right),\ t\in\mathbb{R}.$$

By Hadamard's inequality, one has

$$\Gamma^2\left(x_1,...,x_m\right)\leq\prod_{i=1}^{m}\|x_i\|^2=\left(\prod_{i=1}^{k}\|x_i\|^4\right)\left(\prod_{i=k+1}^{m}\|x_i\|^4\right)$$

which gives

$$\left(\prod_{i=1}^{k}\|x_i\|^4\right)t^2-2\Gamma\left(x_1,...,x_m\right)t+\prod_{i=k+1}^{m}\|x_i\|^4\geq 0 \text{ for all } t\in\mathbb{R}.$$

Since

$$\phi_1(t_0)=-\left(\left(\prod_{i=1}^{k}\|x_i\|^4\right)t_0^2-2\Gamma\left(x_1,...,x_m\right)t_0+\prod_{i=k+1}^{m}\|x_i\|^4\right)\leq 0,$$

we conclude that ϕ has at least one solution in $\mathbb{R}$, i.e.,

$$0\leq\frac{1}{4}\Delta_1=\left(b\pm\Gamma\left(x_1,...,x_m\right)\right)^2-\left(a-\prod_{i=1}^{k}\|x_i\|^4\right)\left(c-\prod_{i=k+1}^{m}\|x_i\|^4\right)$$

and the theorem is proved. ∎

The following converses of Hadamard's inequality hold:

Corollary 5. *Suppose that a, b, c and $x_i \in H$ $\left(i = \overline{1, m}\right)$ are as above. Then one has the following inequality:*

$$\begin{aligned} 0 &\leq \prod_{i=1}^{m} \|x_i\|^4 - \Gamma^2(x_1, ..., x_m) \\ &\leq b^2 - ac + a \prod_{i=k+1}^{m} \|x_i\|^4 + c \prod_{i=1}^{k} \|x_i\|^4 \pm 2\Gamma(x_1, ..., x_m)\, b. \end{aligned}$$

Corollary 6. *Suppose that $M > 0$ and $x_i \in H$ $\left(i = \overline{1, m}\right)$, with the property that*

$$\prod_{i=1}^{k} \|x_i\|^2 \leq M \text{ or } \prod_{i=k+1}^{m} \|x_i\|^2 \leq M.$$

Then one has the inequality

$$0 \leq \prod_{i=1}^{m} \|x_i\|^4 - \Gamma^2(x_1, ..., x_m) \leq M^2 \left(\prod_{i=1}^{k} \|x_i\|^4 + \prod_{i=k+1}^{m} \|x_i\|^4 - 2\Gamma(x_1, ..., x_m) \right).$$

By a similar argument as in the proof of the last theorem, we also have:

Theorem 4. *Let $(H; (\cdot, \cdot))$ be an inner product space over the real or complex field $\mathbb{K}$ and α, β, γ real numbers with $\alpha, \gamma > 0$ and $\beta^2 \geq \alpha\gamma$. Then for all $x_i \in H$ $\left(i = \overline{1, m}\right)$ such that*

$$\prod_{i=1}^{k} \|x_i\|^2 \leq \alpha \text{ or } \prod_{i=k+1}^{m} \|x_i\|^2 \leq \gamma \ (1 \leq k \leq m),$$

we have the inequality

$$\left(\alpha - \prod_{i=1}^{k} \|x_i\|^2 \right) \left(\gamma - \prod_{i=k+1}^{m} \|x_i\|^2 \right) \leq \left(\beta \pm [\Gamma(x_1, ..., x_m)]^{\frac{1}{2}} \right)^2.$$

Now, by the use of the above theorem we can also state the following converses of Hadamard's inequality:

Corollary 7. *Suppose that α, β, γ and $x_i \in H$ $\left(i = \overline{1, m}\right)$ are as above. Then*

$$\begin{aligned} 0 \leq \prod_{i=1}^{m} \|x_i\|^2 - \Gamma(x_1, ..., x_m) &\leq \beta^2 - \alpha\gamma + \alpha \prod_{i=k+1}^{m} \|x_i\|^2 \\ &+ \gamma \prod_{i=1}^{k} \|x_i\|^2 \pm 2\beta [\Gamma(x_1, ..., x_m)]^{\frac{1}{2}}. \end{aligned}$$

Corollary 8. *Let $M > 0$ and $x_i \in H$ $\left(i = \overline{1, m}\right)$ be such that*

$$\prod_{i=1}^{k} \|x_i\|^2 \leq M \text{ or } \prod_{i=k+1}^{m} \|x_i\|^2 \leq M.$$

Then one has the inequality

$$\begin{aligned} 0 &\leq \prod_{i=1}^{m} \|x_i\|^2 - \Gamma(x_1,...,x_m) \\ &\leq M\left(\prod_{i=1}^{k} \|x_i\|^2 + \prod_{i=k+1}^{m} \|x_i\|^2 - 2\left[\Gamma(x_1,...,x_m)\right]^{\frac{1}{2}}\right). \end{aligned}$$

In addition, we note that the following inequality improving Hadamard's result holds:

$$\Gamma(x_1,...,x_m) \leq \Gamma(x_1,...,x_k)\,\Gamma(x_{k+1},...,x_m) \tag{2.5}$$

(see [9, p. 597]), where $x_i \in H$ $\left(i=\overline{1,m}\right)$ and $1 \leq k \leq m$.

By the use of this inequality and a similar argument as above, we can obtain the following converses of (2.5).

(i) If $a \geq \Gamma^2(x_1,...,x_k)$ or $c \geq \Gamma^2(x_{k+1},...,x_m)$ and $b^2 \geq ac > 0$, then

$$\left[a - \Gamma^2(x_1,...,x_k)\right]\left[c - \Gamma^2(x_{k+1},...,x_m)\right] \leq \left(b \pm \Gamma(x_1,...,x_m)\right)^2$$

from which one easily obtains

$$\begin{aligned} 0 &\leq \Gamma^2(x_1,...,x_k)\,\Gamma^2(x_{k+1},...,x_m) - \Gamma^2(x_1,...,x_m) \\ &\leq b^2 - ac + a\Gamma^2(x_{k+1},...,x_m) + c\Gamma^2(x_1,...,x_k) \pm 2\Gamma(x_1,...,x_m)\,b. \end{aligned}$$

If $\Gamma(x_1,...x_k) \leq M$ or $\Gamma(x_{k+1},...,x_m) \leq M$, then also

$$\begin{aligned} 0 &\leq \Gamma^2(x_1,...,x_k)\,\Gamma^2(x_{k+1},...,x_m) - \Gamma^2(x_1,...,x_m) \\ &\leq M^2\left[\Gamma^2(x_1,...,x_k) + \Gamma^2(x_{k+1},...,x_m) - 2\Gamma(x_1,...,x_m)\right]. \end{aligned}$$

(ii) If $\alpha, \gamma > 0$ and $\beta^2 \leq \alpha\gamma$, then for all $x_i \in H$ $\left(i=\overline{1,m}\right)$ with

$$\Gamma(x_1,...,x_k) \leq \alpha \text{ or } \Gamma(x_{k+1},...,x_m) \leq \gamma,$$

one has the inequality:

$$\left[\alpha - \Gamma(x_1,...,x_k)\right]\left[\gamma - \Gamma(x_{k+1},...,x_m)\right] \leq \left(\beta \pm \left[\Gamma(x_1,...,x_m)\right]^{\frac{1}{2}}\right)^2$$

which gives the following converse of (2.5):

$$\begin{aligned} 0 &\leq \Gamma(x_1,...,x_k)\,\Gamma(x_{k+1},...,x_m) - \Gamma(x_1,...,x_m) \\ &\leq \beta^2 - \alpha\gamma + \alpha\Gamma(x_{k+1},...,x_m) + \gamma\Gamma(x_1,...,x_k) \pm 2\beta\left[\Gamma(x_1,...,x_m)\right]^{\frac{1}{2}}. \end{aligned}$$

If $\Gamma(x_1,...x_k) \leq M$ or $\Gamma(x_{k+1},...,x_m) \leq M$, then also

$$\begin{aligned} 0 &\leq \Gamma(x_1,...,x_k)\,\Gamma(x_{k+1},...,x_m) - \Gamma(x_1,...,x_m) \\ &\leq M\left[\Gamma(x_1,...,x_k) + \Gamma(x_{k+1},...,x_m) - 2\left[\Gamma(x_1,...,x_m)\right]^{\frac{1}{2}}\right]. \end{aligned}$$

3. Some Applications

1. Suppose that $(H;(\cdot,\cdot))$ is an inner product space over the real or complex number field $\mathbb{K}$. If $x, y \in H$ and M_1, M_2 are real numbers such that

$$\|x\| \leq |M_1| \text{ or } \|y\| \leq |M_2|,$$

then the following generalization of Aczél's inequality in inner product spaces holds:

$$\begin{aligned}&\left(M_1^2-\|x\|^2\right)\left(M_2^2-\|y\|^2\right)\\ \leq\ &\min\Big\{\left(M_1M_2-\operatorname{Re}(x,y)\right)^2;\left(M_1M_2-|\operatorname{Re}(x,y)|\right)^2;\\ &\left(M_1M_2-\operatorname{Im}(x,y)\right)^2;\left(M_1M_2-|\operatorname{Im}(x,y)|\right)^2;\left(M_1M_2-|(x,y)|\right)^2\Big\}.\end{aligned}$$

(See also the paper [5])

This inequality is obvious from Theorem 1. We omit the details.

2. Suppose that $x,y\in H$ and $M_1,M_2\in\mathbb{R}$ are as above. Then by the use of Theorem 2, we have the following interesting inequality of Aczél type:

$$\left(M_1-\|x\|\right)^{\frac{1}{2}}\left(M_2-\|y\|\right)^{\frac{1}{2}}\leq|M_1M_2|^{\frac{1}{2}}-|(x,y)|^{\frac{1}{2}},$$

provided that $\|x\|\leq|M_1|$ and $\|y\|\leq|M_2|$.

Note that if $a=(a_1,a_2,...,a_n)$, $b=(b_1,...,b_n)\in\mathbb{R}^n$ satisfy

$$a_1^2-a_2^2-...-a_n^2\geq0 \text{ and } b_1^2-b_2^2-...-b_n^2\geq0,$$

then we have the inequality

$$\begin{aligned}&\left[|a_1|-\left(a_2^2+...+a_n^2\right)^{\frac{1}{2}}\right]\left[|b_1|-\left(b_2^2+...+b_n^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\\ \leq\ &|a_1b_1|^{\frac{1}{2}}-|a_2b_2+...+a_nb_n|^{\frac{1}{2}}.\end{aligned}$$

This is a new inequality of Aczél type for real numbers (see also [5]).

3. By the use of Corollary 2, we also have the following converse of Schwartz's inequality in inner product spaces:

$$0\leq\|x\|^2\|y\|^2-[\operatorname{Re}(x,y)]^2\leq M^2\min\left\{\|x-y\|^2,\|x+y\|^2\right\}$$

provided that $x,y\in H$ with $\|x\|\leq M$ or $\|y\|\leq M$.

For other inequalities of Aczél type in inner product spaces, as well as some applications for real or complex numbers and for integrals, see the recent paper [5] where further references are given.

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