An N-Dimensional Version of Ostrowski's Inequality For Mappings of the Hölder Type

AN N-DIMENSIONAL VERSION OF OSTROWSKI'S INEQUALITY FOR MAPPINGS OF THE HÖLDER TYPE

S.S. DRAGOMIR, N.S. BARNETT, AND P. CERONE

Abstract. Generalizations for multiple integrals and mappings of the Hölder type, of the celebrated Ostrowski's inequality which extend Milovanović result from 1975, are given.

1. Introduction

In 1938, A. Ostrowski proved the following integral inequality [6, p. 468].

Let $f : [a, b] \to \mathbf{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbf{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty \tag{1.1}$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible.

For some generalizations of this see the book [6, p. 468 - p484] by Mitrinović, Pečarić and Fink.

Some applications of the above results in Numerical Integration and for special means have been given in a recent paper [4] by S.S. Dragomir and S. Wang. For other results of Ostrowski's type see [2, 3 and 5].

In 1975, G.N. Milovanović generalized Ostrowski's result for a function of several variables [6, p. 468]:

Let $f : \mathbf{R}^m \to \mathbf{R}$ be a differentiable function defined on

$$D = \{(x_1, ..., x_m) : a_i \le x_i \le b_i \, (i = 1, ..., m)\}$$

and let

$$\left| \frac{\partial f}{\partial x_i} \right| \le M_i \;\; (M_i > 0, i = 1, ..., m)$$

in D. Furthermore, let the function $x \longmapsto p(x)$ be integrable and $p(x) > 0$ for every $x \in D$. Then for every $x \in D$, we have the inequality:

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$$\left|f(x)-\frac{\int_D p(y) f(y)\,dy}{\int_D p(y)\,dy}\right| \leq \frac{\sum_{i=1}^{m} M_i \int_D p(y)\left|x_i-y_i\right| dy}{\int_D p(y)\,dy}.$$

In [1], N.S. Barnett and S.S. Dragomir established the following result for real mappings of two variables.

Let $f:[a,b]\times[c,d]\to\mathbf{R}$ be such that $f(\cdot,\cdot)$ is continuous, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,

$$\left\|f''_{s,t}\right\|_{\infty}:=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty.$$

Then we have the inequality:

$$\Big|\int_a^b\int_c^d f(s,t)\,dsdt-\Big[(b-a)\int_c^d f(x,t)\,dt+(d-c)\int_a^b f(s,y)\,ds$$

$$-(d-c)(b-a)f(x,y)\Big]\Big|$$

$$\leq\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|f''_{s,t}\right\|_{\infty}$$

for all $(x,y)\in[a,b]\times[c,d]$.

In this paper we point out an Ostrowski inequality for multiple integrals and Hölder type mappings thus generalizing the Milovanović result as follows.

2. A generalization for mappings of the Hölder type

We start with the following Ostrowski type inequality for mappings of the $r-$Hölder type.

Theorem 2.1. *Assume that the mapping $f:[a_1,b_1]\times\ldots\times[a_n,b_n]\to\mathbf{R}$ satisfies the following $r-$Hölder type condition:*

$$\text{(H)}\qquad \left|f(\bar{\mathbf{x}})-f(\bar{\mathbf{y}})\right|\leq\sum_{i=1}^{n}L_i\left|x_i-y_i\right|^{r_i}\ (L_i\geq 0,i=1,\ldots,n)$$

for all $\bar{\mathbf{x}}=(x_1,\ldots,x_n)$, $\bar{\mathbf{y}}=(y_1,\ldots,y_n)\in[\bar{\mathbf{a}},\bar{\mathbf{b}}]:=[a_1,b_1]\times\ldots\times[a_n,b_n]$, where $r_i\in(0,1]$, $i=1,\ldots,n$. We have then the Ostrowski type inequality:

$$\left|f(\bar{\mathbf{x}})-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}f(\bar{\mathbf{t}})\,d\bar{\mathbf{t}}\right|$$

$$\leq\sum_{i=1}^{n}\frac{L_i}{r_i+1}\left[\left(\frac{x_i-a_i}{b_i-a_i}\right)^{r_i+1}+\left(\frac{b_i-x_i}{b_i-a_i}\right)^{r_i+1}\right](b_i-a_i)^{r_i}$$

$$\leq \sum_{i=1}^{n} \frac{L_i (b_i - a_i)^{r_i}}{r_i + 1} \tag{2.1}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$, *where* $\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}})\, d\bar{\mathbf{t}} = \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(t_1, ..., t_n)\, dt_n ... dt_1$.

Proof. By (H) we have

$$|f(\bar{\mathbf{x}}) - f(\bar{\mathbf{t}})| \leq \sum_{i=1}^{n} L_i |x_i - t_i|^{r_i}$$

for all $\bar{\mathbf{x}}, \bar{\mathbf{t}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Integrating over $\bar{\mathbf{t}}$ on $[\bar{\mathbf{a}}, \bar{\mathbf{b}}]$ and using the modulus properties we get

$$\left| f(\bar{\mathbf{x}}) \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} d\bar{\mathbf{t}} - \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}})\, d\bar{\mathbf{t}} \right| \leq \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |\, (f(\bar{\mathbf{x}}) - f(\bar{\mathbf{t}}))\, |\, d\bar{\mathbf{t}}$$

$$\leq \sum_{i=1}^{n} L_i \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} |x_i - t_i|^{r_i}\, dt_n ... dt_1. \tag{2.2}$$

As

$$\int_{a_1}^{b_1} ... \int_{a_n}^{b_n} dt_n ... dt_1 = \prod_{i=1}^{n} (b_i - a_i)$$

and

$$\int_{a_1}^{b_1} ... \int_{a_n}^{b_n} |x_i - t_i|^{r_i}\, dt_n ... dt_1 = \prod_{\substack{j=1 \\ j \neq i}}^{n} (b_j - a_j) \int_{a_i}^{b_i} |x_i - t_i|^{r_i}\, dt_i$$

$$= \prod_{\substack{j=1 \\ j \neq i}}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{r_i + 1} + (x_i - a_i)^{r_i + 1}}{r_i + 1} \right]$$

$$= \prod_{j=1}^{n} (b_j - a_j) \frac{1}{r_i + 1} \left[\left(\frac{b_i - x_i}{b_i - a_i} \right)^{r_i + 1} + \left(\frac{x_i - a_i}{b_i - a_i} \right)^{r_i + 1} \right] (b_i - a_i)^{r_i},$$

then, dividing (2.2) by $\prod_{j=1}^{n} (b_j - a_j)$ we get the first part of (2.1).

Using the elementary inequality

$$(y - \alpha)^{p+1} + (\beta - y)^{p+1} \leq (\beta - \alpha)^{p+1}$$

for all $\alpha \leq y \leq \beta$ and $p > 0$, we get

$$\left(\frac{b_i - x_i}{b_i - a_i} \right)^{r_i + 1} + \left(\frac{x_i - a_i}{b_i - a_i} \right)^{r_i + 1} \leq 1, i = 1, ..., n$$

and the last part of (2.1) is also proved. ∎

Some particular cases are interesting.

Corollary 2.2. *Under the above assumptions, we have the mid-point inequality:*

$$\left|f\left(\frac{a_1+b_1}{2},...,\frac{a_n+b_n}{2}\right)-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(t_1,...,t_n)\,dt_n...dt_1\right|$$

$$\leq\sum_{i=1}^{n}\frac{L_i(b_i-a_i)^{r_i}}{2^{r_i}(r_i+1)} \tag{2.3}$$

which is the best inequality we can get from (2.1).

Proof. Note that the mapping $h_p:[\alpha,\beta]\to\mathbf{R}$, $h_p(y)=(y-\alpha)^{p+1}+(\beta-y)^{p+1}$ $(p>0)$ has its infimum at $y_0=\frac{\alpha+\beta}{2}$ and

$$\inf_{y\in[\alpha,\beta]}h_p(y)=\frac{(\beta-\alpha)^{p+1}}{2^p}.$$

Consequently, the best inequality we can get from (2.1) is the one for which $x_i=\frac{a_i+b_i}{2}$ giving the desired inequality (2.3). ∎

The following trapezoid type inequaltiy also holds

Corollary 2.3. *Under the above assumptions, we have:*

$$\left|\frac{f(a_1,...,a_n)+f(b_1,...,b_n)}{2}-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(t_1,...,t_n)\,dt_n...dt_1\right|$$

$$\leq\sum_{i=1}^{n}\frac{L_i(b_i-a_i)^{r_i}}{r_i+1} \tag{2.4}$$

Proof. Put in (2.1) $\bar{\mathbf{x}}=\bar{\mathbf{a}}$ and then $\bar{\mathbf{x}}=\bar{\mathbf{b}}$, add the obtained inequalities and use the triangle inequality to get (2.4). ∎

An important particular case is one for which the mapping f is Lipschitzian, i.e.,

$$|f(x_1,...,x_n)-f(y_1,...,y_n)|\leq\sum_{i=1}^{n}L_i|x_i-y_i| \tag{2.5}$$

for all $\bar{\mathbf{x}},\bar{\mathbf{y}}\in\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$.

Thus, we have following corollary.

Corollary 2.4. *Let f be a Lipschitzian mapping with the constants L_i. Then we have*

$$\left|f(\bar{\mathbf{x}})-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}f(\bar{\mathbf{t}})\,d\bar{\mathbf{t}}\right|$$

$$\leq\sum_{i=1}^{n}L_i\left[\frac{1}{4}+\left(\frac{x_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i) \tag{2.6}$$

for all $\bar{\mathbf{x}}\in\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$.

The constant $\frac{1}{4}$, *in all the brackets, is the best possible.*

Proof. Choose $r_i = 1\,(i = 1, ..., n)$ in (2.1) to get

$$\left| f(x_1, ..., x_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(t_1, ..., t_n)\, dt_n ... dt_1 \right|$$

$$\leq \frac{1}{2}\sum_{i=1}^{n} L_i \left[\left(\frac{x_i - a_i}{b_i - a_i}\right)^2 + \left(\frac{b_i - x_i}{b_i - a_i}\right)^2 \right](b_i - a_i)$$

A simple computation shows that

$$\frac{1}{2}\left[\left(\frac{x_i - a_i}{b_i - a_i}\right)^2 + \left(\frac{b_i - x_i}{b_i - a_i}\right)^2 \right] = \frac{1}{4} + \left(\frac{x_i - \frac{a_i + b_i}{2}}{b_i - a_i}\right)^2, i = 1, ..., n$$

giving the desired inequality (2.6).

To prove the sharpness of the constants $\frac{1}{4}$, assume that the inequality (2.6) holds for some positive constants $c_i > 0$, i.e.,

$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}})\, d\bar{\mathbf{t}} \right|$$

$$\leq \sum_{i=1}^{n} L_i \left[c_i + \left(\frac{x_i - \frac{a_i + b_i}{2}}{b_i - a_i}\right)^2 \right](b_i - a_i), \tag{2.7}$$

for all $\bar{\mathbf{x}} \in \left[\bar{\mathbf{a}}, \bar{\mathbf{b}}\right]$.

Choose $f(x_1, ..., x_n) = x_i\ (i = 1, ..., n)$. Then, by (2.7), we get

$$\left| x_i - \frac{a_i + b_i}{2} \right| \leq \left[c_i + \frac{\left(x_i - \frac{a_i + b_i}{2}\right)^2}{(b_i - a_i)^2} \right](b_i - a_i)$$

for all $x_i \in [a_i, b_i]$. Put $x_i = a_i$, to get

$$\frac{b_i - a_i}{2} \leq \left(c_i + \frac{1}{4}\right)(b_i - a_i)$$

from which we deduce $c_i \geq \frac{1}{4}$, and the sharpness of $\frac{1}{4}$ is proved. ∎

Corollary 2.5. *If f is as in Corollary 4, then we get*

a) the mid-point formula

$$\left| f\left(\frac{a_1 + b_1}{2}, ..., \frac{a_n + b_n}{2}\right) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(t_1, ..., t_n)\, dt_n ... dt_1 \right|$$

$$\leq \frac{1}{4}\sum_{i=1}^{n} L_i (b_i - a_i) \tag{2.8}$$

b) the trapezoid formula

$$\left|\frac{f(a_1,...,a_n)+f(b_1,...,b_n)}{2}-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(t_1,...,t_n)\,dt_n...dt_1\right|$$

$$\leq \frac{1}{2}\sum_{i=1}^{n}L_i(b_i-a_i).$$

Remark 2.1. *In practical applications, we assume that the mapping* $f:[\bar{\mathbf{a}},\bar{\mathbf{b}}]\to\mathbf{R}$ *has the partial derivatives* $\frac{\partial f}{\partial x_i}$ *and* $\frac{\partial f}{\partial x_i}$ *bounded on* $[\bar{\mathbf{a}},\bar{\mathbf{b}}]$*. That is,*

$$\left\|\frac{\partial f}{\partial x_i}\right\|_\infty := \sup_{\bar{\mathbf{x}}\in(\bar{\mathbf{a}},\bar{\mathbf{b}})}\left|\frac{\partial f(x_1,...,x_n)}{\partial x_i}\right|<\infty.$$

With this assumption, we have the Ostrowski type inequality

$$\left|f(\bar{\mathbf{x}})-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}f(\bar{\mathbf{t}})\,d\bar{\mathbf{t}}\right|$$

$$\leq\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\left[\frac{1}{4}+\left(\frac{x_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i).$$

The constants, $\frac{1}{4}$, are sharp.

3. A weighted version

The following generalization of Theorem 1 holds

Theorem 3.1. *Let* $f,w:[\bar{\mathbf{a}},\bar{\mathbf{b}}]\to\mathbf{R}$ *be such that* f *is of the* $r-$*Hölder type with the constants* L_i *and* $r_i\in(0,1]\,(i=1,...,n)$ *and where* w *is integrable on* $[\bar{\mathbf{a}},\bar{\mathbf{b}}]$, *nonnegative on this interval and*

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w(\bar{\mathbf{x}})\,d\bar{\mathbf{x}}:=\int_{a_1}^{b_1}...\int_{a_n}^{b_n}w(x_1,...,x_n)\,dx_n...dx_1>0.$$

Then we have the inequality:

$$\left|f(\bar{\mathbf{x}})-\frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w(\bar{\mathbf{y}})\,d\bar{\mathbf{y}}}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w(\bar{\mathbf{y}})f(\bar{\mathbf{y}})\,d\bar{\mathbf{y}}\right|$$

$$\leq\sum_{i=1}^{n}L_i\frac{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}|x_i-y_i|^{r_i}w(\bar{\mathbf{y}})\,d\bar{\mathbf{y}}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w(\bar{\mathbf{y}})\,d\bar{\mathbf{y}}}. \tag{3.1}$$

for all $\bar{\mathbf{x}}\in[\bar{\mathbf{a}},\bar{\mathbf{b}}]$.

Proof. The proof is similar to that of Theorem 1 .

As f is of the $r-$Hölder type with the constants L_i and r_i $(i=1,...,n)$, we can write

$$\left|f\left(\bar{\mathbf{x}}\right)-f\left(\bar{\mathbf{y}}\right)\right|\leq\sum_{i=1}^{n}L_i\left|x_i-y_i\right|^{r_i}$$

for all $\bar{\mathbf{x}},\bar{\mathbf{y}}\in\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$.

Multiplying by $w\left(\bar{\mathbf{y}}\right)\geq 0$ and integrating over $\bar{\mathbf{y}}$ on $\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$, we get

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}\left|f\left(\bar{\mathbf{x}}\right)-f\left(\bar{\mathbf{y}}\right)\right|w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}\leq\sum_{i=1}^{n}L_i\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}\left|x_i-y_i\right|^{r_i}w\left(y_i,...,y_n\right)dy_n...dy_1. \tag{3.2}$$

On the other hand, we have

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}\left|f\left(\bar{\mathbf{x}}\right)-f\left(\bar{\mathbf{y}}\right)\right|w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}\geq\left|\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}\left(f\left(\bar{\mathbf{x}}\right)-f\left(\bar{\mathbf{y}}\right)\right)w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}\right|$$

$$=\left|f\left(\bar{\mathbf{x}}\right)\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}-\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}f\left(\bar{\mathbf{y}}\right)w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}\right| \tag{3.3}$$

Combining (3.2) with (3.3) and dividing by $\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}>0$ we get the desired inequality (3.1). ∎

Remark 3.1. *If we assume that the mapping f is Lipschitzian with constants L_i then we get,*

$$\left|f\left(\bar{\mathbf{x}}\right)-\frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w\left(\bar{\mathbf{y}}\right)f\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}\right|$$

$$\leq\sum_{i=1}^{n}L_i\frac{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}\left|x_i-y_i\right|^{r_i}w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}},$$

which generalizes Milovanović result from 1975 (see [6, p. 468]*).*

The following corollaries hold.

Corollary 3.2. *With the assumptions from Theorem 6 we have:*

$$\left|f\left(\bar{\mathbf{x}}\right)-\frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}w\left(\bar{\mathbf{y}}\right)f\left(\bar{\mathbf{y}}\right)d\bar{\mathbf{y}}\right|$$

$$\leq \sum_{i=1}^{n} L_i \left[\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right] \tag{3.4}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Proof. We have

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}} \leq \sup_{\bar{y} \in [\bar{a}, \bar{b}]} |x_i - y_i|^{r_i} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}$$

$$= \max\{|x_i - a_i|, |x_i - b_i|\} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}$$

$$= \max\{x_i - a_i, b_i - x_i\} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}$$

$$= \left[\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right] \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}.$$

Then

$$\sum_{i=1}^{n} L_i \frac{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}} \leq \sum_{i=1}^{n} L_i \left[\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right]$$

and by (3.1), we get the desired estimation (3.4). ∎

Another type of estimation is as follows

Corollary 3.3. *With the assumptions from Theorem 6, we have*

$$\left| f(\bar{x}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}}) d\bar{\mathbf{y}} \right|$$

$$\leq \frac{\sup_{\bar{\mathbf{y}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]} w(\bar{\mathbf{y}})}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}} \prod_{j=1}^{n} (b_j - a_j) \sum_{i=1}^{n} L_i \left[\frac{(b_i - x_i)^{r_i+1} + (x_i - a_i)^{r_i+1}}{(r_i + 1)(b_i - a_i)} \right] \tag{3.5}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Proof. We observe that

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \le \sup_{\bar{\mathbf{y}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]} w(\bar{\mathbf{y}}) \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i}\, d\bar{\mathbf{y}}$$

$$= \prod_{\substack{j=1 \\ j \ne i}}^{n} (b_j - a_j) \int_{a_i}^{b_i} |x_i - y_i|^{r_i}\, dy_i$$

$$= \prod_{\substack{j=1 \\ j \ne i}}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{r_i+1} + (x_i - a_i)^{r_i+1}}{r_i + 1} \right]$$

$$= \prod_{j=1}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{r_i+1} + (x_i - a_i)^{r_i+1}}{(r_i + 1)(b_i - a_i)} \right].$$

Now, using (3.1) we get (3.5).

Finally, by Hölder's integral inequality we can also state the following corollary:

Corollary 3.4. *With the assumptions from Theorem 6, we have*

$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}}) d\bar{\mathbf{y}} \right|$$

(3.6)

$$\le \frac{\left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} [w(\bar{\mathbf{y}})]^q d\bar{\mathbf{y}} \right)^{1/q}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \prod_{j=1}^{n} (b_i - a_i)^{\frac{1}{p}} \sum_{i=1}^{n} L_i \left[\frac{(b_i - x_i)^{pr_i+1} + (x_i - a_i)^{pr_i+1}}{(pr_i + 1)(b_i - a_i)} \right]^{\frac{1}{p}}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Proof. Using Hölder's integral inequality for multiple integrals, we get

(3.7)

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \le \left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} [w(\bar{\mathbf{y}})]^q d\bar{\mathbf{y}} \right)^{1/q} \left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{pr_i}\, d\bar{\mathbf{y}} \right)^{\frac{1}{p}}.$$

But

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{pr_i}\, d\bar{\mathbf{y}} = \prod_{\substack{j=1 \\ j \ne i}}^{n} (b_j - a_j) \int_{a_i}^{b_i} |x_i - y_i|^{r_i}\, dy_i$$

$$= \prod_{j=1}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{pr_i+1} + (x_i - a_i)^{pr_i+1}}{(pr_i + 1)(b_i - a_i)} \right]$$

and then, by (3.7), we get

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{pr_i} \, d\bar{\mathbf{y}}$$

$$\leq \prod_{j=1}^{n} (b_j - a_j)^{\frac{1}{p}} \left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} [w(\bar{\mathbf{y}})]^q d\bar{\mathbf{y}} \right)^{\frac{1}{q}} \left[\frac{(b_i - x_i)^{pr_i+1} + (x_i - a_i)^{pr_i+1}}{(pr_i + 1)(b_i - a_i)} \right]^{\frac{1}{p}}.$$

Using (3.1), we deduce the desired estimation(3.6). ∎

References

[1] N.S. BARNETT and S.S. DRAGOMIR, An Ostrowski type inequality for double integrals and applications for cubature formulae , *submitted.*

[2] S.S. DRAGOMIR and S. WANG , A new inequality of Ostrowski's type in L_1 norm and applictions to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[3] S.S. DRAGOMIR and S. WANG, An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.,* **33**(1997), 15-20.

[4] S.S. DRAGOMIR and S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[5] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_p norm and applications to some special means and to some numerical quadrature rules, *submitted for publications*

[6] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, MC Melbourne City, Victoria 8001, Australia.

E-mail address: `{sever, neil, pc}@matilda.vu.edu.au`