Lobatto Type Quadrature Rules for Functions with Bounded Derivative

LOBATTO TYPE QUADRATURE RULES FOR FUNCTIONS WITH BOUNDED DERIVATIVE

P. CERONE AND S.S. DRAGOMIR

Abstract. Inequalities are obtained for quadrature rules in terms of upper and lower bounds of the first derivative of the integrand. Bounds of Ostrowski type quadrature rules are obtained and the classical Iyengar inequality for the trapezoidal rule is recaptured as a special case. Applications to numerical integration are demonstrated.

1. Introduction

In 1938, Iyengar proved the following theorem obtaining bounds for a trapezoidal quadrature rule for functions whose derivative is bounded (see for example [3, p. 471]).

Theorem 1. *Let f be a differentiable function on (a,b) and assume that there is a constant $M>0$ such that $|f'(x)| \leq M, \forall x \in (a,b)$. Then we have*

$$\left| \int_a^b f(x)\,dx - (b-a)\frac{f(a)+f(b)}{2} \right| \leq \frac{M(b-a)^2}{4} - \frac{1}{4M}(f(a)-f(b))^2. \tag{1.1}$$

Using the classical inequality due to Hayashi (see for example, [2, pp. 311-312]), Agarwal and Dragomir proved in [1] the following generalization of Theorem 1.

Theorem 2. *Let $f : I \subseteq \mathbb{R} \mapsto \mathbb{R}$ be a differentiable mapping in $\mathring{I}$, the interior of I, and let $a,b \in \mathring{I}$ with $a<b$. Let $M = \sup_{x\in[a,b]} f'(x) < \infty$ and $m = \inf_{x\in[a,b]} f'(x) > -\infty$. If $m<M$ and f' is integrable on $[a,b]$, then we have*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - (b-a)\frac{f(a)+f(b)}{2} \right| \\ \leq\ & \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)}. \end{aligned} \tag{1.2}$$

Thus, by placing $m=-M$ in (1.2) then Iyengar's result (1.1) is recovered.

In 1976, G. V. Milovanović and J. E. Pečarić proved the following Ostrowki type inequality:

Date: April, 1999.
1991 *Mathematics Subject Classification.* Primary 26D15, 26D99; Secondary 41A55.
Key words and phrases. Hayashi, Iyengar and Ostrowski inequalities, Quadrature formulae.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) and let $\|f''\|_\infty=\sup_{x\in(a,b)}|f''(x)|<\infty$. Then*

$$\left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}\left[f(\xi)+\left(\frac{\xi-a}{b-a}\right)f(a)+\left(\frac{b-\xi}{b-a}\right)f(b)\right]\right| \tag{1.3}$$
$$\leq \|f''\|_\infty\frac{(b-a)}{4}\left[\frac{1}{3}\left(\frac{b-a}{2}\right)^2+\left(\xi-\frac{a+b}{2}\right)^2\right]$$

for all $\xi\in[a,b]$.

Placing $\xi=a$ or b would produce a bound for the trapezoidal rule, namely $\|f''\|_\infty\frac{(b-a)^3}{12}$.

S.S. Dragomir, Y.J. Cho and S.S. Kim [5] obtained a bound for the quadrature rule of Milovanović and Pečarić but with the less restrictive assumption on $\|f'\|_\infty$ rather than $\|f''\|_\infty$.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b) and let $\|f'\|_\infty=\sup_{x\in(a,b)}|f'(x)|<\infty$ then*

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{2}\left[f(\xi)+\left(\frac{\xi-a}{b-a}\right)f(a)+\left(\frac{b-\xi}{b-a}\right)f(b)\right]\right| \tag{1.4}$$
$$\leq \frac{\|f'\|_\infty}{2}\left[\left(\frac{b-a}{2}\right)^2+\left(\xi-\frac{a+b}{2}\right)^2\right]-\frac{1}{8\|f'\|_\infty}(f(b)-f(a))^2.$$

In this paper a number of generalizations, simplifications and an extension of the above results are presented.

Firstly, the result (1.2) of Agarwal and Dragomir [1] is proved by utilizing a generalization that simplifies the working and, it is argued, is more enlightening. Secondly, the development leads naturally to obtaining non-symmetric bounds on a generalized trapezoidal rule of the form

$$(\theta-a)f(a)+(b-\theta)f(b).$$

The result (1.2) is recaptured when $\theta=\frac{a+b}{2}$.

Thirdly, the interval is subdivided and the trapezoidal rule is applied to the two intervals separately to obtain a Lobatto type quadrature rule of the form

$$\frac{b-a}{2}\left[f(\xi)+\left(\frac{\xi-a}{b-a}\right)f(a)+\left(\frac{b-\xi}{b-a}\right)f(b)\right],$$

involving an interior point ξ and the end points. This result includes (1.4) as a special case.

Finally, application of the results in numerical integration is demonstrated.

2. Integral Inequalities

The following theorem due to Hayashi [2, pp. 311-312] will be required and thus it is stated for convenience.

Theorem 5. *Let $h:[a,b]\longrightarrow\mathbb{R}$ be a nonincreasing mapping on $[a,b]$ and $g:[a,b]\longrightarrow\mathbb{R}$ an integrable mapping on $[a,b]$ with*

$$0\leq g(x)\leq A,\ \textit{for all } x\in[a,b],$$

then

$$A\int_{b-\lambda}^{b}h\left(x\right)dx\leq\int_{a}^{b}h\left(x\right)g\left(x\right)dx\leq A\int_{a}^{a+\lambda}h\left(x\right)dx \tag{2.1}$$

where

$$\lambda=\frac{1}{A}\int_{a}^{b}g\left(x\right)dx.$$

Theorem 6. *Let $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I) and $[a,b]\subset\mathring{I}$ with $M=\sup_{x\in[a,b]}f'(x)<\infty$, $m=\inf_{x\in[a,b]}f'(x)>-\infty$ and $M>m$. If f' is integrable on $[a,b]$, then the following inequalities hold:*

$$\left|\int_{a}^{b}f\left(x\right)dx-\frac{b-a}{2}\left[f\left(a\right)+f\left(b\right)\right]\right|\leq\frac{\left(b-a\right)^{2}}{2\left(M-m\right)}\left(S-m\right)\left(M-S\right) \tag{2.2}$$

$$\leq\frac{M-m}{2}\left(\frac{b-a}{2}\right)^{2} \tag{2.3}$$

where $S=\frac{f(b)-f(a)}{b-a}$.

Proof. Let $h\left(x\right)=\theta-x$, $\theta\in[a,b]$ and $g\left(x\right)=f'\left(x\right)-m$. Then, from Hayashi's inequality (2.1)

$$L\leq I\leq U \tag{2.4}$$

where

$$I=\int_{a}^{b}\left(\theta-x\right)\left(f'\left(x\right)-m\right)dx,$$

$$\lambda=\frac{1}{M-m}\int_{a}^{b}\left(f'\left(x\right)-m\right)dx,$$

and

$$L=\left(M-m\right)\int_{b-\lambda}^{b}\left(\theta-x\right)dx,$$

$$U=\left(M-m\right)\int_{a}^{a+\lambda}\left(\theta-x\right)dx.$$

It is now a straight-forward matter to evaluate and simplify the above expansions to give

$$I=\int_{a}^{b}f\left(u\right)du-\left[m\left(b-a\right)\left(\theta-\frac{b+a}{2}\right)+\left(b-\theta\right)f\left(b\right)+\left(\theta-a\right)f\left(a\right)\right], \tag{2.5}$$

$$\lambda=\frac{1}{M-m}\left[f\left(b\right)-f\left(a\right)-m\left(b-a\right)\right]=\frac{b-a}{M-m}\left(S-m\right), \tag{2.6}$$

$$L=\frac{\left(M-m\right)}{2}\lambda\left[\lambda+2\left(\theta-b\right)\right], \tag{2.7}$$

and

$$U=\frac{\left(M-m\right)}{2}\lambda\left[2\left(\theta-a\right)-\lambda\right]. \tag{2.8}$$

In addition, it may be noticed from (2.4), that

$$\left|I-\frac{U+L}{2}\right|\leq\frac{U-L}{2}, \tag{2.9}$$

where, upon using (2.7) and (2.8),

$$\frac{U+L}{2}=(M-m)\,\lambda\left(\theta-\frac{b+a}{2}\right) \tag{2.10}$$

and

$$\frac{U-L}{2}=\frac{(M-m)}{2}\lambda\,(b-a-\lambda)\,. \tag{2.11}$$

Equation (2.9) is then, (2.2) upon using (2.5), (2.6), (2.10) and (2.11) together with some routine simplification.
Now, for inequality (2.3). Consider the right hand side of (2.2). Completing the square gives

$$\begin{aligned} &\frac{(b-a)^2}{2\,(M-m)}\,(S-m)\,(M-S) \qquad (2.12)\\ &= \frac{2}{M-m}\left(\frac{b-a}{2}\right)^2\times\left[\left(\frac{M-m}{2}\right)^2-\left(S-\frac{M+m}{2}\right)^2\right]\end{aligned}$$

and (2.3) is readily determined by neglecting the negative term. ∎

Remark 1. *The above theorem was proved independently of the value of* θ. *Agarwal and Dragomir* [1] *proved an equivalent result with effectively* $\theta=a$. *It may be noticed from the above development however, that if* $\theta=\frac{a+b}{2}$ *then there is some simplification for* I *and* $\frac{U+L}{2}=0$.

Remark 2. *For* $\|f'\|_\infty=\sup_{x\in[a,b]}|f'(x)|<\infty$ *and let* $m=-\|f'\|_\infty$, $M=\|f'\|_\infty$ *in* (2.2). *Then the result obtained by Iyengar* [3, p. 471] *using geometrical means, is recovered. It should also be noted that if either both* m *and* M *are positive or both negative, then the bound obtained here is tighter than that of Iyengar as given by* (1.1).

Bounds for the generalized trapezoidal rule will now be developed in the following theorem.

Theorem 7. *Let* f *satisfy the conditions of Theorem 6, then the following result holds*

$$\beta_L\leq\int_a^b f(x)\,dx-(b-a)\left[\left(\frac{\theta-a}{b-a}\right)f(a)+\left(\frac{b-\theta}{b-a}\right)f(b)\right]\leq\beta_U \tag{2.13}$$

where

$$\beta_U=\frac{(b-a)^2}{2\,(M-m)}\left[S\,(2\gamma_U-S)-mM\right], \tag{2.14}$$

$$\beta_L=\frac{(b-a)^2}{2\,(M-m)}\left[S\,(S-2\gamma_L)+mM\right], \tag{2.15}$$

$$\gamma_U=\left(\frac{\theta-a}{b-a}\right)M+\left(\frac{b-\theta}{b-a}\right)m,\ \ \gamma_L=M+m-\gamma_U, \tag{2.16}$$

and

$$S = \frac{f(b) - f(a)}{b - a}. \tag{2.17}$$

Proof. From (2.4) and (2.5) it may be readily seen that

$$\beta_U = U + m(b - a)\left(\theta - \frac{a + b}{2}\right) \tag{2.18}$$

and

$$\beta_L = L + m(b - a)\left(\theta - \frac{a + b}{2}\right). \tag{2.19}$$

Now, from (2.18) and using (2.8), (2.6) gives

$$\begin{aligned}
\beta_U &= \frac{1}{2(M - m)}\Big\{(b - a)(S - m)[2(M - m)(\theta - a) - (b - a)(S - m)] \\
&\quad + 2m(b - a)(M - m)\left(\theta - \frac{a + b}{2}\right)\Big\} \\
&= \frac{(b - a)^2}{2(M - m)}\Big\{(S - m)\left[S - m + 2(M - m)\left(\frac{\theta - a}{b - a}\right)\right] \\
&\quad + 2m\left(\frac{M - m}{b - a}\right)\left(\theta - \frac{a + b}{2}\right)\Big\}.
\end{aligned}$$

Expanding in powers of S and after simplification we produce the expression (2.14). In a similar fashion, (2.15) may be derived from (2.19) and using (2.7), (2.6) gives

$$\begin{aligned}
\beta_L &= \frac{1}{2(M - m)}\{(b - a)(S - m)[(b - a)(S - m) + 2(M - m)(\theta - b)] \\
&\quad + 2m(b - a)(M - m)\left(\theta - \frac{a + b}{2}\right)\Big\} \\
&= \frac{(b - a)^2}{2(M - m)}\Big\{(S - m)\left[S - m + 2(M - m)\left(\frac{\theta - b}{b - a}\right)\right] \\
&\quad + 2m\left(\frac{M - m}{b - a}\right)\left(\theta - \frac{a + b}{2}\right)\Big\}.
\end{aligned}$$

Again, expanding in powers of S produces (2.15) after some algebra and thus the proof of the theorem is complete. ∎

Remark 3. *Allowing* $\theta = \frac{a+b}{2}$ *gives*

$$\beta_L = -\beta_U = \frac{(b - a)^2}{2(M - m)}(S - m)(M - S),$$

thus reproducing the result of Theorem 6.

Remark 4. *It may be shown from* (2.14) *and* (2.15) *that for any* $\theta \in [a, b]$, *the size of the bound interval for the generalized trapezoidal rule is:*

$$\beta_U - \beta_L = \frac{(b - a)^2}{(M - m)}\left[\left(\frac{M - m}{2}\right)^2 - \left(S - \frac{M + m}{2}\right)^2\right].$$

This is the same size as that for the symmetric bounds for the trapezoidal rule of Theorem 6 which seems, at first, surprising though on observing (2.18) *and* (2.19) *may be less so.*

Remark 5. *The difference between the upper and lower bounds is always positive since*

$$\beta_U - \beta_L = \frac{(b-a)^2}{M-m}(S-m)(M-S) \geq 0$$

where S, from (2.17)*, is the slope of the secant and $m \leq S \leq M$.*

Remark 6. *For $\|f'\|_\infty = \sup_{x\in[a,b]} |f'(x)| < \infty$, let $m = -\|f'\|_\infty$ and $M = \|f'\|$ in* (2.13) − (2.17) *then an Iyengar type result for the generalized trapezoidal rule will be obtained.*

Corollary 1. *Let f satisfy the conditions of Theorems 6 and 7. Then*

$$\begin{aligned}
&\frac{(b-a)^2}{2(M-m)}\left(mM - \gamma_L^2\right) \\
\leq\ &\int_a^b f(u)\,du - (b-a)\left[\left(\frac{\theta-a}{b-a}\right)f(a) + \left(\frac{b-\theta}{b-a}\right)f(b)\right] \\
\leq\ &\frac{(b-a)^2}{2(M-m)}\left[\gamma_U^2 - mM\right]
\end{aligned} \tag{2.20}$$

where γ_U and γ_L are as defined in (2.15)*.*

Proof. From (2.13) and (2.14) it may be shown by completing the square that

$$\beta_U = \frac{(b-a)^2}{2(M-m)}\left[\gamma_U^2 - mM - (S-\gamma_U)^2\right]$$

and

$$\beta_L = \frac{(b-a)^2}{2(M-m)}\left[(S-\gamma_L)^2 + mM - \gamma_L^2\right].$$

The result (2.20) follows from neglecting the negative term from β_U and the positive term from β_L. ∎

A composite quadrature rule of Lobatto type will now be developed that involves the end points and an interior point. It relies on the first derivative being bounded.

The following theorem develops bounds for a generalized trapezoidal type rule.

Theorem 8. *Let f satisfy the assumptions of Theorem 6, then the following inequality holds*

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(\xi) + \left(\frac{b-\xi}{b-a}\right)f(b) + \left(\frac{\xi-a}{b-a}\right)f(a)\right]\right| \\
\leq\ &\frac{1}{M-m}\left\{(M+m)\left[\left(\xi - \frac{a+b}{2}\right)f(\xi) + \frac{b-\xi}{2}f(b) - \frac{\xi-a}{2}f(a)\right]\right. \\
&-mM\left[\left(\frac{b-a}{2}\right)^2 + \left(\xi - \frac{a+b}{2}\right)^2\right] \\
&\left.-\left[\left(\frac{f(b)-f(a)}{2}\right)^2 + \left(f(\xi) - \frac{f(b)+f(a)}{2}\right)^2\right]\right\}
\end{aligned} \tag{2.21}$$

where $\xi \in [a,b]$.

Proof. Applying (2.2) on the intervals $[a,\xi]$ and $[\xi,b]$ gives

$$\begin{aligned} & \left| \int_a^\xi f(x)\,dx - \frac{\xi-a}{2}\left[f(a)+f(\xi)\right] \right| \\ \leq & \frac{1}{2(M-m)}\left[A-m(\xi-a)\right]\left[M(\xi-a)-A\right] \end{aligned}$$

and

$$\begin{aligned} & \left| \int_\xi^b f(x)\,dx - \frac{b-\xi}{2}\left[f(\xi)+f(b)\right] \right| \\ \leq & \frac{1}{2(M-m)}\left[B-m(b-\xi)\right]\left[M(b-\xi)-B\right], \end{aligned}$$

where $A = f(\xi) - f(a)\,,\ B = f(b) - f(\xi)\,.$
Summing the above two inequalities we have

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(\xi) + \left(\frac{b-\xi}{b-a}\right) f(b) + \left(\frac{\xi-a}{b-a}\right) f(a) \right] \right| \leq \beta, \tag{2.22}$$

where

$$\begin{aligned} \beta &= \frac{1}{2(M-m)}\left\{(M+m)\left[(\xi-a)A+(b-\xi)B\right]\right. \\ & \left. -\, mM\left[(\xi-a)^2+(b-\xi)^2\right] - \left(A^2+B^2\right)\right\}. \end{aligned}$$

Now, on substituting for A and B it is easily shown that

$$\frac{1}{2}\left[(\xi-a)A+(b-\xi)B\right] = \left(\xi - \frac{a+b}{2}\right) f(\xi) + \frac{b-\xi}{2} f(b) - \frac{\xi-a}{2} f(a)\,. \tag{2.23}$$

Further, using the algebraic fact that

$$\frac{X^2+Y^2}{2} = \left(\frac{X+Y}{2}\right)^2 + \left(\frac{X-Y}{2}\right)^2,$$

then

$$\frac{1}{2}\left[(\xi-a)^2+(b-\xi)^2\right] = \left(\frac{b-a}{2}\right)^2 + \left(\xi - \frac{a+b}{2}\right)^2 \tag{2.24}$$

and

$$\begin{aligned} \frac{1}{2}\left[A^2+B^2\right] &= \frac{1}{2}\left[(f(\xi)-f(a))^2+(f(b)-f(\xi))^2\right] \\ &= \left(\frac{f(b)-f(a)}{2}\right)^2 + \left(f(\xi) - \frac{f(a)+f(b)}{2}\right)^2. \end{aligned} \tag{2.25}$$

Substituting (2.23) – (2.25) into (2.22) gives the desired result (2.21), and thus, the theorem is proved. ∎

Remark 7. *Neglecting either one or both of the negative terms in the last square bracket in* (2.21) *would give a coarser upper bound. If m and M are either both positive or both negative, then another upper bound may be readily obtained.*

Remark 8. *Substitution of either $\xi = a$ or b would reproduce the results in Theorem* 6.

Corollary 2. *With the conditions on f as in Theorem 6 and 8, the following inequality holds*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{2}\right]\right| \tag{2.26}$$
$$\leq \frac{1}{M-m}\left\{(M+m)\frac{b-a}{2}\cdot\frac{f(b)-f(a)}{2} - mM\left(\frac{b-a}{2}\right)^2 - \left[\left(\frac{f(b)-f(a)}{2}\right)^2 + \left(f\left(\frac{a+b}{2}\right) - \frac{f(b)+f(a)}{2}\right)^2\right]\right\}.$$

Proof. Setting $\xi = \frac{a+b}{2}$ in (2.21) immediately gives the result. ∎

Corollary 3. *With f satisfying the conditions of Theorem 6 and 8, then for $\|f'\|_\infty = \sup_{x\in[a,b]}|f'(x)| < \infty$, the following inequality holds.*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(\xi) + \left(\frac{b-\xi}{b-a}\right)f(b) + \left(\frac{\xi-a}{b-a}\right)f(a)\right]\right| \tag{2.27}$$
$$\leq \frac{1}{2}\left\{\|f'\|_\infty\left[\left(\frac{b-a}{2}\right)^2 + \left(\xi - \frac{a+b}{2}\right)^2\right] - \frac{1}{\|f'\|_\infty}\left[\left(\frac{f(b)-f(a)}{2}\right)^2 + \left(f(\xi) - \frac{f(a)+f(b)}{2}\right)^2\right]\right\}.$$

Proof. Put $m = -\|f'\|_\infty$ and $M = \|f'\|_\infty$ in Theorem 8.
The result (2.27) is similar to that obtained by Dragomir, Cho and Kim [5]. Their result neglected the last square term and it contained a small error. The corrected result is given in (1.4). ∎

Theorem 9. *Let f satisfy the assumptions of Theorem 6, then the following inequality holds.*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(\xi) + \left(\frac{b-\xi}{b-a}\right)f(b) + \left(\frac{\xi-a}{b-a}\right)f(a)\right]\right|$$
$$\leq \frac{M-m}{4}\left[\left(\frac{b-a}{2}\right)^2 + \left(\xi - \frac{a+b}{2}\right)^2\right] \tag{2.28}$$
$$\leq \frac{M-m}{2}\cdot\left(\frac{b-a}{2}\right)^2. \tag{2.29}$$

Proof. We use the procedure followed in the proof of Theorem 8 and apply Theorem 6 on the intervals $[a,x]$ and $[x,b]$, except that the bound is used in the form of a difference of two squares as given by (2.12). Thus,

$$\left|\int_a^\xi f(x)\,dx - \frac{\xi-a}{2}[f(a)+f(\xi)]\right|$$
$$\leq \frac{2}{M-m}\left(\frac{\xi-a}{2}\right)^2\left[\left(\frac{M-m}{2}\right)^2 - \left(S_1 - \frac{M+m}{2}\right)^2\right]$$

and

$$\left|\int_{\xi}^{b} f(x)\,dx - \frac{b-\xi}{2}\left[f(\xi)+f(b)\right]\right|$$
$$\leq \frac{2}{M-m}\left(\frac{b-\xi}{2}\right)^{2}\left[\left(\frac{M-m}{2}\right)^{2}-\left(S_2-\frac{M+m}{2}\right)^{2}\right].$$

Where $S_1 = \frac{f(\xi)-f(a)}{\xi-a}$ and $S_2 = \frac{f(b)-f(\xi)}{b-\xi}$.
Summing the above two inequalities results in

$$\left|\int_{a}^{b} f(x)\,dx - \frac{b-a}{2}\left[f(\xi)+\left(\frac{b-\xi}{b-a}\right)f(b)+\left(\frac{\xi-a}{b-a}\right)f(a)\right]\right|$$
$$\leq \frac{2}{M-m}\left\{\left(\frac{M-m}{2}\right)^{2}\left[\left(\frac{\xi-a}{2}\right)^{2}+\left(\frac{b-\xi}{2}\right)^{2}\right]\right.$$
$$\left. -\left(\frac{\xi-a}{2}\right)^{2}\left(S_1-\frac{M+m}{2}\right)^{2}-\left(\frac{b-\xi}{2}\right)^{2}\left(S_2-\frac{M+m}{2}\right)^{2}\right\}.$$

Neglecting the last two negative terms and simplifying gives the result (2.28). Equation (2.29) is obtained by simply noting that a coarser upper bound is obtained at $\xi = a$ or b. ∎

A development similar to this may be accomplished by taking Theorem 7 as a starting point and applying it separately to the intervals $[a,\xi]$ and $[\xi,b]$ in the same manner as Theorem 9. This will, however, not be presented.

3. Application in Numerical Integration

Theorem 10. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b) with $M=\sup_{x\in(a,b)} f'(x)<\infty$, $m=\inf'_{x\in(a,b)} f(x)>-\infty$ and $M>m$. Then for any partition $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ of $[a,b]$ and any intermediate mid-point vectors $\xi=\left(\xi_0,\xi_1,...,\xi_{n-1}\right)$ such that $\xi_i\in[x_i,x_{i+1}]$ for $i=0,1,...,n-1$, we have*

$$\int_{a}^{b} f(x)\,dx = A_C(f,I_n,\xi)+R_C(f,I_n,\xi), \tag{3.1}$$

where $A_C(f,I_n,\xi)$ is a generalized Riemann sum given by

$$\begin{aligned} A_R(f,I_n,\xi) &= \frac{1}{2}\left[\sum_{i=0}^{n-1} f(\xi_i)\,h_i+\sum_{i=0}^{n-1}(\xi_i-x_i)\,f(x_i)+\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)\,f(x_{i+1})\right] \\ &= \frac{1}{2}\left[\sum_{i=0}^{n-1} f(\xi_i)\,h_i+\sum_{i=0}^{n-1}\xi_i\left(f(x_i)-f(x_{i+1})\right)+bf(b)-af(a)\right], \end{aligned}$$

and the remainder term $R_C(f,I_n,\xi)$ satisfies

$$\begin{aligned} |R_C(f,I_n,\xi)| &\leq \frac{M-m}{4}\sum_{i=0}^{n-1}\left[\left(\frac{h_i}{2}\right)^{2}+\left(\xi_i-\frac{x_{i+1}+x_i}{2}\right)^{2}\right] \\ &\leq \frac{M-m}{8}\sum_{i=0}^{n-1} h_i^{2}. \end{aligned}$$

Proof. Applying inequality (2.28) on the interval $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ we have

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{1}{2}\left[f(\xi_i) h_i + (\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right] \right| \\ \leq & \frac{M-m}{4}\left[\left(\frac{h_i}{2}\right)^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right] \\ \leq & \frac{M-m}{2} \cdot \left(\frac{h_i}{2}\right)^2 . \end{aligned}$$

Summing over i for $i = 0$ to $n-1$ we may deduce (3.1) and its subsequent representations. ∎

Remark 9. *In practice, the number of function evaluations is minimized. Therefore,* $\sum_{i=0}^{n-1} \xi_i (f(x_i) - f(x_{i+1}))$ *would be written as*

$$\xi_0 f(x_0) - \xi_{n-1} f(x_n) + \sum_{i=1}^{n-1} \left(\xi_i - \xi_{i-1}\right) f(x_i) .$$

Corollary 4. *With the assumptions as in Theorem 10, we have*

$$\int_a^b f(x)\,dx = A_A(f, I_n) + R_A(f, I_n) ,$$

where

$$A_A(f, I_n) = \frac{1}{2}\left[\sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \sum_{i=0}^{n-1} \frac{h_i}{2}\left(f(x_i) + f(x_{i+1})\right)\right]$$

which is the average of a midpoint and trapezoidal quadrature rule and the remainder $R_A(f, I_n)$ *satisfies the following relation*

$$|R_A(f, I_n)| \leq \frac{M-m}{16} \sum_{i=0}^{n-1} h_i^2 .$$

Proof. Similar to Theorem 10 with $\xi_i = \frac{x_{i+1} + x_i}{2}$. ∎

Corollary 5. *Let the conditions be as in Theorem 10 and the partition be equidistant so that* $I_{2k} : x_i = a + hi,\ i = 0, 1, ..., 2k$ *with* $h = \frac{b-a}{2k}$. *Then,*

$$\int_a^b f(x)\,dx = A(f, I_{2k}) + R(f, I_{2k})$$

where

$$A(f, I_{2k}) = \frac{h}{4}\left[f(a) + f(b) + 2\sum_{i=1}^{2k-1} f(x_i)\right]$$

and

$$|R(f, I_{2k})| \leq \frac{M-m}{16k}(b-a)^2 .$$

Proof. From Corollary 4

$$A\left(f, I_{2k}\right)=\frac{h}{2} \sum_{i=0}^{k-1} f\left(\frac{x_{2i}+x_{2(i+1)}}{2}\right)+\frac{h}{4} \sum_{i=0}^{k-1}\left(f\left(x_{2i}\right)+f\left(x_{2(i+1)}\right)\right)$$

where $\frac{x_{2i}+x_{2(i+1)}}{2}=a+h\left(2i+1\right)=x_{2i+1}$.
Now

$$\begin{aligned} \sum_{i=0}^{k-1}\left[f\left(x_{2i}\right)+f\left(x_{2(i+1)}\right)\right] &= f\left(x_0\right)+f\left(x_{2k}\right)+\sum_{i=1}^{k-1} f\left(x_{2i}\right)+\sum_{i=0}^{k-2} f\left(x_{2(i+1)}\right) \\ &= f\left(x_0\right)+f\left(x_{2k}\right)+\sum_{i=1}^{k-1} f\left(x_{2i}\right). \end{aligned}$$

Therefore, on noting that $x_0=a$ and $x_{2k}=b$, $A\left(f, I_{2k}\right)$ is obtained as given in the corollary.
Now, the remainder from Corollary 4 with $h_i=\left(\frac{b-a}{2k}\right)$ for $i=0,1,...,2k-1$

$$\begin{aligned} \left|R\left(f, I_{2k}\right)\right| &\leq \frac{M-m}{8} \sum_{i=0}^{2k-1}\left(\frac{b-a}{2k}\right)^2 \\ &= \frac{M-m}{8} \cdot\left(\frac{b-a}{2k}\right)^2 \cdot 2k \\ &= \frac{M-m}{16} \cdot \frac{(b-a)^2}{k} \end{aligned}$$

and hence the corollary is proved. ∎

Remark 10. *If* $\int_a^b f\left(x\right) dx$ *is to be approximated using the quadrature rule of Corollary 5,* $A\left(f, I_{2k}\right)$ *with an accuracy of* $\varepsilon>0$*, then* $2k_{\varepsilon} \in \mathbb{N}$ *points of the equispaced partition* I_{2k} *is required where*

$$k_{\varepsilon} \geq\left[\frac{M-m}{16} \cdot \frac{(b-a)^2}{\varepsilon}\right]+1,$$

with $[\cdot]$ *denoting the integer part.*

Conclusion 1. *Inequalities have been developed for quadrature rules in which the integrand is bounded from below and above. Previous results have been recaptured as special cases. Results are obtained for a generalized trapezoidal rule. A Lobatto or Ostrowski type rule, involving the end points and an interior point, has also been obtained.*

References

[1] R.P. Agarwal and S.S. Dragomir, *An application of Hayashi's inequality for differentiable functions,* Computers Math. Appl., **32**(6) (1996), 95-99.

[2] D.S. Mitrinović, J.E. Pećarić and A.M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, 1993.

[3] D.S Mitrinović, J.E. Pećarić and A.M. Fink, *Inequalities for Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.

[4] S.S. Dragomir and S. Wang, *Applications of Iyengar's type inequalities to the estimation of error bounds for the trapezoidal quadrature rule,* Tamkang Journal of Mathematics, **29**(1) (1998), 55-58.

[5] S.S. Dragomir, Y.J. Cho and S.S. Kim, *Some remarks on the Milovanović-Pečarić Inequality and in Applications for special means and numerical integration.,* Accepted for publication in Tamkang Journal of Mathematics.

(P. Cerone and S. S. Dragomir), School of Communications and Informatics, Victoria University of Technology, P.O. Box 14428, Melbourne City, MC 8001 Australia
E-mail address: pc@matilda.vu.edu.au

E-mail address: sever@matilda.vu.edu.au