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SOME INEQUALITIES IN 2–INNER PRODUCT SPACES

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Abstract. In this paper we extend some results on the refinement of Cauchy-Buniakowski-Schwarz’s inequality and Ačzel’s inequality in inner product spaces to 2–inner product spaces.

1. Introduction

Let $X$ be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

$(N_1)$ $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent;

$(N_2)$ $\|x, y\| = \|y, x\|$;

$(N_3)$ $\|\alpha x, y\| = |\alpha| \|x, y\|$ for any real number $\alpha$;

$(N_4)$ $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$\|\cdot,\cdot\|$ is called a 2–norm on $X$ and $(X, \|\cdot,\cdot\|)$ a linear 2–normed space cf. [10]. Some of the basic properties of the 2-norms are that they are nonnegative, and $\|x, y + \alpha x\| = \|x, y\|$ for every $x, y$ in $X$ and every real number $\alpha$.

For any non-zero $x_1, x_2, ..., x_n$ in $X$, let $V(x_1, x_2, ..., x_n)$ denote the subspace of $X$ generated by $x_1, x_2, ..., x_n$. Whenever the notation $V(x_1, x_2, ..., x_n)$ is used, we will understand that $x_1, x_2, ..., x_n$ are linearly independent.

A concept which is closely related to linear 2-normed space is that of 2 inner product spaces. For a linear space $X$ of dimension greater than 1 let $(\cdot, \cdot | \cdot)$ be a real-valued function on $X \times X \times X$ which satisfies the following conditions:

$(I_1)$ $(x, x | z) \geq 0; (x, x | z) = 0$ if and only if $x$ and $z$ are linearly dependent;

$(I_2)$ $(x, x | z) = (z, z | x)$;

$(I_3)$ $(x, y | z) = (y, x | z)$;

$(I_4)$ $(\alpha x, y | z) = \alpha (x, y | z)$ for any real number $\alpha$;

$(I_5)$ $(x + x', y | z) = (x, y | z) + (x', y | z)$.

$(\cdot, \cdot | \cdot)$ is called a 2-inner product and $(X, (\cdot, \cdot | \cdot))$ a 2-inner product space ([3]).

These spaces are studied extensively in [1], [2], [4]-[6] and [11]. In [3] it is shown that $\|x, z\| = (x, x | z)^{\frac{1}{2}}$ is a 2–norm on $(X, \|\cdot,\cdot\|)$. Every 2–inner product space will be considered to be a linear 2–normed space with the 2–norm 1991 Mathematics Subject Classification. Primary 26D99; Secondary 46Cxx.

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\[ \|x, z\| = (x, x | z)^{\frac{1}{2}}. \] R. Ehret, [9], has shown that for any 2–inner product space \((X, (\cdot, \cdot | \cdot)), \|x, z\| = (x, x | z)^{\frac{1}{2}}\) defines a 2-norm for which

\begin{equation}
(x, y | z) = \frac{1}{4} \left( \|x + y, z\|^2 - \|x - y, z\|^2 \right),
\end{equation}

\begin{equation}
\|x + y, z\|^2 + \|x - y, z\|^2 = 2 \left( \|x, z\|^2 + \|y, z\|^2 \right).
\end{equation}

Besides, if \((X, \|\cdot, \cdot\|)\) is a linear 2–normed space in which condition (1.2), being a 2-dimensional analogue of the parallelogram law, is satisfied for every \(x, y, z \in X\), then a 2-inner product on \(X\) is defined on by (1.1).

For a 2-inner product space \((X, (\cdot, \cdot | \cdot))\) Cauchy-Schwarz's inequality

\[ |(x, y | z)| \leq (x, x | z)^{\frac{1}{2}} (y, y | z)^{\frac{1}{2}} = \|x, z\| \|y, z\|, \]

a 2–dimensional analogue of Cauchy-Buniakowski-Schwarz's inequality, holds (cf. [3]).

2. Refinements of Cauchy-Schwarz’s Inequality

Throughout this paper, let \((X, (\cdot, \cdot | \cdot))\) denote a 2–inner product space with \(\|x, z\| = (x, x | z)^{\frac{1}{2}}\), \(\mathbb{R}\) the set of real numbers and \(\mathbb{N}\) the set of natural numbers.

**Theorem 2.1.** Let \(x, y, z, u, v \in X\) with \(z \notin V(x, y, u, v)\) be such that

\begin{equation}
\|u, z\|^2 \leq 2 (x, u | z), \quad \|v, z\|^2 \leq 2 (y, v | z).
\end{equation}

Then, we have the inequality

\begin{equation}
2 (x, u | z) - \|u, z\|^2 \leq \left( 2 (x, u | z) - \|u, z\|^2 \right)^{\frac{1}{2}} \left( 2 (y, v | z) - \|v, z\|^2 \right)^{\frac{1}{2}}
\end{equation}

\begin{equation}
+ |(x, y | z) - (x, v | z) - (u, y | z) + (u, v | z)| \leq \|x, z\| \|y, z\|.
\end{equation}

**Proof.** Note that

\begin{equation}
(m^2 - n^2) (p^2 - q^2) \leq (mp - nq)^2
\end{equation}

for every \(m, n, p, q \in \mathbb{R}\). Since

\[ |(x, y | z) - (x, v | z) - (u, y | z) + (u, v | z)|^2 \]

\[ \leq |(x - u, y - v | z)|^2 \leq \|x - u, z\|^2 \|y - v, z\|^2 \]

\[ = \left( \|x, z\|^2 + \|u, z\|^2 - 2 (x, u | z) \right) \left( \|y, z\|^2 + \|v, z\|^2 - 2 (y, v | z) \right), \]

by (2.3), we have

\begin{equation}
|(x, y | z) - (x, v | z) - (u, y | z) + (u, v | z)|^2 \leq \left( \|x, z\| \|y, z\| - \left( 2 (x, u | z) - \|u, z\|^2 \right)^{\frac{1}{2}} \left( 2 (y, v | z) - \|v, z\|^2 \right)^{\frac{1}{2}} \right)^2.
\end{equation}

On the other hand

\[ 0 \leq \left( 2 (x, u | z) - \|u, z\|^2 \right)^{\frac{1}{2}} \leq \|x, z\|, \]

\[ 0 \leq \left( 2 (y, v | z) - \|v, z\|^2 \right)^{\frac{1}{2}} \leq \|y, z\|, \]
which imply
\[
\left(2(x, u | z) - \|u, z\|^{2}\right)^{\frac{1}{2}} \left(2(y, v | z) - \|v, z\|^{2}\right)^{\frac{1}{2}} \leq \|x, z\| \|y, z\|.
\]

Therefore, from (2.4), we have the inequality (2.2). This completes the proof. \( \square \)

**Corollary 2.2.** Let \( x, y, z, e \in X \) be such that \( \|e, z\| = 1 \) and \( z \not\in V(x, y, e) \). Then
\[
(2.5)
\]
\[
|(x, y | z)| \leq |(x, y | z) - (x, e | z)(e, y | z)|
\]
\[
+ |(x, e | z)(e, u | z)| \leq \|x, z\| \|y, z\|.
\]

**Proof.** If we put \( u = (x, e | z)e \) and \( v = (y, e | z)e \), then the conditions (2.1) hold. In fact,
\[
2(x, u | z) - \|u, z\|^{2} = 2(x, (x, e | z)e | z) - \|(x, e | z)e, z\|^{2}
\]
\[
= 2(x, e | z)(x, e | z) - (x, e | z)^{2} = (x, e | z)(x, e | z) \geq 0.
\]
And similarly for the second condition in (2.1).
Moreover,
\[
|(x, y | z) - (x, v | z) - (u, y | z) + (u, v | z)|
\]
\[
= |(x, y | z) - (x, e | z)(y, e | z) - (x, e | z)(e, y | z) + (x, e | z)(y, e | z)|
\]
\[
= |(x, y | z) - (x, e | z)(e, y | z)|
\]
so, by Theorem 2.1, we have (2.5). \( \square \)

**Corollary 2.3.** Let \( x, y, z \in X \) be such that \( \|x, z\|^{2} \leq 2, \|y, z\|^{2} \leq 2 \) and \( z \not\in V(x, y) \). Then
\[
(2.6)
\]
\[
| (x, y | z) |^{2} \left(2 - \|x, z\|^{2}\right)^{\frac{1}{2}} \left(2 - \|y, z\|^{2}\right)^{\frac{1}{2}}
\]
\[
+ | (x, y | z) | \left|1 - \|x, z\|^{2} - \|y, z\|^{2} + (x, y | z)\right| \leq \|x, z\| \|y, z\|.
\]

**Proof.** If we put \( u = (x, y | z)y \) and \( v = (y, x | z)x \), then the inequality (2.3) holds. Moreover, we have
\[
\left(2(x, u | z) - \|u, z\|^{2}\right)^{\frac{1}{2}} \left(2(y, v | z) - \|v, z\|^{2}\right)^{\frac{1}{2}}
\]
\[
= \left|(x, y | z)^{2} \left(2 - \|x, z\|^{2}\right)^{\frac{1}{2}} \left(2 - \|y, z\|^{2}\right)^{\frac{1}{2}},
\]
\[
\left|(x, y | z) - (x, v | z) - (u, y | z) + (u, v | z)\right|
\]
\[
= \left|(x, y | z) \left|1 - \|x, z\|^{2} - \|y, z\|^{2} + |(x, y | z)|\right|\right|.
\]
Therefore, by Theorem 2.1, we have the inequality (2.6). \( \square \)

**Theorem 2.4.** Let \( x, y, z, e \in X \) be such that \( \|e, z\| = 1 \) and \( z \not\in V(x, y, e) \). Then
\[
(2.7)
\]
\[
| (x, y | z) - (x, e | z)(e, y | z) |^{2}
\]
\[
\leq \left(\|x, z\|^{2} - |(x, e | z)|^{2}\right) \left(\|y, z\|^{2} - |(y, e | z)|^{2}\right).
\]

**Proof.** Consider a mapping \( P : X \times X \times X \to \mathbb{R} \) defined by \( P(x, y, z) = (x, y | z) - (x, e | z)(e, y | z) \) for every \( x, y, z, e \in X \), having the properties:
Corollary 2.5. Let

\[(x, y, z) = P(x, y, z) + \beta P(x', y, z),\]

\[P(x, y, z) = P(y, x, z).\]

Then Cauchy-Schwarz's inequality

\[(2.8) \quad |P(x, y, z)|^2 \leq P(x, x, z) P(y, y, z)\]

holds.

Indeed, we observe that

\[0 \leq P(x + \alpha P(x, y, z), y + \alpha P(x, x, z) y, z)\]

\[= P(x, x, z) + 2\alpha P(x, y, z)^2 + \alpha^2 P(x, y, z)^2 P(y, y, z) \quad (\forall) \alpha \in \mathbb{R}.
\]

It is well known that if \(a \geq 0\) and

\[aa^2 + \beta a + c \geq 0 \quad \text{for all } a \in \mathbb{R},\]

then \(\Delta = b^2 - 4ac \leq 0\).

Then by the above inequality we deduce

\[(2.9) \quad P(x, y, z)^d \leq P(x, x, z) P(y, y, z) P(x, y, z)^2.\]

If \(P(x, y, z) = 0\) then (2.8) holds.

If \(P(x, y, z) \neq 0\) then we can divide in (2.9) by \(P(x, y, z)\) and obtain (2.8).

The theorem is thus proved.

Remark 2.1. By the inequalities (2.3) and (2.7), we have

\[|(x, y \mid z) - (x, e \mid z) (e, y \mid z)|^2\]

\[\leq \left(\|x, z\|^2 - |(x, e \mid z)|^2\right) \left(\|y, z\|^2 - |(y, e \mid z)|^2\right)\]

\[\leq \|x, z\| \|y, z\| - |(x, e \mid z) (e, y \mid z)|^2.\]

Since \(\|x, z\| \|y, z\| \geq |(x, e \mid z) (e, y \mid z)|\), we get

\[|(x, y \mid z) - (x, e \mid z) (e, y \mid z)| \leq \|x, z\| \|y, z\| - |(x, e \mid z) (e, y \mid z)|,
\]

which yields the inequality (2.5).

Corollary 2.5. Let \((x, y, z, e) \in X\) be such that \(\|e, z\| = 1\) and \(z \notin V(x, y, e)\). Then

\[(2.10) \quad \left(\|x, y \mid z\|^2 - |(x, e \mid z)|^2\right)^{\frac{1}{2}}\]

\[\leq \left(\|x, z\|^2 - |(x, e \mid z)|^2\right)^{\frac{1}{2}} + \left(\|y, z\|^2 - |(y, e \mid z)|^2\right)^{\frac{1}{2}}.\]

Proof. If we define \(S : X \times X \to \mathbb{R}\) by \(S(x, z) = P(x, x, z)^{\frac{1}{2}}\) for every \(x, y \in X\) and use the triangle inequality for \(S(x, z)\), then we have (2.10).

Corollary 2.6. For every non-zero \(x, y, z, u \in X\), with \(z \notin V(x, y, u)\), we have

\[(2.11) \quad \frac{(x, y \mid z)}{\|x, z\| \|y, z\|} + \frac{(y, u \mid z)}{\|y, z\| \|y, z\|} + \frac{(u, x \mid z)}{\|u, z\| \|x, z\|}\]

\[\leq 1 + 2 \frac{(x, y \mid z) (y, u \mid z) (u, x \mid z)}{\|x, z\|^2 \|y, z\|^2 \|u, z\|^2}.\]
For the proof of next theorem, we need the following lemma:

**Lemma 2.7.** For every non-zero \( x, y, z \in X \) with \( z \notin V(x, y) \), we have

\[
(\|x, z\| + \|y, z\|) \left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|} \right\| \leq 2 \|x - y, z\|. \tag{2.12}
\]

Proof. Since

\[
\frac{\|x, z\|}{\|y, z\|} + \frac{\|y, z\|}{\|x, z\|} \geq 2,
\]

we have the inequality

\[
(\|x, z\| + \|y, z\|)^2 - (x, y | z) \left( \frac{\|x, z\|}{\|y, z\|} + \frac{\|y, z\|}{\|x, z\|} \right) - 2 (x, y | z)
\]

\[
\leq 2 \|x, z\|^2 + \|y, z\|^2 - 4 (x, y | z)
\]

which implies (2.12). \( \blacksquare \)

**Theorem 2.8.** For every non-zero \( x, y, z \in X \) with \( z \notin V(x, y) \) we have

\[
(\|x, z\| + \|y, z\|)^2 \left( \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|} \right)^2 + \left( \frac{x}{\|x, z\|} + \frac{y}{\|y, z\|} \right)^2 \right) \leq 8 \left( \|x, z\|^2 + \|y, z\|^2 \right). \tag{2.13}
\]

Proof. By (2.12) we have

\[
(\|x, z\| + \|y, z\|)^2 \left( \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|} \right)^2 + \left( \frac{x}{\|x, z\|} + \frac{y}{\|y, z\|} \right)^2 \right) \leq 4 \left( \|x - y, z\|^2 + \|x + y, z\|^2 \right)
\]

and, by a 2-dimensional analogue of the parallelogram law, we get (2.13). \( \blacksquare \)

**Remark 2.2.** For some similar results in inner product spaces, see [7].

### 3. Ačzel’s Inequality

In this section, we shall point out some results in 2-inner product spaces in connection to Ačzel’s inequality [12]. For some other similar results in inner products, see [8]. We note that the results obtained here, in 2-inner product spaces used different techniques as those in [8].

**Theorem 3.1.** Let \((X, \langle \cdot, \cdot \rangle)\) be a 2-inner product space, \( M_1, M_2 \in \mathbb{R} \) and \( x, y, z \in X \) such that

\[
\|x, z\| \leq |M_1|, \quad \|y, z\| \leq |M_2|,
\]

then

\[
(M_1^2 - \|x, z\|^2) \left( M_2^2 - \|y, z\|^2 \right) \leq (|M_1 M_2| - (x, y | z))^2. \tag{3.1}
\]

Proof. Using the elementary inequality (2.3), we get
\[ 0 \leq \left( M_1^2 - \|x, z\|^2 \right) \left( M_2^2 - \|y, z\|^2 \right) \leq (\|M_1 M_2\| - \|x, y \| \|y, z\|)^2 , \]
and by Cauchy-Schwarz’s inequality,
\[ 0 \leq \|M_1 M_2\| - \|x, y \| \|y, z\| \leq |M_1 M_2| - (x, y | z) \]
implying (3.1).

Corollary 3.2. If \( x, y, z \in X \), are such that \( \|x, z\|, \|y, z\| \leq M, M > 0 \), then we have the inequality
\[ 0 \leq \|x, z\|^2 \|y, z\|^2 - (x, y | z)^2 \leq M^2 \|x - y, z\|^2 \]
which is a counterpart of Cauchy-Schwarz’s inequality.

Another similar result to the generalization (3.1) of Aézcl’s inequality is the following one

Theorem 3.3. Let \((X, (\cdot, \cdot | \cdot))\) be a 2–inner product space, and \(M_1, M_2 \in \mathbb{R}\) and \(x, y, z \in X\) such that \( \|x, z\| \leq |M_1|, \quad \|y, z\| \leq |M_2| \). Then
\[ (|M_1| - \|x, z\|)^{1/2} (|M_2| - \|y, z\|)^{1/2} \leq |M_1 M_2|^{1/2} - (x, y | z)^{1/2} . \]

Proof. Applying (2.3) for \( m = \sqrt{|M_1|} \), \( p = \sqrt{|M_2|} \), \( n = \sqrt{\|x, z\|} \), \( q = \sqrt{\|y, z\|} \) and using Cauchy-Schwarz’s inequality for 2–inner products we deduce (3.3).

Corollary 3.4. Suppose that \( x, y, z \in X \) and \( M > 0 \) are such that \( \|x, z\|, \|y, z\| \leq M \). Then we have the following converse of Cauchy-Schwarz’s inequality
\[ 0 \leq \|x, z\| \|y, z\| - (x, y | z) \]
\[ \leq M \left( \|x, z\| + \|y, z\| - 2 |(x, y | z)|^{1/2} \right) . \]

Theorem 3.5. Let \((\cdot, \cdot | \cdot)\) be a 2–inner product and \( \{ (\cdot, \cdot | \cdot) \}_{i \in \mathbb{N}}\) a sequence of 2–inner products satisfying
\[ \|x, z\|^2 > \sum_{i=0}^{\infty} \|x, z\|^2_i \]
for all \( x, z \), being linearly independent. Then we have the following refinement of Cauchy-Schwarz’s inequality
\[ \begin{array}{c}
\|x, z\| \|y, z\| - |(x, y | z)| \\
\geq \left[ \sum_{i=0}^{\infty} \|x, z\|_i \sum_{i=0}^{\infty} \|y, z\|_i - |(x, y | z)| \right] \geq 0
\end{array} \]
for all \( x, y, z \in X \).

Proof. Let \( n \in \mathbb{N} \) and \( n \geq 1 \). Define the mapping
\[ (x, y | z)_n = (x, y | z) - \sum_{i=0}^{n} (x, y | z)_i , \quad x, y, z \in X . \]
We observe, by (3.5), that the mapping \((\cdot, \cdot | \cdot)_n\) satisfies the properties
(i) \((x, x | z)_n \geq 0, \)
(ii) \((\alpha x + \beta x', y | z)_n = \alpha (x, y | z)_n + \beta (x', y | z)_n \),

\[ \text{for all } x, y, z, \text{ and } \alpha, \beta \in \mathbb{R} . \]
(iii) \((x, y \mid z)_n = (y, x \mid z)_n\) 

for every \(x, x', y, z \in X\) and \(\alpha, \alpha' \in \mathbb{R}\).

By a similar proof to that in Theorem 2.4, we can state Cauchy-Schwarz’s inequality
\[
(x, x \mid z)_n (y, y \mid z)_n \geq |(x, y \mid z)_n|^2, \quad x, y, z \in X,
\]
that is
\[
(x, x \mid z)_n (y, y \mid z)_n \geq (x, y \mid z)_n - \sum_{i=0}^{n} (x, y \mid z)_i.
\]

(3.8)

Using Aćzel’s inequality \([12]\)
\[
\left(a^2 - m \sum_{i=0}^{m} a_i^2\right) \left(b^2 - m \sum_{i=0}^{m} b_i^2\right) \leq \left(ab - \sum_{i=0}^{m} a_i b_i\right)^2,
\]
where \(a, b, a_i, b_i \in \mathbb{R}\) for \(i = 0, \ldots, m\); we can prove that
\[
(x, x \mid z)_n (y, y \mid z)_n \geq (x, y \mid z)_n - \sum_{i=0}^{m} (x, y \mid z)_i.
\]

(3.9)

for all \(x, y, z \in X\). Since, by Cauchy-Buniakowski-Schwarz’s inequality
\[
\|x, z\| \|y, z\| \geq \left(\sum_{i=0}^{n} \|x, z\|^2 \sum_{i=0}^{n} \|y, z\|^2\right)^{1/2} \geq \sum_{i=0}^{n} \|x, z\|_i \|y, z\|_i,
\]
then by (3.8) and (3.9) we deduce
\[
\|x, z\| \|y, z\| - \sum_{i=0}^{n} \|x, z\|_i \|y, z\|_i
\]
\[
= \|x, z\| \|y, z\| - \sum_{i=0}^{n} \|x, z\|_i \|y, z\|_i \geq |(x, y \mid z)_n - \sum_{i=0}^{n} (x, y \mid z)_i|,
\]
which implies (3.6), by using the inequality
\[
\|x, z\|_i \|y, z\|_i - |(x, y \mid z)_i| \geq 0.
\]

The theorem is thus proved. 

The following corollaries are interesting as refinements of the triangle inequality for 2-norms generated by 2-inner products.

**Corollary 3.6.** With the assumptions from Theorem, we have the following refinement of the triangle inequality
\[
\left(\|x, z\| + \|y, z\|\right)^2 - \|x + y, z\|^2
\]
\[ \geq \sum_{i=0}^{\infty} \left[ (\|x,z\|_1 + \|y,z\|_1)^2 - \|x+y,z\|_1^2 \right] \geq 0, x, y, z \in X. \]

**Corollary 3.7.** Let \((\cdot, \cdot |_1), (\cdot, \cdot |_2)\) be two 2-inner products such that
\[ \|x,z\|_2 > \|x,z\|_1 \]
for all \(x, z\) being linearly independent in \(X\). Then
\[ \|x,z\|_2 \|y,z\|_2 - |(x,y|z)_2| \geq \|x,z\|_1 \|y,z\|_1 - |(x,y|z)_1| \geq 0, x, y, z \in X. \]

**Corollary 3.8.** Let \((\cdot, \cdot |_1), (\cdot, \cdot |_2)\) be as above. Then
\[ \left( \|x,z\|_2 + \|y,z\|_2 \right)^2 - \|x+y,z\|_2^2 \geq \left( \|x,z\|_1 + \|y,z\|_1 \right)^2 - \|x+y,z\|_1^2 \geq 0, x, y, z \in X. \]

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