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*New Inequalities for Convex Functions with  
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Variable*

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**NEW INEQUALITIES FOR CONVEX FUNCTIONS WITH  
APPLICATIONS FOR THE N-ENTROPY OF A DISCRETE  
RANDOM VARIABLE**

S. S. DRAGOMIR

ABSTRACT. New inequalities for convex mappings of a real variable and applications in Information Theory for Shannon's entropy are given.

1. INTRODUCTION

The following converse of Jensen's discrete inequality for convex mappings of a real variable has been proved in 1994 by S.S. Dragomir and N.M. Ionescu [?]:

**Theorem 1.** *Let  $f : I \subseteq R \rightarrow R$  be a convex function on the interval  $I$ ,  $x_i \in \overset{\circ}{I}$  ( $\overset{\circ}{I}$  is the interior of  $I$ ),  $p_i \geq 0$ , ( $i = 1, \dots, n$ ) and  $\sum_{i=1}^n p_i = 1$ . Then we have the inequality*

$$(1.1) \quad \begin{aligned} 0 &\leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\ &\leq \sum_{i=1}^n p_i x_i f'_+(x_i) - \sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i f'_+(x_i) \end{aligned}$$

where  $f'_+$  is the right derivative of  $f$  on  $\overset{\circ}{I}$ .

They also pointed out some natural application of (??) in connection to the arithmetic mean - geometric mean inequality, to the generalized polygonal inequality etc.

A generalization of (??) for differentiable convex mappings of several variables has been obtained in 1996 by S.S. Dragomir and C.J. Goh in [?].

**Theorem 2.** *Let  $f : K \subseteq R^m \rightarrow R$  be a differentiable convex mapping on the convex set  $K$ ,  $\bar{x}_i \in K$  ( $i = 1, \dots, n$ ),  $p_i \geq 0$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ . Then*

$$(1.2) \quad \begin{aligned} 0 &\leq \sum_{i=1}^n p_i f(\bar{x}_i) - f\left(\sum_{i=1}^n p_i \bar{x}_i\right) \\ &\leq \sum_{i=1}^n p_i \langle \nabla f(\bar{x}_i), \bar{x}_i \rangle - \left\langle \sum_{i=1}^n p_i \nabla f(\bar{x}_i), \sum_{i=1}^n p_i \bar{x}_i \right\rangle \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $R^m$  and  $\nabla f(\bar{x}) = \left( \frac{\delta f(\bar{x})}{\delta x_1}, \dots, \frac{\delta f(\bar{x})}{\delta x_m} \right)$ .

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The authors applied the inequality (??) in Information Theory for the entropy mapping, conditional entropy, mutual information etc.

An integral version of (??) has been employed by S.S. Dragomir and C.J. Goh [?] to obtain different bounds for the entropy, conditional entropy and mutual information of continuous random variable. Also, some applications in torsion theory of a similar result which counterparts Hadamard's inequality, has been given by S.S. Dragomir and G. Keady in 1996 [?].

For recent generalizations, both for the discrete case and continuous case as well as extensions for mappings defined on normed linear spaces, we recommend M. Matic's Ph.D. Dissertation [?], where further applications in Information Theory have been given.

In paper [?], the authors obtained the following upper bound for the first member in (??),

**Theorem 3.** *Let  $f : K \subseteq R^m \rightarrow R$  be a differentiable convex mapping on the convex set  $K$ ,  $\bar{x}_i \in K$  ( $i = 1, \dots, n$ ),  $p_i \geq 0$ , ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ . Then*

$$\begin{aligned}
 (1.3) \\
 0 &\leq \sum_{i=1}^n p_i f(\bar{x}_i) - f\left(\sum_{i=1}^n p_i \bar{x}_i\right) \\
 &\leq \begin{cases} \max_{1 \leq i < j \leq n} \|\bar{x}_i - \bar{x}_j\| \sum_{1 \leq i < j \leq n} p_i p_j \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\|; \\ \left( \sum_{1 \leq i < j \leq n} p_i p_j \|\bar{x}_i - \bar{x}_j\|^p \right)^{1/p} \left( \sum_{1 \leq i < j \leq n} p_i p_j \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\|^q \right)^{1/q}, \\ \max_{1 \leq i < j \leq n} \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\| \sum_{1 \leq i < j \leq n} p_i p_j \|\bar{x}_i - \bar{x}_j\| \end{cases} \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\
 &\leq \frac{1}{2} \max_{1 \leq i < j \leq n} \|\bar{x}_i - \bar{x}_j\| \max_{1 \leq i < j \leq n} \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\|
 \end{aligned}$$

where  $\|\cdot\|$  is the usual Euclidean norm on  $R^m$ .

and applied it for the entropy mapping in Information Theory.

In [?], the authors pointed out other converse inequalities which are more useful in applications.

**Theorem 4.** Let  $f : K \subseteq R^m \rightarrow R$  be a differentiable convex mapping on the convex set  $K$ ,  $\bar{x}_i \in K$  ( $i = 1, \dots, n$ ),  $p_i \geq 0$ , ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ . Then

$$(1.4) \quad \begin{aligned} 0 &\leq \sum_{i=1}^n p_i f(\bar{x}_i) - f\left(\sum_{i=1}^n p_i \bar{x}_i\right) \\ &\leq \begin{cases} \max_{i=1, \dots, n} \left\| \bar{x}_i - \sum_{j=1}^n p_j \bar{x}_j \right\| \left\| \sum_{i=1}^n p_i \left\| \nabla f(\bar{x}_i) - \sum_{j=1}^n p_j \nabla f(\bar{x}_j) \right\| \right\|; \\ \left( \sum_{i=1}^n p_i \left\| \bar{x}_i - \sum_{j=1}^n p_j \bar{x}_j \right\|^p \right)^{1/p} \left( \sum_{i=1}^n p_i \left\| \nabla f(\bar{x}_i) - \sum_{j=1}^n p_j \nabla f(\bar{x}_j) \right\|^q \right)^{1/q}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{i=1, \dots, n} \left\| \nabla f(\bar{x}_i) - \sum_{j=1}^n p_j \nabla f(\bar{x}_j) \right\| \left\| \sum_{i=1}^n p_i \left\| \bar{x}_i - \sum_{j=1}^n p_j \bar{x}_j \right\| \right\| \end{cases}, \\ &\leq \max_{i=1, \dots, n} \left\| \bar{x}_i - \sum_{j=1}^n p_j \nabla f(\bar{x}_j) \right\| \max_{i=1, \dots, n} \left\| \sum_{i=1}^n \nabla f(\bar{x}_i) - \sum_{j=1}^n p_j \nabla f(\bar{x}_j) \right\|. \end{aligned}$$

In this paper, we point out other general inequalities for convex mappings of a real variable and apply them in Information Theory for the entropy mapping.

## 2. SOME RESULTS FOR CONVEX FUNCTIONS OF A REAL VARIABLE

We shall define the following sequence of functionals:

$$\mathfrak{J}_n : Conv(I) \times S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\mathring{I}) \longrightarrow \mathbb{R}$$

given by:

$$\mathfrak{J}_n(f, p, H, x) := \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f\left(\frac{x_{i_1} + \dots + x_{i_n}}{n}\right),$$

where

$$Conv(I) := \{f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is convex on the interval } I\};$$

$$S_+^*(\mathbb{R}) := \{p = (p_i)_{i \in \mathbb{N}} \mid p_i > 0 \text{ for all } i \in \mathbb{N}\};$$

$$\mathfrak{P}_f^*(\mathbb{N}) := \{H \subset \mathbb{N} \mid H \text{ is finite and } H \neq \emptyset\};$$

$$S(\mathring{I}) := \left\{x = (x_i)_{i \in \mathbb{N}} \mid x_i \in \mathring{I} \text{ for all } i \in \mathbb{N}, \mathring{I} \text{ is the interior of } I\right\}.$$

The following estimation holds:

**Theorem 5.** Let  $f'_+ : \mathring{I} \rightarrow \mathbb{R}$  be the right derivative of the convex function  $f$ . Define the sequence of mappings:

$$\begin{aligned} E_n(f, p, H, x) &:= \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) x_{i_1} \\ &\quad - \frac{1}{P_H} \sum_{i \in H} p_i x_i \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \end{aligned}$$

for  $(f, p, H, x) \in \text{Conv}(I) \times S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\mathring{I})$ . Then we have the inequalities:

$$(2.1) \quad 0 \leq \mathfrak{J}_n(f, p, H, x) - \mathfrak{J}_{n+1}(f, p, H, x) \leq \frac{1}{n+1} E_n(f, p, H, x)$$

and

$$(2.2) \quad 0 \leq \mathfrak{J}_n(f, p, H, x) - f\left(\frac{1}{P_H} \sum_{i \in H} p_i x_i\right) \leq E_n(f, p, H, x)$$

for all  $n \geq 1$  and  $(f, p, H, x) \in \text{Conv}(I) \times S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\mathring{I})$ .

*Proof.* By the convexity of  $f$  on  $I$ , we have the inequality:

$$(2.3) \quad f'_+(y)(x-y) \leq f(x) - f(y) \leq f'_+(x)(x-y)$$

for all  $x, y \in \mathring{I}$ .

If we choose in (??)

$$x = \frac{x_{i_1} + \dots + x_{i_n}}{n} \in \mathring{I} \text{ and } y = \frac{x_{i_1} + \dots + x_{i_{n+1}}}{n+1} \in \mathring{I},$$

where  $i_1, \dots, i_{n+1} \in H$ , we deduce:

$$(2.4) \quad \begin{aligned} & f'_+\left(\frac{x_{i_1} + \dots + x_{i_{n+1}}}{n+1}\right) \left[\frac{x_{i_1} + \dots + x_{i_n} - nx_{i_{n+1}}}{n(n+1)}\right] \\ & \leq f\left(\frac{x_{i_1} + \dots + x_{i_n}}{n}\right) - f\left(\frac{x_{i_1} + \dots + x_{i_{n+1}}}{n+1}\right) \\ & \leq f'_+\left(\frac{x_{i_1} + \dots + x_{i_n}}{n}\right) \left[\frac{x_{i_1} + \dots + x_{i_n} - nx_{i_{n+1}}}{n(n+1)}\right] \end{aligned}$$

for all  $i_1, \dots, i_{n+1} \in H$ .

If we multiply (??) by  $p_{i_1} \dots p_{i_{n+1}} \geq 0$  and summing over  $i_1, \dots, i_{n+1} \in H$  we deduce:

$$(2.5) \quad \begin{aligned} A & : = \frac{1}{P_H^{n+1}} \sum_{i_1, \dots, i_{n+1} \in H} p_{i_1} \dots p_{i_{n+1}} f'_+\left(\frac{x_{i_1} + \dots + x_{i_{n+1}}}{n+1}\right) \\ & \quad \times \left[\frac{x_{i_1} + \dots + x_{i_n} - nx_{i_{n+1}}}{n(n+1)}\right] \\ & \leq \mathfrak{J}_n(f, p, H, x) - \mathfrak{J}_{n+1}(f, p, H, x) \\ & \leq \frac{1}{P_H^{n+1}} \sum_{i_1, \dots, i_{n+1} \in H} p_{i_1} \dots p_{i_{n+1}} f'_+\left(\frac{x_{i_1} + \dots + x_{i_n}}{n}\right) \\ & \quad \times \left[\frac{x_{i_1} + \dots + x_{i_n} - nx_{i_{n+1}}}{n(n+1)}\right] \\ & : = B. \end{aligned}$$

As, it is clear that

$$\begin{aligned} & \frac{1}{P_H^{n+1}} \sum_{i_1, \dots, i_{n+1} \in H} p_{i_1} \dots p_{i_{n+1}} f'_+\left(\frac{x_{i_1} + \dots + x_{i_{n+1}}}{n+1}\right) x_{i_k} \\ & = \frac{1}{P_H^{n+1}} \sum_{i_1, \dots, i_{n+1} \in H} p_{i_1} \dots p_{i_{n+1}} f'_+\left(\frac{x_{i_1} + \dots + x_{i_{n+1}}}{n+1}\right) x_{i_j} \end{aligned}$$

for all  $k, j \in \{1, \dots, n+1\}$ , we deduce that  $A = 0$ .

On the other hand, since

$$\begin{aligned} & \frac{1}{P_H^{n+1}} \sum_{i_1, \dots, i_{n+1} \in H} p_{i_1} \dots p_{i_{n+1}} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) (x_{i_k} - x_{i_{n+1}}) \\ &= \frac{1}{P_H^{n+1}} \sum_{i_1, \dots, i_{n+1} \in H} p_{i_1} \dots p_{i_{n+1}} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) (x_{i_1} - x_{i_{n+1}}) \end{aligned}$$

for all  $k \in \{1, \dots, n\}$ , we get that

$$\begin{aligned} B &= \frac{1}{n+1} \cdot \frac{1}{P_H^{n+1}} \sum_{i_1, \dots, i_{n+1} \in H} p_{i_1} \dots p_{i_{n+1}} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) (x_{i_1} - x_{i_{n+1}}) \\ &= \frac{1}{n+1} \left[ \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) x_{i_1} \right. \\ &\quad \left. - \frac{1}{P_H} \sum_{i \in H} p_i x_i \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \right] \end{aligned}$$

and by (??), the inequality (??) is proved.

Now, if we choose in (??):

$$x = \frac{x_{i_1} + \dots + x_{i_n}}{n} \in \mathring{I} \text{ and } y = \frac{1}{P_H} \sum_{i \in H} p_i x_i \in \mathring{I},$$

where  $i_1, \dots, i_n \in H$ , we deduce:

$$\begin{aligned} (2.6) \quad & f'_+ \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \left[ \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right] \\ &\leq f \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) - f \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \\ &\leq f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \left[ \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right] \end{aligned}$$

for all  $i_1, \dots, i_n \in H$ .

If we multiply the inequality (??) by  $p_{i_1} \dots p_{i_n} \geq 0$  and summing over  $i_1, \dots, i_n \in H$

we deduce:

$$\begin{aligned}
(2.7) \quad C & : = \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \\
& \quad \times \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \\
& \leq \mathfrak{J}_n(f, p, H, x) - f \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \\
& \leq \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \\
& \quad \times \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \\
& : = D.
\end{aligned}$$

As it is clear that:

$$\frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} x_{i_k} = \frac{1}{P_H} \sum_{i \in H} p_i x_i$$

for all  $k \in \{1, \dots, n\}$ , then  $C = 0$ .

In addition, because

$$\begin{aligned}
& \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) x_{i_k} \\
& = \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) x_{i_1}
\end{aligned}$$

for all  $k \in \{1, \dots, n\}$ , then

$$\begin{aligned}
D & = \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) x_{i_1} \\
& \quad - \frac{1}{P_H} \sum_{i \in H} p_i x_i \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right).
\end{aligned}$$

Consequently, by (??) we deduce the inequality (??) and the theorem is thus proved. ■

**Remark 1.** From the above theorem we have the inequalities:

$$\begin{aligned}
(2.8) \quad f \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) & \leq \dots \leq \mathfrak{J}_{n+1}(f, p, H, x) \leq \mathfrak{J}_n(f, p, H, x) \\
& \leq \dots \leq \mathfrak{J}_1(f, p, H, x) = \frac{1}{P_H} \sum_{i \in H} p_i f(x_i)
\end{aligned}$$

which have been established for the first time by S. S. Dragomir and J. E. Pečarić in 1987 [?] for the general case of convex mappings  $f : C \subseteq X \rightarrow \mathbb{R}$  defined on convex subsets from linear spaces. Note that, in the paper [?] the authors used

Jensen's discrete inequality for multiple sums, i.e., a different argument that we used before.

In what follows we shall estimate the sequence of functionals  $E_n$ .

**Theorem 6.** *With the assumptions of Theorem ??, and the conditions that  $s > 1$  and  $\frac{1}{s} + \frac{1}{p} = 1$ , we have the inequality:*

$$(2.9) \quad 0 \leq E_n(f, p, H, x) \leq \left( \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left| f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \right|^s \right)^{\frac{1}{s}} \times \left( \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left| \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right|^q \right)^{\frac{1}{q}}$$

for all  $(f, p, H, x) \in \text{Conv}(I) \times S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\hat{I})$  and  $n \geq 1$ .

*Proof.* Using an identity from Theorem ?? we have that

$$0 \leq E_n(f, p, H, x) = |E_n(f, p, H, x)| \leq \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left| f'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \right| \left| \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right|.$$

Applying Hölder's inequality for multiple sums, i.e.,

$$\left| \sum_{i_1, \dots, i_n \in H} p_{i_1, \dots, i_n} a_{i_1, \dots, i_n} b_{i_1, \dots, i_n} \right| \leq \left( \sum_{i_1, \dots, i_n \in H} p_{i_1, \dots, i_n} |a_{i_1, \dots, i_n}|^s \right)^{\frac{1}{s}} \times \left( \sum_{i_1, \dots, i_n \in H} p_{i_1, \dots, i_n} |b_{i_1, \dots, i_n}|^q \right)^{\frac{1}{q}},$$

where  $\frac{1}{s} + \frac{1}{q} = 1$ ,  $s > 1$ ,  $a_{i_1, \dots, i_n}, b_{i_1, \dots, i_n} \in \mathbb{R}$ ,  $p_{i_1}, \dots, p_{i_n} > 0$  for all  $i_1, \dots, i_n \in H$ , we deduce the desired inequality (??). ■

The above theorem admits the following corollary which is more useful in applications.

**Corollary 1.** *With the above assumptions and if the condition*

$$M := \sup_{x \in \hat{I}} |f'_+(x)| < \infty,$$

holds, then we have the estimate:

$$(2.10) \quad 0 \leq E_n(f, p, H, x) \leq \frac{\sqrt{2}M}{2\sqrt{n}P_H} \left[ \sum_{i, j \in H} p_i p_j (x_i - x_j)^2 \right]^{\frac{1}{2}}$$

for all  $n \geq 1$  and  $(p, H, x) \in S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\hat{I})$ .



*Proof.* For  $s = p = 2$  and by the boundedness of  $f'_+$  on  $\mathring{I}$  we get:

$$E_n(f, p, H, x) \leq M \left( \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left| \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right|^2 \right)^{\frac{1}{2}}.$$

Let us compute:

$$\begin{aligned} F & : = \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left| \frac{x_{i_1} + \dots + x_{i_n}}{n} - \frac{1}{P_H} \sum_{i \in H} p_i x_i \right|^2 \\ & = \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \\ & \quad \times \left| \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right)^2 - 2 \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \right. \\ & \quad \left. + \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right)^2 \right|. \end{aligned}$$

We also have:

$$(x_{i_1} + \dots + x_{i_n})^2 = \sum_{j=1}^n x_{i_j}^2 + 2 \sum_{1 \leq j < l \leq n} x_{i_j} x_{i_l},$$

then

$$\begin{aligned} & \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right)^2 \\ & = \frac{1}{n^2} \cdot \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left[ \sum_{j=1}^n x_{i_j}^2 + 2 \sum_{1 \leq j < l \leq n} x_{i_j} x_{i_l} \right] \\ & = \frac{1}{n^2} \left[ n \cdot \frac{1}{P_H} \sum_{i \in H} p_i x_i^2 + 2 \cdot \frac{n(n-1)}{2} \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right)^2 \right] \\ & = \frac{1}{n} \left[ \frac{1}{P_H} \sum_{i \in H} p_i x_i^2 + (n-1) \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right)^2 \right]. \end{aligned}$$

On the other hand we have:

$$\frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) = \frac{1}{P_H} \sum_{i \in H} p_i x_i.$$

Thus,

$$\begin{aligned}
F &= \frac{1}{n} \left[ \frac{1}{P_H} \sum_{i \in H} p_i x_i^2 + (n-1) \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right)^2 \right] - 2 \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right)^2 \\
&\quad + \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right)^2 \\
&= \frac{1}{n} \left[ \frac{1}{P_H} \sum_{i \in H} p_i x_i^2 - \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right)^2 \right] \\
&= \frac{1}{nP_H^2} \left[ \sum_{i \in H} p_i \sum_{i \in H} p_i x_i^2 - \left( \sum_{i \in H} p_i x_i \right)^2 \right] \\
&= \frac{1}{2nP_H^2} \sum_{i,j \in H} p_i p_j (x_i - x_j)^2
\end{aligned}$$

and the corollary is proved. ■

We have the following corollary of Theorem ??:

**Corollary 2.** *With the assumptions of Corollary ?? we have the estimate:*

$$\begin{aligned}
(2.11) \quad 0 &\leq \mathfrak{J}_n(f, p, H, x) - \mathfrak{J}_{n+1}(f, p, H, x) \\
&\leq \frac{\sqrt{2}M}{2(n+1)\sqrt{n}} \cdot \frac{1}{P_H} \left[ \sum_{i,j \in H} p_i p_j (x_i - x_j)^2 \right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
(2.12) \quad 0 &\leq \mathfrak{J}_n(f, p, H, x) - f \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \\
&\leq \frac{\sqrt{2}M}{2\sqrt{n}P_H} \left[ \sum_{i,j \in H} p_i p_j (x_i - x_j)^2 \right]^{\frac{1}{2}}
\end{aligned}$$

for all  $(p, H, x) \in S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\mathring{I})$ .

We can conclude with the following theorems of convergence:

**Theorem 7.** (Convergence) *If  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is a convex function having a bounded derivative on  $\mathring{I}$ , then for all  $(p, H, x) \in S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\mathring{I})$  we have:*

$$\lim_{n \rightarrow \infty} n^\alpha [\mathfrak{J}_n(f, p, H, x) - \mathfrak{J}_{n+1}(f, p, H, x)] = 0 \text{ for } \alpha \in \left[ 0, \frac{3}{2} \right)$$

and

$$\lim_{n \rightarrow \infty} n^\beta \left[ \mathfrak{J}_n(f, p, H, x) - f \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \right] = 0 \text{ for } \beta \in \left[ 0, \frac{1}{2} \right)$$

The proof is obvious by the inequalities (??) and (??).  
Finally, we have the following.

**Theorem 8.** (Uniform Convergence) *If  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is a convex function having a bounded derivative on  $\overset{\circ}{I}$ , then*

$$\lim_{n \rightarrow \infty} n^\alpha [\mathfrak{J}_n(f, p, H, x) - \mathfrak{J}_{n+1}(f, p, H, x)] = 0 \text{ for } \alpha \in \left[0, \frac{3}{2}\right)$$

and

$$\lim_{n \rightarrow \infty} n^\beta \left[ \mathfrak{J}_n(f, p, H, x) - f \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \right] = 0 \text{ for } \beta \in \left[0, \frac{1}{2}\right)$$

uniformly on  $S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S([a, b])$  where  $[a, b] \subset \overset{\circ}{I}$ .

*Proof.* Let  $(p, H, x) \in S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S([a, b])$ . Then  $|x_i - x_j| \leq b - a$  ( $i, j \in H$ ) and, by (??) and (??) we get that:

$$0 \leq \mathfrak{J}_n(f, p, H, x) - \mathfrak{J}_{n+1}(f, p, H, x) \leq \frac{\sqrt{2}M(b-a)}{2(n+1)\sqrt{n}}$$

and

$$0 \leq \mathfrak{J}_n(f, p, H, x) - f \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \leq \frac{\sqrt{2}M(b-a)}{2\sqrt{n}},$$

which proves the theorem. ■

**Remark 2.** *Similar results can be stated for differentiable mappings of several variables by using the inequality*

$$\langle \nabla f(y), x - y \rangle \leq f(x) - f(y) \leq \langle \nabla f(x), x - y \rangle$$

for any  $x, y \in \mathbb{R}^m$ . We omit the details.

### 3. APPLICATIONS FOR THE MAPPING $\log_b$

If we assume that  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is a concave mapping on  $I$  and  $(p, H, x) \in S_+^*(\mathbb{R}) \times \mathfrak{P}_f^*(\mathbb{N}) \times S(\overset{\circ}{I})$  we have that:

$$(3.1) \quad \begin{aligned} \mathfrak{J}_1(g, p, H, x) &\leq \dots \leq \mathfrak{J}_n(g, p, H, x) \\ &\leq \mathfrak{J}_{n+1}(g, p, H, x) \leq \dots \leq g \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) \end{aligned}$$

and

$$(3.2) \quad 0 \leq \mathfrak{J}_{n+1}(g, p, H, x) - \mathfrak{J}_n(g, p, H, x) \leq \frac{1}{n+1} L_n(g, p, H, x)$$

and

$$(3.3) \quad 0 \leq g \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) - \mathfrak{J}_n(g, p, H, x) \leq L_n(g, p, H, x)$$

respectively, where:

$$\begin{aligned} L_n(g, p, H, x) &: = \frac{1}{P_H} \sum_{i \in H} p_i x_i \cdot \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} g'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \\ &\quad - \frac{1}{P_H^n} \sum_{i_1, \dots, i_n \in H} p_{i_1} \dots p_{i_n} g'_+ \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) x_{i_1}. \end{aligned}$$

If  $M := \sup_{x \in \mathring{I}} |g'_+(x)| < \infty$ , then we have:

$$(3.4) \quad 0 \leq \mathfrak{J}_{n+1}(g, p, H, x) - \mathfrak{J}_n(g, p, H, x) \leq \frac{\sqrt{2}M}{2(n+1)\sqrt{n}} \cdot \frac{1}{P_H} \left[ \sum_{i,j \in H} p_i p_j (x_i - x_j)^2 \right]^{\frac{1}{2}}$$

and

$$(3.5) \quad 0 \leq g \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) - \mathfrak{J}_n(g, p, H, x) \leq \frac{\sqrt{2}M}{2\sqrt{n}} \cdot \frac{1}{P_H} \left[ \sum_{i,j \in H} p_i p_j (x_i - x_j)^2 \right]^{\frac{1}{2}}.$$

If  $x_i \in [a, b] \subset \mathring{I}$ ,  $i \in H$ , we have that:

$$(3.6) \quad 0 \leq \mathfrak{J}_{n+1}(g, p, H, x) - \mathfrak{J}_n(g, p, H, x) \leq \frac{\sqrt{2}M(b-a)}{2(n+1)\sqrt{n}}$$

and

$$(3.7) \quad 0 \leq g \left( \frac{1}{P_H} \sum_{i \in H} p_i x_i \right) - \mathfrak{J}_n(g, p, H, x) \leq \frac{\sqrt{2}M(b-a)}{2\sqrt{n}}.$$

Now, for the concave mapping  $g : (0, \infty) \rightarrow \mathbb{R}$ ,  $g(x) = \log_b x$ ,  $b > 1$ , we define the sequence of mappings:

$$H_n(p, x, b) := \frac{1}{P_s^n} \sum_{i_1, \dots, i_n=1}^s p_{i_1} \dots p_{i_n} \log_b \left( \frac{x_{i_1} + \dots + x_{i_n}}{n} \right), \quad n \geq 1$$

where  $s \geq 1$  is given in  $\mathbb{N}$  and  $p_i, x_i > 0$  for all  $i \in \mathbb{N}$ ,  $i \geq 1$  and  $P_s := \sum_{i=1}^s p_i$ .

Note that for this case we have:

$$\begin{aligned} 0 &\leq L_n(p, x, b) \\ &= \frac{n}{\ln b} \left[ \frac{1}{P_s} \sum_{i=1}^s p_i x_i \frac{1}{P_s^n} \sum_{i_1, \dots, i_n=1}^s \frac{p_{i_1} \dots p_{i_n}}{x_{i_1} + \dots + x_{i_n}} \right. \\ &\quad \left. - \frac{1}{P_s^n} \sum_{i_1, \dots, i_n=1}^s \frac{p_{i_1} \dots p_{i_n}}{x_{i_1} + \dots + x_{i_n}} x_{i_1} \right] \\ &= \frac{n}{P_s^{n+1} \ln b} \sum_{i_1, \dots, i_{n+1}=1}^s \frac{p_{i_1} \dots p_{i_{n+1}} (x_{i_{n+1}} - x_{i_1})}{x_{i_1} + \dots + x_{i_n}} \\ &= \frac{n}{P_s^n \ln b} \sum_{i_1, \dots, i_n=1}^s \frac{p_{i_1} \dots p_{i_n}}{x_{i_1} + \dots + x_{i_n}} \left( \frac{1}{P_s} \sum_{i=1}^s p_i x_i - \frac{x_{i_1} + \dots + x_{i_n}}{n} \right) \\ &= \frac{n}{P_s^n \ln b} \sum_{i_1, \dots, i_n=1}^s \frac{p_{i_1} \dots p_{i_n}}{x_{i_1} + \dots + x_{i_n}} \left( \frac{1}{P_s} \sum_{i=1}^s p_i x_i - x_{i_1} \right). \end{aligned}$$

For  $n = 1$  we have

$$\begin{aligned} L_1(p, x, b) &= \frac{1}{\ln b P_s} \sum_{j=1}^s \frac{p_j}{x_j} \left( \frac{1}{P_s} \sum_{i=1}^s p_i x_i - x_j \right) \\ &= \frac{1}{\ln b} \left[ \frac{1}{P_s} \sum_{i=1}^s p_i x_i \cdot \frac{1}{P_s} \sum_{i=1}^s \frac{p_i}{x_i} - 1 \right] \end{aligned}$$

On the other hand we have:

$$\frac{1}{2P_s^2} \sum_{i,j=1}^s \frac{p_i p_j}{x_i x_j} (x_i - x_j)^2 = \frac{1}{P_s^2} \sum_{i=1}^s p_i x_i \sum_{i=1}^s \frac{p_i}{x_i} - 1$$

and then:

$$L_1(p, x, b) = \frac{1}{2 \ln b} \sum_{i,j=1}^s \frac{p_i p_j}{x_i x_j} (x_i - x_j)^2.$$

If  $x_i \geq m > 0$ ,  $i \in \{1, \dots, s\}$ , then we have:

$$g'(x) = \frac{1}{\ln b} \cdot \frac{1}{x} \leq \frac{1}{\ln b} \cdot \frac{1}{m}, \quad x \in [m, \infty).$$

By the inequality (??) we have:

(3.8)

$$0 \leq H_{n+1}(p, x, b) - H_n(p, x, b) \leq \frac{1}{m(n+1)\sqrt{n}} \cdot \frac{1}{P_s} \left[ \sum_{1 \leq i < j \leq s} p_i p_j (x_i - x_j)^2 \right]^{\frac{1}{2}},$$

and by (??) we derive

(3.9)

$$0 \leq \log_b \left( \frac{1}{P_s} \sum_{i=1}^s p_i x_i \right) - H_n(p, x, b) \leq \frac{1}{m\sqrt{n}} \cdot \frac{1}{P_s} \left[ \sum_{1 \leq i < j \leq s} p_i p_j (x_i - x_j)^2 \right]^{\frac{1}{2}}$$

for all  $p \in S_+^*(\mathbb{R})$ ,  $b > 1$  and  $s \geq 1$ .

Finally, if  $x_i \in [m, M] \subset (0, \infty)$ , then we have:

$$(3.10) \quad 0 \leq H_{n+1}(p, x, b) - H_n(p, x, b) \leq \frac{\sqrt{2}(M-m)}{2m(n+1)\sqrt{n}}$$

and

$$(3.11) \quad 0 \leq \log_b \left( \frac{1}{P_s} \sum_{i=1}^s p_i x_i \right) - H_n(p, x, b) \leq \frac{\sqrt{2}(M-m)}{2m\sqrt{n}}$$

for all  $s \geq 1$ , for all  $x_i \in [m, M] \subset (0, \infty)$ ,  $i \in \{1, \dots, s\}$ ,  $b > 1$ ,  $n \geq 1$  and  $p_i > 0$  for all  $i \in \mathbb{N}$ .

These two inequalities enable us to state the following results of convergence:

$$\lim_{n \rightarrow \infty} n^\alpha [H_{n+1}(p, x, b) - H_n(p, x, b)] = 0 \text{ for } \alpha \in \left[0, \frac{3}{2}\right)$$

and

$$\lim_{n \rightarrow \infty} n^\beta \left[ \log_b \left( \frac{1}{P_s} \sum_{i=1}^s p_i x_i \right) - H_n(p, x, b) \right] = 0 \text{ for } \beta \in \left[0, \frac{1}{2}\right)$$

uniformly for  $(p, s, x, b) \in S_+^*(\mathbb{R}) \times \mathbb{M}^* \times S[m, M] \times (1, \infty)$ .

#### 4. SOME ESTIMATION FOR THE $n - b$ -ENTROPY MAPPING

Suppose that  $X$  is a discrete random variable whose range is finite  $R = \{x_1, \dots, x_s\}$ . Let  $p_i = P\{X = x_i\}$ .

We can introduce the following concept.

**Definition 1.** *The  $n - b$ -entropy mapping associated to the random variable  $X$  can be defined by:*

$$(4.1) \quad H_n(X; b) := \sum_{i_1, \dots, i_n=1}^s p_{i_1} \dots p_{i_n} \log_b \left( \frac{\frac{1}{p_{i_1}} + \dots + \frac{1}{p_{i_n}}}{n} \right)$$

where  $n \geq 1$ .

Note that for  $n = 1$  we recapture the usual  $b$ -entropy, i.e.,

$$H_n(X; b) = H(X, b) = \sum_{i=1}^s p_i \log_b \frac{1}{p_i}.$$

Using the results from section 3, we have

$$(4.2) \quad \begin{aligned} 0 &\leq H_1(X; b) \leq H_2(X; b) \leq \dots \leq H_n(X; b) \\ &\leq H_{n+1}(X; b) \leq \dots \leq \log_b s \quad \text{for all } n \geq 1; \end{aligned}$$

and

$$(4.3) \quad 0 \leq \log_b r - H_n(X; b) \leq h_n(X; b) \quad \text{for all } n \geq 1$$

and

$$(4.4) \quad 0 \leq H_{n+1}(X; b) - H_n(X; b) \leq \frac{1}{n+1} h_n(X; b) \quad \text{for all } n \geq 1$$

where:

$$\begin{aligned} h_n(X; b) &= \frac{n}{\ln b} \sum_{i_1, \dots, i_n=1}^s \frac{p_{i_1} \dots p_{i_n}}{\frac{1}{p_{i_1}} + \dots + \frac{1}{p_{i_n}}} \left( \sum_{i=1}^s p_i \cdot \frac{1}{p_i} - \frac{\frac{1}{p_{i_1}} + \dots + \frac{1}{p_{i_n}}}{n} \right) \\ &= \frac{n}{\ln b} \sum_{i_1, \dots, i_n=1}^s \frac{p_{i_1} \dots p_{i_n}}{\frac{1}{p_{i_1}} + \dots + \frac{1}{p_{i_n}}} \left( s - \frac{\frac{1}{p_{i_1}} + \dots + \frac{1}{p_{i_n}}}{n} \right) \\ &= \frac{n}{\ln b} \left[ s \sum_{i_1, \dots, i_n=1}^s \frac{p_{i_1} \dots p_{i_n}}{\frac{1}{p_{i_1}} + \dots + \frac{1}{p_{i_n}}} - \frac{1}{n} \right]. \end{aligned}$$

For  $n = 1$ , we have:

$$\begin{aligned} h_1(X; b) &= \frac{1}{2 \ln b} \sum_{i, j=1}^s \frac{p_i p_j}{\frac{1}{p_i} \cdot \frac{1}{p_j}} \left( \frac{1}{p_i} - \frac{1}{p_j} \right)^2 \\ &= \frac{1}{2 \ln b} \sum_{i, j=1}^s (p_i - p_j)^2 \\ &= \frac{1}{\ln b} \sum_{1 \leq i < j \leq s} (p_i - p_j)^2. \end{aligned}$$

Let us denote  $p_M := \max_{i=\overline{1,s}} \{p_i\}$ ,  $p_m := \min_{i=\overline{1,s}} \{p_i\}$ . Then  $\frac{1}{p_i} \in \left[ \frac{1}{p_M}, \frac{1}{p_m} \right]$ , and by (??) and (??) we have that:

$$(4.5) \quad \begin{aligned} 0 &\leq H_{n+1}(X; b) - H_n(X; b) \\ &\leq \frac{p_M}{(n+1)\sqrt{n}} \left[ \sum_{1 \leq i < j \leq s} \frac{(p_i - p_j)^2}{p_i p_j} \right]^{\frac{1}{2}} \\ &\leq \frac{1}{(n+1)\sqrt{n}} \left[ \sum_{1 \leq i < j \leq s} \frac{(p_i - p_j)^2}{p_i p_j} \right]^{\frac{1}{2}}, \quad n \geq 1 \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} 0 &\leq \log_b s - H_n(X; b) \\ &\leq \frac{p_M}{\sqrt{n}} \left[ \sum_{1 \leq i < j \leq s} \frac{(p_i - p_j)^2}{p_i p_j} \right]^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{n}} \left[ \sum_{1 \leq i < j \leq s} \frac{(p_i - p_j)^2}{p_i p_j} \right]^{\frac{1}{2}}. \end{aligned}$$

Now, if we use the inequalities (??) and (??) we have that:

$$(4.7) \quad 0 \leq H_{n+1}(X; b) - H_n(X; b) \leq \frac{\sqrt{2}(p_M - p_m)}{2p_m(n+1)\sqrt{n}}, \quad n \geq 1$$

and

$$(4.8) \quad 0 \leq \log_b s - H_n(X; b) \leq \frac{\sqrt{2}(p_M - p_m)}{2p_m(n+1)\sqrt{n}}, \quad n \geq 1.$$

The following result of convergence also holds:

$$(4.9) \quad \lim_{n \rightarrow \infty} n^\alpha [H_{n+1}(X; b) - H_n(X; b)] = 0, \quad \alpha \in \left[ 0, \frac{3}{2} \right)$$

and

$$(4.10) \quad \lim_{n \rightarrow \infty} n^\beta [\log_b s - H_n(X; b)] = 0, \quad \beta \in \left[ 0, \frac{1}{2} \right)$$

uniformly by rapport of  $b > 1$  and  $s \geq 1$ .

We can state the following proposition.

**Proposition 1.** *With the above assumptions we have the estimation:*

$$(4.11) \quad 0 \leq \log_b s - H_b(X) \leq \frac{1}{\ln b} \sum_{1 \leq i < j \leq s} (p_i - p_j)^2.$$

*Proof.* Follows from the inequality (??) taking into account that

$$h_1(X; b) = \frac{1}{\ln b} \sum_{1 \leq i < j \leq s} (p_i - p_j)^2.$$

■

**Remark 3.** *The result we obtained above is exactly Theorem 4.3 from [?] written in another form.*

**Corollary 3.** *With the above assumptions and if*

$$\max_{1 \leq i < j \leq s} |p_i - p_j| \leq \sqrt{\frac{2\varepsilon \ln s}{s(s-1)}}$$

for  $\varepsilon > 0$ , then:

$$(4.12) \quad 0 \leq \log_b s - H_b(X) \leq \varepsilon.$$

The following theorem holds.

**Theorem 9.** *Let  $\varepsilon > 0$ . If  $p_i > 0$  ( $i = \overline{1, s}$ ) ( $s \geq 2$ ) are such that:*

$$\frac{p_i}{p_j} \in \left[ \frac{2+q-\sqrt{q(q+4)}}{2}, \frac{2+q+\sqrt{q(q+4)}}{2} \right], \quad 1 \leq i < j \leq s$$

where:

$$q = \frac{2\varepsilon^2 n(n+1)^2}{s(s-1)}, \quad n \geq 1.$$

Then we have the estimate:

$$(4.13) \quad 0 \leq H_{n+1}(X; b) - H_n(X; b) \leq \varepsilon, \quad n \geq 1.$$

*Proof.* By the inequality (??) we have that

$$0 \leq H_{n+1}(X; b) - H_n(X; b) \leq \frac{1}{(n+1)\sqrt{n}} \left[ \sum_{1 \leq i < j \leq s} \frac{(p_i - p_j)^2}{p_i p_j} \right]^{\frac{1}{2}}.$$

Let us consider the inequality:

$$\frac{(p_i - p_j)^2}{p_i p_j} \leq q, \quad 1 \leq i < j \leq s, \quad q > 0,$$

which is equivalent to

$$p_i^2 - (2+q)p_i p_j + p_j^2 \leq 0, \quad 1 \leq i < j \leq s, \quad q > 0$$

or

$$\left( \frac{p_i}{p_j} \right)^2 - (2+q) \frac{p_i}{p_j} + 1 \leq 0, \quad 1 \leq i < j \leq s,$$

or, additionally,

$$\frac{p_i}{p_j} \in \left[ \frac{2+q-\sqrt{q(q+4)}}{2}, \frac{2+q+\sqrt{q(q+4)}}{2} \right], \quad 1 \leq i < j \leq s.$$

Now, if we choose

$$q = \frac{2\varepsilon^2 n(n+1)^2}{s(s-1)},$$



then

$$\sum_{1 \leq i < j \leq s} \frac{(p_i - p_j)^2}{p_i p_j} \leq \frac{s(s-1)}{2} q = \varepsilon^2 n (n+1)^2$$

and

$$0 \leq H_{n+1}(X; b) - H_n(X; b) \leq \frac{1}{(n+1)\sqrt{n}} \left[ \varepsilon^2 n (n+1)^2 \right]^{\frac{1}{2}} = \varepsilon.$$

The theorem is thus proved. ■

Similarly, we can prove the following theorem too.

**Theorem 10.** *Let  $\varepsilon > 0$ . If  $p_i > 0$  ( $i = \overline{1, s}$ ) ( $s \geq 2$ ) are such that:*

$$\frac{p_i}{p_j} \in \left[ \frac{2+q-\sqrt{q(q+4)}}{2}, \frac{2+q+\sqrt{q(q+4)}}{2} \right], \quad 1 \leq i < j \leq s,$$

where

$$q = \frac{2\varepsilon^2 n}{s(s-1)},$$

then we have the estimate;

$$0 \leq \log_b s - H_b(X) \leq \varepsilon, \quad n \geq 1.$$

The proof goes by the inequality (??) and we shall omit the details.

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SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO  
BOX 14428, MELBOURNE CITY MCMC, 8001 VICTORIA, AUSTRALIA.

*E-mail address:* `sever@matilda.vu.edu.au`

*URL:* `http://matilda.vu.edu.au/~rgmia/dragomirweb.html`