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INEQUALITIES AND MONOTONICITY OF SEQUENCES INVOLVING $\sqrt[n]{(n+k)!/k!}$

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ABSTRACT. Using Stirling's formula, for all nonnegative integers k and natural numbers n and m , we prove that

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} / \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \leq \sqrt{\frac{n+k}{n+m+k}}.$$

From this, some monotonicity results of sequences involving $\sqrt[n]{(n+k)!/k!}$ are deduced, and the related inequalities are refined.

1. INTRODUCTION

In [4], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{1/r}$, then

$$(1) \quad 1 < \phi(r+1)/\phi(r) < (r+1)/r.$$

In [1, 3], H. Alzer and J. S. Martins refined the right inequality in (1) and showed that, if n is a positive integer, then we have for all positive real numbers r ,

$$(2) \quad \frac{n}{n+1} < \left(\frac{1}{n} \sum_{i=1}^n i^r / \frac{1}{n+1} \sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}.$$

Both bounds in (2) are the best possible.

Let n and m be natural numbers, k a nonnegative integer. The author generalized in [8] the left inequality in (2) and obtained

$$(3) \quad \frac{n+k}{n+m+k} < \left(\frac{1}{n} \sum_{i=k+1}^{n+k} i^r / \frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r\right)^{1/r},$$

where r is a given positive real number. The lower bound in (3) is the best possible.

By mathematical induction, N. Elezović and J. Pečarić [2] and the author [6, 7] further generalized the left side of inequalities in (2) and the inequality (3) to a large class of positive, increasing and convex sequences and of positive, increasing and logarithmically concave sequences in different directions, respectively.

Recently, the inequalities in (1) were generalized by the author and Q.-M. Luo in [10] and the following inequalities were obtained:

$$(4) \quad \frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < 1,$$

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where n and m are natural numbers, k is a nonnegative integer. Meanwhile, some monotonicity results of the sequences involving $\sqrt[n]{(n+k)!/k!}$ were presented.

In this article, using inequalities deduced from Stirling's formula, we give a refinement of the left inequality in (1) and the right inequality in (4), that is

Theorem 1. *Let k be a nonnegative integer, n and m be natural numbers, then*

$$(5) \quad \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} \bigg/ \left(\prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)} \leq \sqrt{\frac{n+k}{n+m+k}}.$$

When $n = m = 1$, the equality in (5) is valid.

Theorem 2. *The sequences*

$$(6) \quad \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} \bigg/ \sqrt{n+k}$$

are strictly increasing with n and k , respectively. The sequences

$$(7) \quad \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} \bigg/ \left(\prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)}$$

are strictly increasing in k for all given natural numbers n and m .

2. LEMMAS

In order to verify our theorems, we need some lemmas.

Lemma 1. *For all natural numbers n , we have*

$$(8) \quad n \ln n - n \leq \ln n! \leq \left(n + \frac{1}{2} \right) \ln n - n + 1,$$

$$(9) \quad \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp \frac{1}{12n+1} < n! < \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp \frac{1}{12n}.$$

For details about inequalities (8) and (9), please refer to [5, p. 184 and p. 194].

Lemma 2. *For all natural numbers k and $n > 1$, we have*

$$(10) \quad \frac{n(n-1)}{2} \ln \left(1 + \frac{1}{n+k} \right) + \frac{2k+1}{2} \ln \left(1 + \frac{n}{k} \right) \leq n + \frac{12n-1}{12(n+k)(12k+1)}.$$

Proof. Let

$$\varphi(x, y) = \frac{x(x-1)}{2} \ln \left(1 + \frac{1}{x+y} \right) + \frac{2y+1}{2} \ln \left(1 + \frac{x}{y} \right) - \frac{12x-1}{12(x+y)(12y+1)} - x, \quad x \geq 2, \quad y \geq 1.$$

Differentiating and simplifying produces

$$\frac{\partial \varphi(x, y)}{\partial x} = \frac{2x-1}{2} \ln \left(1 + \frac{1}{x+y} \right) + \frac{6y^2 + 5y - 1 + (5 + 6y - 12y^2)x - 30yx^2 - 18x^3}{12(x+y)^2(x+y+1)}.$$

Using the inequality $\ln(1+t) \leq \frac{t(2+t)}{2(1+t)}$ for $t \geq 0$ in [9], we have

$$\frac{\partial \varphi(x, y)}{\partial x} \leq \frac{2y-1+2x-6yx^2-6x^3}{12(x+y)^2(x+y+1)} \leq 0,$$

therefore $\varphi(x, y)$ is decreasing with $x \geq 2$.

It is easy to see that

$$\begin{aligned}\varphi(2, y) &= \ln\left(1 + \frac{1}{y+2}\right) + \frac{2y+1}{2} \ln\left(1 + \frac{2}{y}\right) - \frac{23}{12(y+2)(12y+1)} - 2, \\ \varphi(2, 1) &= 2 \ln 2 + \frac{1}{2} \ln 3 - \frac{23}{468} - 2 < 0.\end{aligned}$$

Denote

$$(11) \quad \psi(y) = \varphi(2, y) + \frac{23}{12(y+2)(12y+1)},$$

then

$$\begin{aligned}\psi'(y) &= \ln\left(1 + \frac{2}{y}\right) - \frac{2y^2 + 8y + 3}{y(y+2)(y+3)}, \\ \psi''(y) &= \frac{18 - 6y - 5y^2}{[y(y+2)(y+3)]^2}.\end{aligned}$$

Thus $\psi'(y)$ is decreasing with $y \geq 2$. Since $\lim_{y \rightarrow +\infty} \psi'(y) = 0$, then $\psi'(y) \geq 0$ for $y \geq 2$ and $\psi(y)$ increases for $y \geq 2$.

Also since $\lim_{y \rightarrow +\infty} \psi(y) = 0$, we get $\psi(y) < 0$ for $y \geq 2$ and $\varphi(2, y) < 0$ for $y \geq 2$. Hence the sequence $\{\varphi(2, k)\}_{k=1}^{+\infty}$ is negative, and so is the sequence $\{\varphi(n, k)\}_{n=2, k=1}^{+\infty}$. Lemma 2 follows. ■

3. PROOFS OF THEOREMS

Proposition 1. *Let n be a natural number, then*

$$(12) \quad \frac{\sqrt[n]{n!}}{^{n+1}\sqrt{(n+1)!}} \leq \sqrt{\frac{n}{n+1}}.$$

When $n = 1$, the equality in (12) holds.

Proof. Let

$$f(x) = x(x-1) \ln(1+x) - (x^2 - x - 1) \ln x - 2x + 2, \quad x \geq 1.$$

By standard arguments, we get

$$\begin{aligned}f'(x) &= (2x-1) \ln\left(1 + \frac{1}{x}\right) + \frac{x(x-1)}{1+x} - \frac{x^2 - x - 1}{x} - 2 \\ &\leq \frac{2x-1}{x} + \frac{x(x-1)}{1+x} - \frac{x^2 - x - 1}{x} - 2 \\ &= \frac{1-x}{1+x} \leq 0, \\ f(1) &= 0.\end{aligned}$$

Therefore, $f(x)$ is decreasing and $f(x) \leq 0$ for $x \geq 1$. And for $n \geq 1$, we obtain

$$n(n-1) \ln(1+n) + 2 \leq (n^2 - n - 1) \ln n + 2n,$$

this is equivalent to

$$\left(n + \frac{1}{2}\right) \ln n - n + 1 \leq \frac{n(n+1)}{2} \ln n - \frac{n(n-1)}{2} \ln(1+n).$$

Thus, from the right-hand side of inequality (8), we obtain

$$\ln n! \leq \frac{n(n+1)}{2} \ln n - \frac{n(n-1)}{2} \ln(1+n),$$

that is,

$$\begin{aligned} n! &\leq \frac{n^{n(n+1)/2}}{(n+1)^{n(n-1)/2}}, \\ \frac{n!}{(n+1)^n} &\leq \frac{n^{n(n+1)/2}}{(n+1)^{n(n+1)/2}}, \\ \sqrt{\frac{n}{n+1}} &\geq \left(\frac{(n!)^{n+1}}{[(n+1)!]^n} \right)^{1/n(n+1)}. \end{aligned}$$

The proof is complete. ■

Proposition 2. *Let k and n be natural numbers, then*

$$(13) \quad \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} \bigg/ \left(\prod_{i=k+1}^{n+k+1} i \right)^{1/(n+1)} \leq \sqrt{\frac{n+k}{n+k+1}}.$$

If $n = 1$, the equality in (13) is valid for all $k \geq 1$.

Proof. For $n > 1$, the inequality (10) may be rewritten as

$$\begin{aligned} \frac{n(n-1)}{2} \ln \frac{n+k+1}{n+k} + \frac{2k+1}{2} \ln \frac{n+k}{k} &\leq n + \frac{12n-1}{12(n+k)(12k+1)}, \\ \frac{n(n-1)}{2} \ln(n+k+1) + \frac{n-n^2+2k+1}{2} \ln(n+k) & \\ &\leq \left(k + \frac{1}{2}\right) \ln k + n + \frac{12n-1}{12(n+k)(12k+1)}, \\ \left(n+k + \frac{1}{2}\right) \ln(n+k) - \left(k + \frac{1}{2}\right) \ln k - n + \frac{1}{12(n+k)} - \frac{1}{12k+1} & \\ &\leq \frac{n(n+1)}{2} \ln(n+k) - \frac{n(n-1)}{2} \ln(n+k+1). \end{aligned}$$

Then, substituting inequalities in (9) into the final inequality above yields

$$\ln(n+k)! - \ln k! \leq \frac{n(n+1)}{2} \ln(n+k) - \frac{n(n-1)}{2} \ln(n+k+1),$$

this inequality can be rearranged as

$$\begin{aligned} \frac{(n+k)!}{k!} &\leq \frac{(n+k)^{n(n+1)/2}}{(n+k+1)^{n(n-1)/2}}, \\ \prod_{i=k+1}^{n+k} i \bigg/ (n+k+1)^n &\leq \left(\frac{n+k}{n+k+1} \right)^{n(n+1)/2}, \\ \sqrt{\frac{n+k}{n+k+1}} &\geq \left[\left(\prod_{i=k+1}^{n+k} i \right)^n \bigg/ \left(\prod_{i=k+1}^{n+k+1} i \right)^n \right]^{1/n(n+1)}. \end{aligned}$$

From this, the inequality (13) follows. ■

Proofs of Theorem 1 and Theorem 2. The inequality (5) can be rewritten as

$$\frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}} \leq \frac{\left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)}}{\sqrt{n+m+k}},$$

which is equivalent to

$$\frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}} \leq \frac{\left(\prod_{i=k+1}^{n+k+1} i\right)^{1/(n+1)}}{\sqrt{n+k+1}},$$

that is,

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \bigg/ \left(\prod_{i=k+1}^{n+k+1} i\right)^{1/(n+1)} \leq \sqrt{\frac{n+k}{n+k+1}}.$$

This follows from the combination of Proposition 1 and Proposition 2. Thus, Theorem 1 and the monotonicity of the sequences $\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} / \sqrt{n+k}$ being increasing with n are proved.

It is easy to see that

$$\frac{\left(\prod_{i=k+2}^{n+k+1} i\right)^{1/n}}{\sqrt{n+k+1}} = \sqrt{\frac{n+k}{n+k+1}} \left(\frac{n+k+1}{k+1}\right)^{1/n} \frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}}.$$

Let

$$g(x) = \left(\frac{1}{x} - \frac{1}{2}\right) \ln(x+k+1) - \frac{1}{x} \ln(k+1) + \frac{1}{2} \ln(x+k), \quad x \geq 1.$$

Using $\frac{2t}{2+t} \leq \ln(1+t)$ for $t > 0$ in [5, pp. 273–274] and [9] and differentiating yields

$$\begin{aligned} g'(x) &= \frac{1}{x^2} \left[\frac{x(3x+2k)}{2(x+k)(x+k+1)} + \ln \frac{k+1}{x+k+1} \right] \\ &\leq \frac{1}{x^2} \left[\frac{x(3x+2k)}{2(x+k)(x+k+1)} - \frac{2x}{x+2(k+1)} \right] \\ &= \frac{2-x}{2(x+k)(x+k+1)[x+2(k+1)]}. \end{aligned}$$

Thus, $g(x)$ is decreasing with $x \geq 2$. Since $\lim_{x \rightarrow +\infty} g(x) = 0$ and $g(1) = \frac{1}{2} \ln \frac{k+2}{k+1} > 0$,

then the sequence $\{g(n)\}_{n=1}^{+\infty}$ is positive. Hence

$$\begin{aligned} \sqrt{\frac{n+k}{n+k+1}} \left(\frac{n+k+1}{k+1}\right)^{1/n} &> 1, \\ \frac{\left(\prod_{i=k+2}^{n+k+1} i\right)^{1/n}}{\sqrt{n+k+1}} &> \frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}}. \end{aligned}$$

The sequences $\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} / \sqrt{n+k}$ being increasing with k are verified.

It is clear that

$$\begin{aligned} & \frac{\sqrt[n]{(n+k+1)!/(k+1)!}}{\sqrt[n+m]{(n+m+k+1)!/(k+1)!}} \\ &= \left[\frac{(n+k+1)^{n+m}}{(k+1)^m(n+m+k+1)^n} \right]^{1/n(n+m)} \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}. \end{aligned}$$

Let

$$h(x) = (x+m)\ln(x+k+1) - x\ln(x+m+k+1) - m\ln(k+1), \quad x \geq 0.$$

Direct calculation leads to

$$\begin{aligned} h'(x) &= \frac{m(2x+m+k+1)}{(x+k+1)(x+m+k+1)} - \ln\left(1 + \frac{m}{x+k+1}\right) \\ &\geq \frac{m(2x+m+k+1)}{(x+k+1)(x+m+k+1)} - \frac{m}{x+k+1} \\ &= \frac{mx}{(x+k+1)(x+m+k+1)} \geq 0, \\ h(0) &= 0. \end{aligned}$$

So, the function $h(x) \geq 0$ for $x \geq 0$, and we have

$$\frac{(n+k+1)^{n+m}}{(k+1)^m(n+m+k+1)^n} > 1,$$

$$\left(\prod_{i=k+2}^{n+k+1} \right)^{1/n} \Big/ \left(\prod_{i=k+2}^{n+m+k+1} \right)^{1/(n+m)} > \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} \Big/ \left(\prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)}.$$

The proof of Theorem 2 is complete.

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