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*A New Generalization of the Trapezoid Formula for  
n-Time Differentiable Mappings and Applications*

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# A NEW GENERALIZATION OF THE TRAPEZOID FORMULA FOR $n$ -TIME DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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ABSTRACT. A new generalization of the trapezoid formula for  $n$ -time differentiable mappings and applications in Numerical Analysis are given.

## 1. INTRODUCTION

In the recent paper [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following generalization of the trapezoid rule.

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping such that  $f^{(n-1)}$  is absolutely continuous on  $[a, b]$ . Then we have the equality*

$$(1.1) \quad \begin{aligned} \int_a^b f(t) dt &= \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \\ &\quad + \int_a^b T_n(t) f^{(n)}(t) dt, \end{aligned}$$

where

$$(1.2) \quad T_n(t) := \frac{1}{n!} \left[ \frac{(b-t)^n + (-1)^n (t-a)^n}{2} \right], \quad t \in [a, b].$$

In the same paper, the authors pointed out the following inequality which provides an approximation formula for the integral  $\int_a^b f(t) dt$  whose error can be estimated in terms of the sup-norm of  $f^{(n)}(t)$ .

**Corollary 1.** *Under the above assumptions, we have the inequality*

$$(1.3) \quad \begin{aligned} &\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \\ &\leq \frac{(b-a)^{n+1}}{(n+1)!} \|f^{(n)}\|_{\infty} \times \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1}-1}{2^r} & \text{if } n = 2r+1 \end{cases}. \end{aligned}$$

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If, in the above corollary, we consider  $n = 1$ , then we get the known inequality [2]

$$(1.4) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) \right| \leq \frac{1}{2} (b - a)^2 \|f'\|_\infty.$$

For  $n = 2$ , we obtain

$$(1.5) \quad \begin{aligned} & \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) - \frac{(b - a)^2}{2} \cdot \frac{f'(a) + f'(b)}{2} \right| \\ & \leq \frac{(b - a)^3}{6} \|f''\|_\infty. \end{aligned}$$

For other recent results concerning the trapezoid formula, see the book [11] and the recent papers [1]-[10] and [12]-[13], where further references are given.

The main aim of this paper is to point out a generalization of the trapezoid rule and inequality in a different way. Applications in Numerical Analysis for quadrature formulae will also be provided. A perturbed trapezoidal type rule is presented in Section 4 in which a number of *premature* results are given that provide tighter bounds than the traditional Grüss, Chebychev and Lupas inequalities.

## 2. INTEGRAL IDENTITIES

We start with the following result.

**Theorem 2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping such that the derivative  $f^{(n-1)}$  ( $n \geq 1$ ) is absolutely continuous on  $[a, b]$ . Then*

$$(2.1) \quad \begin{aligned} & \int_a^b f(t) dt \\ &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ (x - a)^{k+1} f^{(k)}(a) + (-1)^k (b - x)^{k+1} f^{(k)}(b) \right] \\ & \quad + \frac{1}{n!} \int_a^b (x - t)^n f^{(n)}(t) dt, \end{aligned}$$

for all  $x \in [a, b]$ .

*Proof.* The proof is by mathematical induction.

For  $n = 1$ , we have to prove that

$$(2.2) \quad \int_a^b f(t) dt = (x - a) f(a) + (b - x) f(b) + \int_a^b (x - t) f^{(1)}(t) dt,$$

which is straightforward as may be seen by the integration by parts formula applied for the integral

$$\int_a^b (x - t) f^{(1)}(t) dt.$$

Assume that (2.1) holds for “ $n$ ” and let us prove it for “ $n + 1$ ”. That is, we wish to show that:

$$\begin{aligned}
 (2.3) \quad & \int_a^b f(t) dt \\
 &= \sum_{k=0}^n \frac{1}{(k+1)!} \left[ (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \\
 & \quad + \frac{1}{(n+1)!} \int_a^b (x-t)^{n+1} f^{(n+1)}(t) dt.
 \end{aligned}$$

For this purpose, we apply formula (2.2) for the mapping  $g(t) := (x-t)^n f^{(n)}(t)$ , which is absolutely continuous on  $[a, b]$ , and then, we can write:

$$\begin{aligned}
 (2.4) \quad & \int_a^b (x-t)^n f^{(n)}(t) dt \\
 &= (x-a)(x-a)^n f^{(n)}(a) + (b-x)(x-b)^n f^{(n)}(b) \\
 & \quad + \int_a^b (x-t) \frac{d}{dt} \left[ (x-t)^n f^{(n)}(t) \right] dt \\
 &= \int_a^b (x-t) \left[ -n(x-t)^{n-1} f^{(n)}(t) + (x-t)^n f^{(n+1)}(t) \right] dt \\
 & \quad + (x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b) \\
 &= -n \int_a^b (x-t)^n f^{(n)}(t) dt + \int_a^b (x-t)^{n+1} f^{(n+1)}(t) dt \\
 & \quad + (x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n+1)}(t) dt.
 \end{aligned}$$

From this identity we can get

$$\begin{aligned}
 & \int_a^b (x-t)^n f^{(n)}(t) dt \\
 &= \frac{1}{n+1} \int_a^b (x-t)^{n+1} f^{(n+1)}(t) dt \\
 & \quad + \frac{1}{n+1} \left[ (x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b) \right].
 \end{aligned}$$

Now, using the induction hypothesis, we have

$$\int_a^b f(t) dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] +$$

$$\begin{aligned}
& + \frac{1}{n!} \left[ \frac{1}{n+1} \int_a^b (x-t)^{n+1} f^{(n+1)}(t) dt \right. \\
& \quad \left. + \frac{1}{n+1} \left[ (x-a)^{n+1} f^{(n)}(a) + (b-x)^{n+1} f^{(n)}(b) \right] \right] \\
= & \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \\
& + \frac{1}{(n+1)!} \int_a^b (x-t)^{n+1} f^{(n+1)}(t) dt
\end{aligned}$$

and the identity (2.3) is proved. This completes the proof.  $\blacksquare$

The following corollary is useful in practice.

**Corollary 2.** *With the above assumptions for  $f$  and  $R$ , we have the particular identities (which can also be obtained by using Taylor's formula with the integral remainder)*

$$\begin{aligned}
(2.5) \quad \int_a^b f(t) dt &= \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} (b-a)^{k+1} f^{(k)}(b) \\
&+ \frac{(-1)^n}{n!} \int_a^b (t-a)^n f^{(n)}(t) dt,
\end{aligned}$$

$$\begin{aligned}
(2.6) \quad \int_a^b f(t) dt &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} (b-a)^{k+1} f^{(k)}(a) \\
&+ \frac{1}{n!} \int_a^b (b-t)^n f^{(n)}(t) dt,
\end{aligned}$$

and the identity (see also [13])

$$\begin{aligned}
(2.7) \quad \int_a^b f(t) dt &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( \frac{b-a}{2} \right)^{k+1} \left[ f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \\
&+ \frac{(-1)^n}{n!} \int_a^b \left( t - \frac{a+b}{2} \right)^n f^{(n)}(t) dt.
\end{aligned}$$

**Remark 1.** a) For  $n = 1$ , we get the identity (2.2) which is a generalization of the trapezoid rule.

i) For  $x = a$  in (2.2), we capture the "right rectangle rule"

$$\int_a^b f(t) dt = (b-a) f(b) - \int_a^b (t-a) f'(t) dt.$$

ii) For  $x = b$  in (2.2), we obtain the “left rectangle rule”

$$(2.8) \quad \int_a^b f(t) dt = (b-a) f(a) - \int_a^b (b-t) f'(t) dt.$$

iii) Finally, for  $x = \frac{a+b}{2}$ , we get [2]

$$(2.9) \quad \int_a^b f(t) dt = \frac{f(a) + f(b)}{2} (b-a) - \int_a^b \left( t - \frac{a+b}{2} \right) f'(t) dt$$

which is the “trapezoid rule”.

b) For  $n = 2$ , we get the identity:

$$(2.10) \quad \begin{aligned} & \int_a^b f(t) dt \\ &= (x-a) f(a) + (b-x) f(b) \\ &+ \frac{1}{2} \left[ (x-a)^2 f'(a) + (b-x)^2 f'(b) \right] + \frac{1}{2} \int_a^b (x-t)^2 f''(t) dt. \end{aligned}$$

i) If in (2.10) we choose  $x = b$ , then we obtain the “perturbed left rectangle rule”

$$(2.11) \quad \int_a^b f(t) dt = (b-a) f(a) + \frac{1}{2} (b-a)^2 f'(a) + \frac{1}{2} \int_a^b (t-a)^2 f''(t) dt,$$

which can also be obtained by using Taylor’s formula with the integral remainder.

ii) If in (2.10) we choose  $x = a$ , we can write the “perturbed right rectangle rule”

$$(2.12) \quad \int_a^b f(t) dt = (b-a) f(b) - \frac{1}{2} (b-a)^2 f'(b) + \frac{1}{2} \int_a^b (t-b)^2 f''(t) dt.$$

iii) Finally, for  $x = \frac{a+b}{2}$ , we capture the “perturbed trapezoid rule” [13]

$$(2.13) \quad \begin{aligned} \int_a^b f(t) dt &= \frac{f(a) + f(b)}{2} (b-a) + \frac{(b-a)^2}{8} \left( f'(a) - f'(b) \right) \\ &+ \frac{1}{2} \int_a^b \left( t - \frac{a+b}{2} \right)^2 f''(t) dt. \end{aligned}$$

### 3. INTEGRAL INEQUALITIES

Using the integral representation by Theorem 1, we can prove the following inequality

**Theorem 3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping so that the derivative  $f^{(n-1)}$  ( $n \geq 1$ ) is absolutely continuous on  $[a, b]$ . Then*

$$(3.1) \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right| \leq \begin{cases} \frac{\|f^{(n)}\|_{\infty}}{(n+1)!} \left[ (x-a)^{n+1} + (b-x)^{n+1} \right] & \text{if } f^{(n)} \in L_{\infty}[a, b]; \\ \frac{\|f^{(n)}\|_p}{n!} \left[ \frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and } f^{(n)} \in L_p[a, b]; \\ \frac{\|f^{(n)}\|_1}{n!} \left[ \frac{1}{2} (b-a) + |x - \frac{a+b}{2}| \right]^n & \end{cases}$$

for all  $x \in [a, b]$ .

*Proof.* Using the representation (2.1) and the properties of the modulus, we have

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{1}{n!} \int_a^b |x-t|^n |f^{(n)}(t)| dt =: R.$$

Observe that

$$\begin{aligned} R &\leq \left[ \frac{1}{n!} \int_a^b |x-t|^n dt \right] \|f^{(n)}\|_{\infty} \\ &= \frac{\|f^{(n)}\|_{\infty}}{n!} \left[ \int_a^b (x-t)^n dt + \int_a^b (t-x)^n dt \right] \\ &= \frac{\|f^{(n)}\|_{\infty}}{n!} \left[ \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n+1} \right] \end{aligned}$$

and the first inequality in (3.1) is proved.

Using Hölder's integral inequality, we also have

$$\begin{aligned} R &\leq \frac{1}{n!} \left( \int_a^b |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}} \left( \int_a^b |x-t|^{nq} dt \right)^{\frac{1}{q}} \\ &= \frac{1}{n!} \|f^{(n)}\|_p \left[ \frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \end{aligned}$$

which proves the second inequality in (3.1).

Finally, let us observe that

$$\begin{aligned}
R &\leq \frac{1}{n!} \sup_{t \in [a, b]} |x - t|^n \int_a^b |f^{(n)}(t)| dt \\
&= \frac{1}{n!} \left[ \sup_{t \in [a, b]} |x - t| \right]^n \|f^{(n)}\|_1 \\
&= \frac{1}{n!} [\max(x - a, b - x)]^n \|f^{(n)}\|_1 \\
&= \frac{1}{n!} \left[ \frac{1}{2} (b - a) + \left| x - \frac{a+b}{2} \right| \right]^n \|f^{(n)}\|_1
\end{aligned}$$

and the theorem is completely proved. ■

The following corollary is useful in practice.

**Corollary 3.** *With the above assumptions for  $f$  and  $n$ , we have the particular inequalities*

$$\begin{aligned}
(3.2) \quad &\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} (b-a)^{k+1} f^{(k)}(b) \right| \\
&\leq M := \begin{cases} \frac{\|f^{(n)}\|_\infty}{(n+1)!} (b-a)^{n+1} & \text{if } f^{(n)} \in L_\infty[a, b]; \\ \frac{\|f^{(n)}\|_p}{n!} \frac{(b-a)^{n+\frac{1}{q}}}{(nq+1)^{1/q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{\|f^{(n)}\|_1}{n!} (b-a)^n, & \text{and } f^{(n)} \in L_p[a, b]; \end{cases}
\end{aligned}$$

and

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} (b-a)^{k+1} f^{(k)}(b) \right| \leq M$$

and (see also [13])

$$\begin{aligned}
(3.3) \quad &\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( \frac{b-a}{2} \right)^{k+1} \left[ f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \\
&\leq \begin{cases} \frac{\|f^{(n)}\|_\infty}{2^n (n+1)!} (b-a)^{n+1} & \text{if } f^{(n)} \in L_\infty[a, b]; \\ \frac{\|f^{(n)}\|_p}{2^n n! (nq+1)^{1/q}} (b-a)^{n+\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{\|f^{(n)}\|_1}{2^n n!} (b-a)^n; & \text{and } f^{(n)} \in L_p[a, b]; \end{cases}
\end{aligned}$$

respectively.

**Remark 2.** If we put  $n = 1$  in (3.1), we capture the inequality

$$(3.4) \quad \begin{aligned} & \left| \int_a^b f(t) dt - (x-a)f(a) + (b-x)f(b) \right| \\ & \leq \begin{cases} \left[ \frac{1}{4}(b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right] \|f^{(1)}\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \|f'\|_p \left[ \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1 & \text{and } f' \in L_p[a, b]; \end{cases} \end{aligned}$$

for all  $x \in [a, b]$ , and, in particular,

a) the “left rectangle” inequality

$$\left| \int_a^b f(t) dt - (b-a)f(a) \right| \leq \begin{cases} \frac{\|f'\|_\infty}{2} (b-a)^2 & \text{if } f' \in L_\infty[a, b]; \\ \frac{\|f'\|_p}{(q+1)^{1/q}} (b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a, b]; \\ \|f'\|_1 (b-a). \end{cases}$$

b) the “right rectangle” inequality

$$\left| \int_a^b f(t) dt - (b-a)f(b) \right| \leq \begin{cases} \frac{\|f'\|_\infty}{2} (b-a)^2 & \text{if } f' \in L_\infty[a, b]; \\ \frac{\|f'\|_p}{(q+1)^{1/q}} (b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a, b]; \\ \|f'\|_1 (b-a). \end{cases}$$

c) the “trapezoid” inequality

$$(3.5) \quad \begin{aligned} & \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b-a) \right| \\ & \leq \begin{cases} \frac{\|f'\|_\infty}{4} (b-a)^2 & \text{if } f' \in L_\infty[a, b]; \\ \frac{\|f'\|_p}{2(q+1)^{1/q}} (b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a, b]; \\ \frac{\|f'\|_1}{2} (b-a). \end{cases} \end{aligned}$$

**Remark 3.** If we put  $n = 2$  in (3.1), we get the inequality

$$(3.6) \quad \begin{aligned} & \left| \int_a^b f(t) dt - (x-a)f(a) - (b-x)f(b) \right. \\ & \left. - \frac{1}{2} \left[ (x-a)^2 f'(a) - (b-x)^2 f'(b) \right] \right| \\ & \leq \begin{cases} \frac{\|f''\|_\infty}{6} \left[ (b-a)^3 + (b-x)^3 \right] & \text{if } f'' \in L_\infty[a, b]; \\ \frac{\|f''\|_p}{2} \left[ \frac{(x-a)^{2q+1} + (b-x)^{2q+1}}{2q+1} \right]^{\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and } f'' \in L_p[a, b]; \\ \frac{\|f''\|_1}{2} \left[ \frac{1}{2}(b-a) + |x - \frac{a+b}{2}| \right]^2; \end{cases} \end{aligned}$$

for all  $x \in [a, b]$ , and, in particular:

a) the “perturbed left rectangle” inequality

$$(3.7) \quad \begin{aligned} & \left| \int_a^b f(t) dt - (b-a)f(a) - \frac{1}{2}(b-a)^2 f'(a) \right| \\ & \leq M_2 := \begin{cases} \frac{\|f''\|_\infty}{6} (b-a)^3 & \text{if } f'' \in L_\infty[a, b]; \\ \frac{\|f''\|_p}{2(2q+1)^{1/q}} (b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a, b]; \\ \frac{\|f''\|_1}{2} (b-a)^2; \end{cases} \end{aligned}$$

b) the “perturbed right rectangle” inequality

$$(3.8) \quad \left| \int_a^b f(t) dt - (b-a)f(b) + \frac{1}{2}(b-a)^2 f'(b) \right| \leq M_2$$

c) the “perturbed trapezoid” inequality

$$(3.9) \quad \begin{aligned} & \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b-a) + \frac{(b-a)^2}{8} \left( f'(b) - f'(a) \right) \right| \\ & \leq \begin{cases} \frac{\|f''\|_\infty}{24} (b-a)^3 & \text{if } f'' \in L_\infty[a, b]; \\ \frac{\|f''\|_p}{8(2q+1)^{1/q}} (b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a, b]; \\ \frac{\|f''\|_1}{8} (b-a)^2. \end{cases} \end{aligned}$$

## 4. A PERTURBED VERSION

A *premature* Grüss inequality is embodied in the following lemma which was proven by Cerone and Dragomir in the papers [14] and [16].

**Lemma 1.** *Let  $f, g$  be integrable functions defined on  $[a, b]$  and let  $d \leq g(t) \leq D$ . Then*

$$(4.1) \quad |T(f, g)| \leq \frac{D-d}{2} [T(f, f)]^{\frac{1}{2}},$$

where

$$T(f, g) = \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

Using the above lemma, the following result may be stated.

**Theorem 4.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  so that the derivative  $f^{(n-1)}$ ,  $n \geq 1$  is absolutely continuous on  $[a, b]$ . Assume that there exist constants  $\gamma, \Gamma \in \mathbb{R}$  such that  $\gamma \leq f^{(n)}(t) \leq \Gamma$  a.e on  $[a, b]$ . Then, the following inequality holds*

$$(4.2) \quad \begin{aligned} |P_T(x)| &:= \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \left( \frac{1}{(k+1)!} \times \right. \right. \\ &\quad \left. \left. \left[ (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right) \right. \\ &\quad \left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} \left[ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \right| \\ &\leq \frac{\Gamma - \gamma}{2} \cdot \frac{1}{n!} I(x, n) \\ &\leq \frac{\Gamma - \gamma}{2} \cdot \frac{n}{n+1} \cdot \frac{(b-a)^{n+1}}{\sqrt{2n+1}}, \end{aligned}$$

where

$$(4.3) \quad I(x, n) = \frac{1}{(n+1) \sqrt{2n+1}} \left\{ n^2 (b-a) \left[ (x-a)^{2n+1} + (b-x)^{2n+1} \right] \right. \\ \left. + (2n+1) (x-a) (b-x) [(x-a)^n - (x-b)^n]^2 \right\}^{\frac{1}{2}}.$$

*Proof.* Applying the premature Grüss result (4.1) on  $(x-t)^n$  and  $f^{(n)}(t)$ , we have

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b (x-t)^n f^{(n)}(t) dt - \frac{1}{b-a} \int_a^b (x-t)^n dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt \right| \\ &\leq \frac{\Gamma - \gamma}{2} \left\{ \frac{1}{b-a} \int_a^b (x-t)^{2n} dt - \left[ \frac{1}{b-a} \int_a^b (x-t)^n dt \right]^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b (x-t)^n f^{(n)}(t) dt \right. \\
& \left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)(b-a)} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right| \\
& \leq \frac{\Gamma - \gamma}{2} \left\{ \frac{(x-a)^{2n+1} + (b-x)^{2n+1}}{(2n+1)(b-a)} - \left[ \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(b-a)(n+1)} \right]^2 \right\}^{\frac{1}{2}}.
\end{aligned}$$

Further simplification of the above result by multiplying throughout by  $\frac{b-a}{n!}$  gives

$$\begin{aligned}
(4.4) \quad & \left| \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t) dt \right. \\
& \left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} \cdot \left[ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \right| \\
& \leq \frac{\Gamma - \gamma}{2} \cdot \frac{1}{n!} J(x, n),
\end{aligned}$$

where

$$\begin{aligned}
(4.5) \quad J^2(x, n) &= \frac{1}{(2n+1)(n+1)^2} \left\{ (n+1)^2 (A+B) (A^{2n+1} + B^{2n+1}) \right. \\
&\quad \left. - (2n+1) (A^{n+1} + (-1)^n B^{n+1})^2 \right\}
\end{aligned}$$

with  $A = x - a$ ,  $B = b - x$ .

Now, from (4.5),

$$\begin{aligned}
& (2n+1)(n+1)^2 J^2(x, n) \\
&= n^2 (A+B) (A^{2n+1} + B^{2n+1}) \\
&\quad + (2n+1) \left[ (A+B) (A^{2n+1} + B^{2n+1}) - (A^{n+1} + (-1)^n B^{n+1})^2 \right] \\
&= n^2 (A+B) (A^{2n+1} + B^{2n+1}) \\
&\quad + (2n+1) [AB (A^{2n} + B^{2n}) - 2A^{n+1} \cdot (-1)^n B^{n+1}] \\
&= n^2 (A+B) [A^{2n+1} + B^{2n+1}] + (2n+1) AB [A^n - (-B)^n]^2
\end{aligned}$$

Now, substitution of  $A = x - a$ ,  $B = b - x$  and the fact that  $A + B = b - a$  gives  $I(x, n) = \frac{J(x, n)}{(n+1)\sqrt{2n+1}}$ , as presented in (4.3). Substitution of identity (2.1) into (4.4) gives (4.2) and the first part of the theorem is thus proved.

The upper bound is obtained by taking either  $I(a, n)$  or  $I(b, n)$  since  $I(x, n)$  is convex. Hence the theorem is completely proved. ■

**Corollary 4.** *Let the conditions of Theorem 4 hold. Then the following result holds*

$$(4.6) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( \frac{b-a}{2} \right)^{k+1} \left[ f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right. \\ & \quad \left. - \left( \frac{b-a}{2} \right)^n \frac{[1 + (-1)^n]}{(n+1)!} \left[ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \right| \\ & \leq \frac{\Gamma - \gamma}{2} \cdot \frac{1}{n!} \left( \frac{b-a}{2} \right)^{n+1} \cdot \frac{1}{\sqrt{2n+1}} \cdot \begin{cases} \frac{2n}{n+1}, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}. \end{aligned}$$

*Proof.* Taking  $x = \frac{a+b}{2}$  in (4.2) gives (4.2), where

$$I \left( \frac{a+b}{2}, n \right) = \frac{1}{(n+1) \sqrt{2n+1}} \left( \frac{b-a}{2} \right)^{n+1} \left\{ 4n^2 + (2n+1) [1 + (-1)^n]^2 \right\}^{\frac{1}{2}}.$$

Examining the above expression for  $n$  even or  $n$  odd readily gives the result (4.6).  $\blacksquare$

**Remark 4.** *For  $n$  even, then the third term in the modulus sign vanishes and thus there is no perturbation to the trapezoidal rule (4.6).*

**Theorem 5.** *Let the conditions of Theorem 4 be satisfied. Further, suppose that  $f^{(n)}$  is differentiable and be such that*

$$\left\| f^{(n+1)} \right\|_{\infty} := \sup_{t \in [a,b]} |f^{n+1}(t)| < \infty.$$

*Then*

$$(4.7) \quad |P_T(x)| \leq \frac{b-a}{\sqrt{12}} \left\| f^{(n+1)} \right\|_{\infty} \cdot \frac{1}{n!} I(x, n),$$

where  $P_T(x)$  is the perturbed trapezoidal type rule given by the left hand side of (4.2) and  $I(x, n)$  is as given by (4.3).

*Proof.* Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous and  $f', g'$  be bounded. Then Chebychev's inequality holds (see [15, p. 207])

$$|T(f, g)| \leq \frac{(b-a)^2}{12} \sup_{t \in [a,b]} |f'(t)| \cdot \sup_{t \in [a,b]} |g'(t)|.$$

Matić, Pečarić and Ujević [16] using a premature Grüss type argument proved that

$$(4.8) \quad |T(f, g)| \leq \frac{(b-a)}{\sqrt{12}} \sup_{t \in [a,b]} |g'(t)| \sqrt{T(f, f)}.$$

Thus, associating  $f^{(n)}(\cdot)$  with  $g(\cdot)$  and  $(x-t)^n$  with  $f$  in (4.8) readily produces (4.7) where  $I(x, n)$  is as given by (4.3).  $\blacksquare$

**Theorem 6.** *Let the conditions of Theorem 4 be satisfied. Further, suppose that  $f^{(n)}$  is locally absolutely continuous on  $(a, b)$  and let  $f^{(n+1)} \in L_2(a, b)$ . Then*

$$(4.9) \quad |P_T(x)| \leq \frac{b-a}{\pi} \left\| f^{(n+1)} \right\|_2 \cdot \frac{1}{n!} I(x, n),$$

where  $P_T(x)$  is the perturbed trapezoidal type rule given by the left hand side of (4.2) and  $I(x, n)$  is as given in (4.3).

*Proof.* The following result was obtained by Lupaş (see [15, p. 210]). For  $f, g : (a, b) \rightarrow \mathbb{R}$  locally absolutely continuous on  $(a, b)$  and  $f', g' \in L_2(a, b)$ , then

$$|T(f, g)| \leq \frac{(b-a)^2}{\pi^2} \|f'\|_2 \|g'\|_2,$$

where

$$\|h\|_2 := \left( \frac{1}{b-a} \int_a^b |h(t)|^2 dt \right)^{\frac{1}{2}} \quad \text{for } h \in L_2(a, b).$$

Matić, Pečarić and Ujević [16] further show that

$$(4.10) \quad |T(f, g)| \leq \frac{(b-a)}{\pi} \|g'\|_2 \sqrt{T(f, f)}.$$

Now, associating  $f^{(n)}(\cdot)$  with  $g(\cdot)$  and  $(x-t)^n$  with  $f$  in (4.10) gives (4.9), where  $I(x, n)$  is found in (4.3). ■

**Remark 5.** Results (4.7) and (4.9) are not readily comparable to that obtained in Theorem 4 since the bound now involves the behaviour of  $f^{(n+1)}(\cdot)$  rather than  $f^{(n)}(\cdot)$ .

## 5. APPLICATION IN NUMERICAL INTEGRATION

Consider the partition  $I_m : a = x_0 < x_1 < \dots < x_{m-1} < x_m = b$  of the interval  $[a, b]$  and the intermediate points  $\xi = (\xi_0, \dots, \xi_{m-1})$ , where  $\xi_j \in [x_j, x_{j+1}]$  ( $j = 0, \dots, m-1$ ). Put  $h_j := x_{j+1} - x_j$  and  $\vartheta(h) = \max \{h_j \mid j = 0, \dots, m-1\}$ .

In [1], the authors considered the following generalization of the trapezoid formula

$$(5.1) \quad T_{m,n}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{h_j^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})}{2} \right]$$

and proved the following theorem:

**Theorem 7.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that it's derivative  $f^{(n-1)}$  is absolutely continuous on  $[a, b]$ . Then we have

$$(5.2) \quad \int_a^b f(t) dt = T_{m,n}(f, I_m) + R_{m,n}(f, I_m),$$

where the remainder  $R_{m,n}(f, I_m)$  satisfies the estimate

$$(5.3) \quad |R_{m,n}(f, I_m)| \leq \frac{C_n}{(n+1)!} \left\| f^{(n)} \right\|_{\infty} \sum_{j=0}^{m-1} h_j^{n+1},$$

and

$$C_n := \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1}-1}{2^{2r}} & \text{if } n = 2r+1. \end{cases}$$

Now, let us define the even more generalized quadrature formula

$$\begin{aligned}\tilde{T}_{m,n}(f, \xi, I_m) &:= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ (\xi_j - x_j)^{k+1} f^{(k)}(x_j) \right. \\ &\quad \left. + (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1}) \right],\end{aligned}$$

where  $x_j, \xi_j$  ( $j = 0, \dots, m-1$ ) are as above.

The following theorem holds.

**Theorem 8.** *Let  $f$  be as in Theorem 7. Then we have the formula*

$$(5.4) \quad \int_a^b f(t) dt = \tilde{T}_{m,n}(f, \xi, I_m) + \tilde{R}_{m,n}(f, \xi, I_m),$$

where the remainder satisfies the estimate

$$(5.5) \quad \begin{aligned} & \left| \tilde{R}_{m,n}(f, \xi, I_m) \right| \\ &:= \begin{cases} \frac{1}{(n+1)!} \|f^{(n)}\|_{\infty} \sum_{j=0}^{m-1} \left[ (\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1} \right], \\ \frac{1}{n!(nq+1)^{1/q}} \|f^{(n)}\|_p \left[ \sum_{j=0}^{m-1} (\xi_j - x_j)^{nq+1} + \sum_{j=0}^{m-1} (x_{j+1} - \xi_j)^{nq+1} \right]^{\frac{1}{q}}, \\ \frac{1}{n!} \|f^{(n)}\|_1 \left[ \frac{1}{2} \vartheta(h) + \max_{j=0, \dots, m-1} \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n. \end{cases}\end{aligned}$$

*Proof.* Apply the inequality (3.1) on the subinterval  $[x_j, x_{j+1}]$  to get

$$\begin{aligned} & \left| \int_{x_j}^{x_{j+1}} f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \right. \\ & \quad \times \left. \left[ (\xi_j - x_j)^{k+1} f^{(k)}(x_j) + (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1}) \right] \right| \\ & \leq \begin{cases} \frac{1}{(n+1)!} \sup_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| \left[ (\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1} \right], \\ \frac{1}{n!} \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} \left[ \frac{(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \\ \frac{1}{n!} \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \right) \left[ \frac{1}{2} h_j + \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n. \end{cases}\end{aligned}$$

Summing over  $j$  from 0 to  $m - 1$  and using the generalized triangle inequality, we have

$$\begin{aligned}
& \left| \tilde{R}_{m,n}(f, \xi, I_m) \right| \\
& \leq \left| \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \right. \\
& \quad \times \left. \left[ (\xi_j - x_j)^{k+1} f^{(k)}(x_j) + (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1}) \right] \right| \\
& = \begin{cases} \frac{1}{(n+1)!} \sum_{j=0}^{m-1} \sup_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| \left[ (\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1} \right], \\ \frac{1}{n!} \sum_{j=0}^{m-1} \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} \left[ \frac{(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \\ \frac{1}{n!} \sum_{j=0}^{m-1} \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \right) \left[ \frac{1}{2} h_j + \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n. \end{cases}
\end{aligned}$$

As  $\sup_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| \leq \|f^{(n)}\|_\infty$ , the first inequality is obvious.

Using the discrete Hölder inequality, we have

$$\begin{aligned}
& \frac{1}{(nq+1)^{1/q}} \sum_{j=0}^{m-1} \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} \left[ (\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1} \right]^{\frac{1}{q}} \\
& \leq \frac{1}{(nq+1)^{1/q}} \left[ \sum_{j=0}^{m-1} \left[ \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \right]^{\frac{1}{p}} \\
& \quad \times \left[ \sum_{j=0}^{m-1} \left[ \left[ (\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1} \right]^{\frac{1}{q}} \right]^q \right]^{\frac{1}{q}} \\
& = \frac{1}{(nq+1)^{1/q}} \|f^{(n)}\|_p \left[ \sum_{j=0}^{m-1} (\xi_j - x_j)^{nq+1} + \sum_{j=0}^{m-1} (x_{j+1} - \xi_j)^{nq+1} \right]^{\frac{1}{q}}
\end{aligned}$$

and the second inequality in (5.5) is proved.

Finally, let us observe that

$$\frac{1}{n!} \sum_{j=0}^{m-1} \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \right) \left[ \frac{1}{2} h_j + \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n \leq$$

$$\begin{aligned}
&\leq \max_{j=0, \dots, m-1} \left[ \frac{1}{2} h_j + \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n \sum_{j=0}^{m-1} \left( \int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \right) \\
&\leq \left[ \frac{1}{2} h_j + \max_{j=0, \dots, m-1} \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n \|f^{(n)}\|_1
\end{aligned}$$

and the last part of (5.5) is proved.  $\blacksquare$

**Remark 6.** As  $(x-a)^\alpha + (b-a)^\alpha \leq (b-a)^\alpha$  for  $\alpha \geq 1$ ,  $x \in [a, b]$ , then we remark that the first branch of (5.5) can be bounded by

$$(5.6) \quad \frac{1}{(n+1)!} \|f^{(n)}\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}.$$

The second branch can be upper bounded by

$$(5.7) \quad \frac{1}{n!(nq+1)^{1/q}} \|f^{(n)}\|_p \left[ \sum_{j=0}^{m-1} h_j^{nq+1} \right]^{\frac{1}{q}}$$

and finally, the last branch in (5.5) can be upper bounded by

$$(5.8) \quad \frac{1}{n!} [\vartheta(h)]^n \|f^{(n)}\|_1.$$

Note that all the bounds provided by (5.6)-(5.8) are uniform bounds for  $\tilde{R}_{m,n}(f, \xi, I_m)$  in terms of the intermediate points  $\xi$ .

The last inequality we can get from (5.5) is that one for which we have  $\xi_j = \frac{x_j + x_{j+1}}{2}$ . Consequently, we can state the following corollary (see also [13]):

**Corollary 5.** Let  $f$  be as in Theorem 8. Then we have the formula

$$(5.9) \quad \int_a^b f(t) dt = \tilde{T}_{m,n}(f, I_m) + \tilde{R}_{m,n}(f, I_m),$$

where

$$(5.10) \quad \tilde{T}_{m,n}(f, I_m) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{1}{2^{k+1}(k+1)!} \left[ f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1}) \right] h_j^{n+1}$$

and the remainder  $\tilde{R}$  satisfies the estimate

$$\left| \tilde{R}_{m,n}(f, I_m) \right| \leq \begin{cases} \frac{1}{2^n(n+1)!} \|f^{(n)}\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}, \\ \frac{1}{2^n n! (nq+1)^{1/q}} \|f^{(n)}\|_p \left[ \sum_{j=0}^{m-1} h_j^{nq+1} \right]^{\frac{1}{q}}, \\ \frac{1}{2^n n!} [\vartheta(h)]^n \|f^{(n)}\|_1. \end{cases}$$

**Remark 7.** Similar results can be stated by using the “perturbed” versions embodied in Theorems 4, 5 and 6, but we omit the details.

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