An Inequality of Ostrowski Type for Twice Differentiable Mappings in Terms of the Lp Norm and Applications

AN INEQUALITY OF OSTROWSKI TYPE FOR TWICE DIFFERENTIABLE MAPPINGS IN TERMS OF THE L_p NORM AND APPLICATIONS

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Abstract. An inequality of the Ostrowski type for twice differentiable mappings whose derivatives belong to $L_p(a,b)$, $1 < p < \infty$, and applications to special means and numerical integration are investigated.

1. Introduction

The following inequality is well known in the literature as Ostrowski's integral inequality (see for example [1, p. 468]).

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I) and let $a, b \in \mathring{I}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded, i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty$$

for all $x \in (a,b)$.
The constant $\frac{1}{4}$ is the best possible.

For a simple proof and some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [2] by S. S. Dragomir and S. Wang.

In [3], the same authors considered another inequality of Ostrowski type for the $\|\cdot\|_p$ −norm $(p > 1)$ as follows:

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ and $a, b \in \mathring{I}$ with $a < b$. If $f' \in L_p(a,b)$ $\left(p > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right)$ then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{b-a}\left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]\|f'\|_p \tag{1.1}$$

for all $x \in [a,b]$, where

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p\,dt \right)^{\frac{1}{p}},$$

is the $L_p(a,b)$ −norm.

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They also pointed out some applications of (1.1) in Numerical Integration as well as for special means.

In 1976, G. V. Milovanović and J. E. Pečarić proved a generalization of the Ostrowski inequality for $n-$times differentiable mappings (see for example [1, p. 468]). The case of twice differentiable mappings [1, p. 470] is as follows:

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that $f'' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{\|f''\|_\infty}{4}(b-a)^2\left[\frac{1}{12} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

for all $x \in [a,b]$.

In 1998, Cerone, Dragomir and Roumeliotis [4] proved the following inequality of Ostrowski type for mappings which are twice differentiable.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) and $f'' \in L_p(a,b)$ $(p > 1)$. Then we have the inequality:*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right)f'(x)\right| &\leq \frac{1}{2(b-a)(2q+1)^{\frac{1}{q}}}\left[(x-a)^{2q+1} + (b-x)^{2q+1}\right]^{\frac{1}{q}}\|f''\|_p \\ &\leq \frac{(b-a)^{1+\frac{1}{q}}\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}\end{aligned}$$

for all $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Dragomir and Sofo [5] proved the following inequality in the case where the second derivative belongs to the $L_\infty(\cdot,\cdot)$ norm.

Theorem 5. *Let $g : [a,b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $g'' \in L_\infty[a,b]$. Then we have the inequality*

$$\left|\int_a^b g(t)\,dt - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2}\right](b-a) + \frac{(b-a)}{2}\left(x - \frac{a+b}{2}\right)g'(x)\right| \leq \|g'\|_\infty\left(\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}\right) \tag{1.2}$$

for all $x \in [a,b]$.

In this paper we point out an inequality of Ostrowski type, different to that of Cerone, Dragomir and Roumeliotis [4], for twice differentiable mappings which is in terms of the $L_p(\cdot,\cdot)$ norm of the second derivative, g'', and apply it to special means.

2. The Main Theorem

The following theorem is now proved and later applied to special means.

Theorem 6. *Let $g : [a,b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a,b]$. If we assume that the second derivative $g'' \in L_p(a,b)$, $1 < p < \infty$, then we have the inequality*

$$\left|\int_a^b g(t)\,dt - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2}\right](b-a) + \frac{(b-a)}{2}\left(x - \frac{a+b}{2}\right)g'(x)\right| \tag{2.1}$$

$$\leq \frac{1}{2}\left(\frac{b-a}{2}\right)^{2+\frac{1}{q}}\|g''\|_p \times \begin{cases} \left[B(q+1,q+1) + B_{x_1}(q+1,q+1) + \Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}} \\ \text{for } x \in \left[a, \frac{a+b}{2}\right], \\ \left[B(q+1,q+1) + B_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}} \\ \text{for } x \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $q>1$, and $B(\cdot,\cdot)$ is the Beta function of Euler given by

$$B(l,s) = \int_0^1 t^{l-1}(1-t)^{s-1}\,dt,\ l,s>0,$$

$$B_r(l,s) = \int_0^r t^{l-1}(1-t)^{s-1}\,dt$$

is the incomplete Beta function,

$$\Psi_r(l,s) = \int_0^r t^{l-1}(1+t)^{s-1}\,dt$$

is a real positive valued integral,

$$x_1 = \frac{2(x-a)}{b-a},\ x_2 = 1-x_1,$$
$$x_3 = x_1 - 1,\ x_4 = 2 - x_1$$

and

$$\|g''\|_p := \left(\int_a^b |g''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

If we assume that $g'' \in L_1(a,b)$, then we have

$$\left|\int_a^b g(t)\,dt - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2}\right](b-a) + \frac{(b-a)}{2}\left(x - \frac{a+b}{2}\right)g'(x)\right| \tag{2.2}$$

$$\leq \frac{\|g''\|_1}{8}(b-a)^2,$$

where

$$\|g''\|_1 := \int_a^b |g''(t)|\,dt.$$

Proof. We begin with the proof of the following integral equality

$$f(x)=\frac{1}{b-a}\left(\int_a^b f(t)\,dt+\int_a^b p(x,t)\,f'(t)\,dt\right) \tag{2.3}$$

$\forall x\in[a,b]$, provided that f is absolutely continuous on $[a,b]$, and the kernel $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t):=\begin{cases} t-a \text{ if } & t\in[a,x], \\ t-b \text{ if } & t\in(x,b]; \end{cases}$$

where $t\in[a,b]$.

A proof of (2.3) may be found in the paper by Dragomir and Wang [2].

Now, if we choose $f(x)=\left(x-\frac{a+b}{2}\right)g'(x)$ and apply it in (2.3), we obtain, after a moderate amount of manipulation, details of which may be seen in a paper by Dragomir and Sofo [5], the integral equality

$$\begin{aligned}\int_a^b g(t)\,dt &= \frac{(b-a)}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right] \\ &-\frac{(b-a)}{2}\left(x-\frac{a+b}{2}\right)g'(x) \\ &+\frac{1}{2}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)\,dt,\end{aligned} \tag{2.4}$$

for all $x\in[a,b]$.

From (2.4), we have the inequality

$$\begin{aligned}&\left|\int_a^b g(t)\,dt-\frac{(b-a)}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right]+\frac{(b-a)}{2}\left(x-\frac{a+b}{2}\right)g'(x)\right| \\ &\le \frac{1}{2}\left|\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)\,dt\right|,\end{aligned} \tag{2.5}$$

whose left hand side is equivalent to that of (1.2).

From the right hand side of (2.5) we have, by Hölder's inequality

$$\begin{aligned}&\left|\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)\,dt\right| \\ &\le \left(\int_a^b |g''(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}} \\ &= \|g''\|_p\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}},\end{aligned}$$

and from (2.5) we obtain the inequality

$$\left|\int_a^b g(t)\,dt-\frac{(b-a)}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right]+\frac{(b-a)}{2}\left(x-\frac{a+b}{2}\right)g'(x)\right| \tag{2.6}$$
$$\le \frac{1}{2}\|g''\|_p\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}.$$

From the right hand side of (2.6) we may define

$$\begin{aligned} I \;:&= \int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt \\ &= \int_a^x (t-a)^q\left|t-\frac{a+b}{2}\right|^q dt+\int_x^b |t-b|^q\left|t-\frac{a+b}{2}\right|^q dt, \end{aligned} \tag{2.7}$$

such that we can identify two distinct cases.

(a) For $x\in\left[a,\frac{a+b}{2}\right]$

$$\begin{aligned} I_A &= \int_a^x (t-a)^q\left(\frac{a+b}{2}-t\right)^q dt+\int_x^{\frac{a+b}{2}} (b-t)^q\left(\frac{a+b}{2}-t\right)^q dt \\ &\quad+\int_{\frac{a+b}{2}}^b (b-t)^q\left(t-\frac{a+b}{2}\right)^q dt. \end{aligned}$$

Investigating the three separate integrals, we may evaluate as follows:

$$I_1=\int_a^x (t-a)^q\left(\frac{a+b}{2}-t\right)^q dt,$$

making the change of variable $t=a+\left(\frac{b-a}{2}\right)w$, we arrive at

$$\begin{aligned} I_1 &= \left(\frac{b-a}{2}\right)^{2q+1}\int_0^{x_1} w^q(1-w)^q\,dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} B_{x_1}(q+1,q+1), \end{aligned}$$

where $B_{x_1}(\cdot,\cdot)$ is the incomplete Beta function and $x_1=\frac{2(x-a)}{b-a}$.

$$I_2=\int_x^{\frac{a+b}{2}} (b-t)^q\left(\frac{a+b}{2}-t\right)^q dt,$$

making the change of variable $t=\frac{a+b}{2}-\left(\frac{b-a}{2}\right)w$, we obtain

$$\begin{aligned} I_2 &= \left(\frac{b-a}{2}\right)^{2q+1}\int_0^{x_2} w^q(1+w)^q\,dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1}\Psi_{x_2}(q+1,q+1), \end{aligned}$$

where

$$\Psi_{x_2}:=\int_0^{x_2} w^q(1+w)^q\,dw$$

and $x_2=\frac{a+b-2x}{b-a}=1-x_1$.

$$I_3=\int_{\frac{a+b}{2}}^{b}(b-t)^q\left(t-\frac{a+b}{2}\right)^q dt,$$

making the change of variable $t=\frac{a+b}{2}+\left(\frac{b-a}{2}\right)w$, we get

$$\begin{aligned} I_3 &= \left(\frac{b-a}{2}\right)^{2q+1}\int_0^{1} w^q(1-w)^q\,dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1}B(q+1,q+1), \end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function.

We may now write

$$\begin{aligned} I_A &= I_1+I_2+I_3 \\ &= \left(\frac{b-a}{2}\right)^{2q+1}\left[B_{x_1}(q+1,q+1)+\Psi_{x_2}(q+1,q+1)+B(q+1,q+1)\right] \end{aligned}$$

for $x\in\left[a,\frac{a+b}{2}\right]$.

(b) For $x\in\left(\frac{a+b}{2},b\right]$

$$\begin{aligned} I_B &= \int_a^{\frac{a+b}{2}}(t-a)^q\left(\frac{a+b}{2}-t\right)^q dt+\int_{\frac{a+b}{2}}^{x}(t-a)^q\left(t-\frac{a+b}{2}\right)^q dt \\ &\quad +\int_x^{b}(b-t)^q\left(t-\frac{a+b}{2}\right)^q dt. \end{aligned}$$

In a similar fashion to the previous case, we have

$$I_4=\int_a^{\frac{a+b}{2}}(t-a)^q\left(\frac{a+b}{2}-t\right)^q dt.$$

Letting $t=a+\left(\frac{b-a}{2}\right)w$, we obtain

$$\begin{aligned} I_4 &= \left(\frac{b-a}{2}\right)^{2q+1}\int_0^{1} w^q(1-w)^q\,dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1}B(q+1,q+1), \end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function.

$$I_5 = \int_{\frac{a+b}{2}}^{x} (t-a)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} + \left(\frac{b-a}{2}\right) w$, we arrive at

$$\begin{aligned} I_5 &= \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_3} w^q (1-w)^q \, dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} B_{x_3}(q+1,q+1), \end{aligned}$$

where $B_{x_3}(\cdot,\cdot)$ is the incomplete Beta function and $x_3 = x_1 - 1$.

$$I_6 = \int_{x}^{b} (b-t)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = b - \left(\frac{b-a}{2}\right) w$, we get

$$\begin{aligned} I_6 &= \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_4} w^q (1-w)^q \, dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} B_{x_4}(q+1,q+1), \end{aligned}$$

where $B_{x_4}(\cdot,\cdot)$ is the incomplete Beta function and $x_4 = 2 - x_1$.
Now

$$\begin{aligned} I_B &= I_4 + I_5 + I_6 \\ &= \left(\frac{b-a}{2}\right)^{2q+1} \left[B(q+1,q+1) + B_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1)\right] \end{aligned}$$

for $x \in \left(\frac{a+b}{2}, b\right]$,
and from (2.7)

$$\begin{aligned} I &= I_A + I_B \\ &= \left(\frac{b-a}{2}\right)^{2q+1} \begin{cases} B_{x_1}(q+1,q+1) + \Psi_{x_2}(q+1,q+1) + B(q+1,q+1) \\ \text{for } x \in \left[a, \frac{a+b}{2}\right], \\ B(q+1,q+1) + B_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1) \\ \text{for } x \in \left(\frac{a+b}{2}, b\right]. \end{cases} \end{aligned}$$

Utilizing (2.6), we obtain the result (2.1).
Using the inequality (2.5), we can also state that

$$\begin{aligned} &\left| \int_a^b g(t)\,dt - \frac{b-a}{2}\left[g(x) + \frac{g(a)+g(b)}{2}\right](b-a) + \frac{(b-a)}{2}\left(x - \frac{a+b}{2}\right) g'(x) \right| \\ \leq\ & \frac{1}{2} \left\|g''\right\|_1 \left\|K(x,t)\right\|_\infty, \end{aligned}$$

where

$$K(x,t) := p(x,t)\left(t - \frac{a+b}{2}\right).$$

As it is easy to see that

$$\|K(x,t)\|_{\infty} = \frac{(b-a)^2}{4},\ x \in [a,b],$$

we deduce (2.2). ∎

Remark 1. *The inequality* (2.1) *may be rewritten as follows*

$$\left| g(x) + \frac{g(a)+g(b)}{2} - \left(x - \frac{a+b}{2}\right) g'(x) - \frac{2}{b-a}\int_a^b g(t)\,dt \right| \tag{2.8}$$
$$\leq \frac{1}{2}\left(\frac{b-a}{2}\right)^{1+\frac{1}{q}} \|g''\|_p \times \begin{cases} \left[B(q+1,q+1) + B_{x_1}(q+1,q+1) + \Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}} \\ \text{for } x \in \left[a, \frac{a+b}{2}\right], \\ \left[B(q+1,q+1) + B_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}} \\ \text{for } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Choosing $x = a$, *we obtain, from* (2.8), $x_1 = 0$ *and* $x_2 = 1$ *so that*

$$\left| \frac{3g(a)+g(b)}{2} + \frac{(b-a)}{2} g'(a) - \frac{2}{b-a}\int_a^b g(t)\,dt \right| \tag{2.9}$$
$$\leq \frac{1}{2}\left(\frac{b-a}{2}\right)^{1+\frac{1}{q}} \|g''\|_p \left[B(q+1,q+1) + \Psi_1(q+1,q+1)\right]^{\frac{1}{q}}.$$

Choosing $x = b$, *we obtain from* (2.8) $x_3 = 1$, *and* $x_4 = 0$ *so that*

$$\left| \frac{g(a)+3g(b)}{2} - \frac{(b-a)}{2} g'(b) - \frac{2}{b-a}\int_a^b g(t)\,dt \right| \tag{2.10}$$
$$\leq \frac{1}{4}(b-a)^{1+\frac{1}{q}} \|g''\|_p B^{\frac{1}{q}}(q+1,q+1).$$

At the midpoint $x = \frac{a+b}{2}$, *we obtain the best estimator so that* $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$ *and*

$$\left| g\left(\frac{a+b}{2}\right) + \frac{g(a)+g(b)}{2} - \frac{2}{b-a}\int_a^b g(t)\,dt \right|$$
$$\leq \frac{1}{4}(b-a)^{1+\frac{1}{q}} \|g''\|_p B^{\frac{1}{q}}(q+1,q+1).$$

Assuming the inequalities (2.9) *and* (2.10) *, using the triangle inequality and dividing by* 4*, we obtain a perturbed trapezoid formula:-*

$$\left|\frac{g(a)+g(b)}{2}-\frac{(b-a)}{8}\left(g'(b)-g'(a)\right)-\frac{1}{b-a}\int_a^b g(t)\,dt\right|$$
$$\leq \frac{1}{8}\left(\frac{b-a}{2}\right)^{1+\frac{1}{q}}\|g''\|_p\left[\left(1+2^{\frac{1}{q}}\right)B^{\frac{1}{q}}(q+1,q+1)+\Psi_1^{\frac{1}{q}}(q+1,q+1)\right].$$

The following particular case for Euclidean norms $p=q=2$ is of particular importance.

Corollary 1. *Let* $g:[a,b]\to\mathbb{R}$ *be as in Theorem 6 and* $g''\in L_2(a,b)$*. Using the result* (2.1)*, we have the inequality*

$$\left|\int_a^b g(t)\,dt-\frac{(b-a)}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right]+\frac{(b-a)}{2}\left(x-\frac{a+b}{2}\right)g'(x)\right| \tag{2.11}$$
$$\leq \frac{(b-a)^{\frac{1}{2}}}{2}\|g''\|_2\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^4+\frac{1}{480}(b-a)^4\right]^{\frac{1}{2}}$$

for all $x\in[a,b]$*.*

Proof. Applying inequality (2.1) for $p=q=2$, we obtain, after a moderate amount of manipulation (or simply by directly integrating the expression (2.7)),

$$\left|\int_a^b g(t)\,dt-\frac{(b-a)}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right]+\frac{(b-a)}{2}\left(x-\frac{a+b}{2}\right)g'(x)\right| \tag{2.12}$$
$$\leq \frac{\|g''\|_2}{2}\left\{\frac{(b-a)}{60}\left[30x^4-60x^3(a+b)+45x^2(a+b)^2\right.\right.$$
$$\left.\left.-15x(a+b)^3+2a^4+7a^3b+12a^2b^2+7ab^3+2b^4\right]\right\}^{\frac{1}{2}}.$$

for $x\in[a,b]$.

Let $x:=\tau+\frac{a+b}{2}$ and $A:=\frac{a+b}{2}$ so that

$$\begin{aligned}30x^4-120x^3A+180x^2A^2-120xA^3 &= 30\tau^4-30A^4\\ &= 30\left(x-\frac{a+b}{2}\right)^4-\frac{15}{8}(a+b)^4.\end{aligned}$$

Now, from the inner bracket of (2.12), we have

$$\begin{aligned}&30x^4-120x^3A+180x^2A^2-120xA^3+2a^4+7a^3b+12a^2b^2+7ab^3+2b^4\\ =\ &30\left(x-\frac{a+b}{2}\right)^4+\frac{1}{8}(b-a)^4.\end{aligned}$$

and the inequality (2.11) follows. ∎

3. Applications For Some Special Means.

Let us recall the following means:

(a) The *Arithmetic mean*:

$$A=A(a,b):=\frac{a+b}{2},\ a,b\geq 0.$$

(b) The *Geometric mean*:

$$G = G(a,b) := \sqrt{ab},\ a,b \geq 0.$$

(c) The *Harmonic mean*:

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}},\ a,b > 0.$$

(d) The *Logarithmic mean*:

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a=b \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b \end{cases},\ a,b>0.$$

(e) The *Identric mean*:

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b \end{cases},\ a,b>0.$$

(f) The $p-$*logarithmic mean*:

$$L_p = L_p(a,b) := \begin{cases} a & \text{if} \quad a=b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \end{cases},\ a,b>0$$

where $p \in \mathbb{R}\setminus\{-1,0\}$. The following is well known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also well known that L_p is monotonically increasing over $p \in \mathbb{R}$ (assuming that $L_0 := I$ and $L_{-1} := L$).

The inequality (2.8) may be rewritten as:

$$\left|g(x) + \frac{g(a)+g(b)}{2} - (x - A(a,b))\, g'(x) - \frac{2}{b-a}\int_a^b g(t)\,dt\right| \tag{3.1}$$

$$\leq \frac{1}{2}\left(\frac{b-a}{2}\right)^{1+\frac{1}{q}} \|g''\|_p$$

$$\times \begin{cases} \left[B(q+1,q+1) + B_{x_1}(q+1,q+1) + \Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}} \\ \text{for } x \in \left[a, \frac{a+b}{2}\right], \\ \left[B(q+1,q+1) + B_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}} \\ \text{for } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

We may now apply (3.1) to deduce some inequalities for special means given above, by the use of some particular mappings as follows.

(i). Consider $g(x)=\ln x,\ x\in[a,b]\subset(0,\infty)$. Then

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(t)\,dt &= \ln I(a,b),\\ \frac{g(a)+g(b)}{2} &= \ln G(a,b)\end{aligned}$$

and

$$\|g''\|_p=\left(\int_a^b |g''(t)|^p\,dt\right)^{\frac{1}{p}}=(b-a)^{\frac{1}{p}}L_{-2p}^{-2}(a,b).$$

From (3.1),

$$\begin{aligned}&\left|\ln x+\ln G(a,b)-\left(1-\frac{A(a,b)}{x}\right)-2\ln I(a,b)\right|\\ \leq\ & 2^{\frac{1}{p}-1}\left(\frac{b-a}{2}\right)^2 L_{-2p}^{-2}(a,b)\\ &\times\begin{cases}\left[B(q+1,q+1)+B_{x_1}(q+1,q+1)+\Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}}\\ \text{for } x\in\left[a,\frac{a+b}{2}\right],\\ \left[B(q+1,q+1)+B_{x_3}(q+1,q+1)+B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}}\\ \text{for } x\in\left(\frac{a+b}{2},b\right].\end{cases}\end{aligned}$$

(ii). Consider $g(x)=\frac{1}{x},\ x\in(a,b)\subset(0,\infty)$

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(t)\,dt &= L^{-1}(a,b),\\ \frac{g(a)+g(b)}{2} &= \frac{A(a,b)}{G^2(a,b)}\end{aligned}$$

and

$$\|g''\|_p=2(b-a)^{\frac{1}{p}}L_{-3p}^{-1}(a,b).$$

From (3.1)

$$\begin{aligned}&\left|\frac{1}{x}\left(2-\frac{A(a,b)}{x}\right)+\frac{A(a,b)}{G^2(a,b)}-2L^{-1}(a,b)\right|\\ \leq\ & 2^{\frac{1}{p}}\left(\frac{b-a}{2}\right)^2 L_{-3p}^{-1}(a,b)\\ &\times\begin{cases}\left[B(q+1,q+1)+B_{x_1}(q+1,q+1)+\Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}}\\ \text{for } x\in\left[a,\frac{a+b}{2}\right],\\ \left[B(q+1,q+1)+B_{x_3}(q+1,q+1)+B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}}\\ \text{for } x\in\left(\frac{a+b}{2},b\right].\end{cases}\end{aligned}$$

(iii). Consider $g(x) = x^r,\ g : (0,\infty) \to \mathbb{R}$ where $r \in \mathbb{R}\backslash\{-1,0\}$. Then, for $a<b$

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(t)\,dt &= L_r^r(a,b,), \\ \frac{g(a)+g(b)}{2} &= A(a^r,b^r)\end{aligned}$$

and

$$\|g''\|_p = |r(r-1)|(b-a)^{\frac{1}{p}} L_{p(r-2)}^{r-2}(a,b).$$

From (3.1),

$$\begin{aligned}&\left|x^{r-1}\{rA(a,b)+(1-r)x\}+A(a^r,b^r)-2L_r^r(a,b)\right| \\ \leq\ & |r(r-1)|\,2^{\frac{1}{p}-1}\left(\frac{b-a}{2}\right)^2 L_{p(r-2)}^{r-2}(a,b) \\ & \times\begin{cases}[B(q+1,q+1)+B_{x_1}(q+1,q+1)+\Psi_{x_2}(q+1,q+1)]^{\frac{1}{q}} \\ \text{for } x\in\left[a,\frac{a+b}{2}\right], \\ [B(q+1,q+1)+B_{x_3}(q+1,q+1)+B_{x_4}(q+1,q+1)]^{\frac{1}{q}} \\ \text{for } x\in\left(\frac{a+b}{2},b\right].\end{cases}\end{aligned}$$

4. Applications in Numerical Integration for the $L_2(a,b)$ Norm

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a subdivision of the interval $\xi_i \in [x_i, x_{i+1}]$, $i = 0,1,...,n-1$. We have the following quadrature formula.

Theorem 7. *Let $g : [a,b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative, $g'' \in L_2(a,b)$. Then the following perturbed Riemann type quadrature formula holds.*

$$\int_a^b g(x)\,dx = A(g,g',\xi,I_n)+R(g,g',\xi,I_n), \tag{4.1}$$

where $A(g,g',\xi,I_n)$ is given by

$$\begin{aligned}&A(g,g',\xi,I_n) \\ =\ & -\frac{1}{2}\sum_{i=0}^{n-1}h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)g'(\xi_i)+\frac{1}{2}\sum_{i=0}^{n-1}h_i\left[g(\xi_i)+\frac{g(x_i)+g(x_{i+1})}{2}\right]\end{aligned}$$

and the remainder $R(g,g',\xi,I_n)$ satisfies the estimation

$$|R(g,g',\xi,I_n)| \leq \frac{\|g''\|_2}{2}\left[\sum_{i=0}^{n-1}\frac{h_i}{2}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^4+\sum_{i=0}^{n-1}\frac{h_i^5}{480}\right]^{\frac{1}{2}}$$

for all $\xi_i \in [x_i, x_{i+1}]$.

Proof. Applying inequality (2.11) on the interval $[x_i, x_{i+1}]$, we obtain

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} g(t)\,dt - \frac{h_i}{2}\left[g(\xi_i) + \frac{g(x_i)+g(x_{i+1})}{2}\right] \right. \\ & \left. + \frac{h_i}{2}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right) g'(\xi_i) \right| \\ \leq & \frac{\|g''\|_2}{2}\left\{\frac{h_i}{60}\left[30\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^4 + \frac{1}{8}h_i^4\right]\right\}^{\frac{1}{2}} \end{aligned}$$

for all $i = 0, 1, ..., n-1$.

Summing over i from 0 to $n-1$, using the triangle inequality and Cauchy-Schwartz's discrete inequality, we obtain

$$\begin{aligned} & |R(g, g', \xi, I_n)| \\ \leq & \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} g(t)\,dt - \frac{h_i}{2}\left[g(\xi_i) + \frac{g(x_i)+g(x_{i+1})}{2}\right] \right. \\ & \left. + \frac{h_i}{2}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right) g'(\xi_i) \right| \\ \leq & \frac{1}{2}\sum_{i=0}^{n-1}\left\{\frac{h_i}{60}\left[30\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^4 + \frac{h_i^4}{8}\right]\right\}^{\frac{1}{2}} \left(\int_{x_i}^{x_{i+1}} |g''(t)|^2\,dt\right)^{\frac{1}{2}} \\ \leq & \frac{1}{2}\left(\sum_{i=0}^{n-1}\left(\left\{\frac{h_i}{60}\left[30\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^4 + \frac{h_i^4}{8}\right]\right\}^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\ & \times \left(\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}} |g''(t)|^2\,dt\right)^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\ = & \frac{\|g''\|_2}{2}\left(\sum_{i=0}^{n-1}\frac{h_i}{2}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^4 + \sum_{i=0}^{n-1}\frac{h_i^5}{480}\right)^{\frac{1}{2}} \end{aligned}$$

and the theorem is proved. ∎

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