Generalization of H. Minc and L. Sathre's Inequality

GENERALIZATION OF H. MINC AND L. SATHRE'S INEQUALITY

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ABSTRACT. In the article, an inequality of H. Minc and L. Sathre (*Proc. Edinburgh Math. Soc.* **14** (1964/65), 41–46) is generalized: Let n and m be natural numbers, k a nonnegative integer, then we have

$$\frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < 1.$$

From this, some corollaries are deduced. At last, an open problem is proposed.

It is known that, for $n \in \mathbb{N}$, the following inequalities were given in [3]:

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1. \tag{1}$$

In [1], the left inequality in (1) was refined by

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{2}$$

for all positive real numbers r. Both bounds are best possible.

In this article, using analytic method, we obtain

Theorem. *Let n and m be natural numbers, k a nonnegative integer. Then we have*

$$\frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < 1. \tag{3}$$

Proof. The upper bound is obtained immediately from

$$\frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} = \left[\left(\prod_{i=k+1}^{n+k} i\right)^m \Bigg/ \left(\prod_{i=n+k+1}^{n+m+k} i\right)^n\right]^{1/n(n+m)} < 1.$$

The left inequality in (3) can be rearranged as

$$\frac{n+k}{\sqrt[n]{(n+k)!/k!}} < \frac{n+m+k}{\sqrt[n+m]{(n+m+k)!/k!}},$$

this is equivalent to

$$\frac{n+k}{\sqrt[n]{(n+k)!/k!}} < \frac{n+k+1}{\sqrt[n+1]{(n+k+1)!/k!}}. \tag{4}$$

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When $k = 0$, inequality (4) follows from the left inequality in (1). When $k1$, the inequality (4) can be rewritten as

$$\left[\frac{(n+k)!}{k!}\right]^{1/n} > \frac{(n+k)^{n+1}}{(n+k+1)^n}. \tag{5}$$

In [4, p. 184], the following inequalities were given for $n \in \mathbb{N}$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\frac{1}{12n}. \tag{6}$$

By substituting the inequalities in (6) into the left term of inequality (5), we see that it is sufficient to prove

$$\left[\sqrt{2\pi(n+k)}\left(\frac{n+k}{e}\right)^{n+k}\right]^{1/n} > \frac{(n+k)^{n+1}}{(n+k+1)^n}\left[\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \exp\frac{1}{12k}\right]^{1/n}. \tag{7}$$

Simplifying (7) directly and standard arguments leads to

$$n\ln\left(1+\frac{1}{n+k}\right) + \frac{2k+1}{2n}\ln\left(1+\frac{n}{k}\right) - \frac{1}{12kn} - 1 > 0. \tag{8}$$

In [2, pp. 367–368], [4, pp. 273–274] and [8], we have for $t > 0$

$$\ln\left(1+\frac{1}{t}\right) > \frac{2}{2t+1}.$$

Thus, to get inequality (8), it suffices to show

$$\frac{2n}{2(n+k)+1} + \frac{2k+1}{2n}\cdot\frac{2n}{2k+n} - \frac{1}{12kn} - 1 > 0.$$

But this is equivalent to

$$2(12k^2-1)n^2 + (12kn-1)n + 4(6n-1)k^2 + 2(3n-1)k > 0.$$

The proof is complete. ∎

Corollary 1. *For any given nonnegative integer k, the sequences*

$$\sqrt[n]{(n+k)!/k!}, \qquad \frac{n+k}{\sqrt[n]{(n+k)!/k!}},$$

$$\frac{(n+k)\sqrt[n+1]{(n+k+1)!/k!}}{\sqrt[n]{(n+k)!/k!}}, \qquad \frac{(n+k+1)\sqrt[n]{(n+k)!/k!}}{\sqrt[n+1]{(n+k+1)!/k!}}$$

are strictly increasing with respect to $n \in \mathbb{N}$.

Corollary 2. *For any given $n \in \mathbb{N}$, the sequences*

$$\sqrt[n]{(n+k)!/k!}, \quad \frac{(n+k)\sqrt[n+1]{(n+k+1)!/k!}}{\sqrt[n]{(n+k)!/k!}}, \quad \frac{(n+k+1)\sqrt[n]{(n+k)!/k!}}{\sqrt[n+1]{(n+k+1)!/k!}}$$

are strictly increasing with respect to the nonnegative integers k.

Remark. Recently, the first author in [5] and [7], among other things, generalized the left inequality in (2) in new directions and got that, if n and m are natural numbers, k is a nonnegative integer, then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{9}$$

where r is any given positive real number. The lower bound is best possible.

In [6], the first author further presented that, let n and m be natural numbers, suppose $a = (a_1, a_2, \dots)$ is a positive and increasing sequence satisfying

$$a_{k+1}^2 a_k a_{k+2}, \tag{10}$$

$$\frac{a_{k+1} - a_k}{a_{k+1}^2 - a_k a_{k+2}} \max\left\{\frac{k+1}{a_{k+1}}, \frac{k+2}{a_{k+2}}\right\} \tag{11}$$

for $k \in \mathbb{N}$, then the inequality

$$\frac{a_n}{a_{n+m}} \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r}. \tag{12}$$

holds for any given positive real number $r \in \mathbb{R}$. The lower bound of (12) is best possible.

Using L'Hospital rule yields

$$\lim_{r \to 0}\left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r} = \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}, \tag{13}$$

thus, we propose the following

Open Problem. *Let n and m be natural numbers, k a nonnegative integer. Then, for all real numbers $r > 0$, we have*

$$\left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}. \tag{14}$$

The upper bound is best possible.

References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.

[2] Ji-Chang Kuang, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[3] H. Minc and L. Sathre, *Some inequalities involving* $(r!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1964/65), 41–46.

[4] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York/Heidelberg/Berlin, 1970.

[5] Feng Qi, *An algebraic inequality*, submitted for publication.

[6] Feng Qi, *Further generalization of H. Alzer's inequality*, submitted for publication.

[7] Feng Qi, *Generalization of H. Alzer's inequality*, submitted for publication.

[8] Feng Qi and Chao-Ping Chen, *Monotonicities of two sequences*, Mathematics and Informatics Quarterly (1999), to appear.

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