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This is the Published version of the following publication

Barnett, Neil S and Dragomir, Sever S (1999) Some Inequalities for Random Variables whose Probability Density Functions are Bounded using a Pre-Grüss Inequality. RGMIA research report collection, 2 (6).

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**SOME INEQUALITIES FOR RANDOM VARIABLES WHOSE
PROBABILITY DENSITY FUNCTIONS ARE BOUNDED USING
A PRE-GRÜSS INEQUALITY**

N. S. BARNETT AND S. S. DRAGOMIR

ABSTRACT. Using the pre-Grüss inequality considered by Matic-Pecari-Ujevic in a recent paper [1] and some related results, we point out some inequalities for random variables whose p.d.f.'s are bounded above and below by the assumed known constants γ and ϕ .

1. INTRODUCTION

In a recent paper [1], Matic, Pecarić and Ujević proved the following inequality, which has been called, in [2], the *pre-Grüss inequality*

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{1}{2}(\phi - \gamma) \left[\frac{1}{b-a} \int_a^b g^2(t) dt - \left(\frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right]^{\frac{1}{2}},$$

provided that $\gamma \leq f(t) \leq \phi$ a.e. on $[a, b]$ and the integrals exist and are finite.

In [1], the authors used (1.1) to obtain some bounds for the remainder in certain Taylor like formulae whilst in [2], the authors applied (1.1) to estimation of the remainder in three point quadrature formulae.

Basically, (1.1) is a pre-Grüss inequality since, if we assume that $\alpha \leq g(t) \leq \beta$ a.e. in $[a, b]$, then, by the well known fact that (see for example [8])

$$(1.2) \quad \frac{1}{b-a} \int_a^b g^2(t) dt - \left(\frac{1}{b-a} \int_a^b g(t) dt \right)^2 \leq \frac{1}{4}(\beta - \alpha)^2,$$

and, by (1.1) and (1.2), we can deduce the original *Grüss inequality*:

$$(1.3) \quad \left| \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{1}{4}(\phi - \gamma)(\beta - \alpha).$$

Date: October 11, 1999.

1991 Mathematics Subject Classification. Primary 26D15; Secondary 65Xxx.

Key words and phrases. Pre-Grüss Inequality, Random Variables, Expectation, Cumulative Distribution Function.

In [1], Matic, Pečarić and Ujević observed that if a factor is known, for example $g(t)$, $t \in [a, b]$, then instead of using (1.3) in estimating the difference

$$\frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt,$$

it is better to use (1.1), as they have shown in their paper that it improves some recent results by the second author [4].

In this paper, by using the same approach, we obtain some inequalities for the expectation $E(X)$ and cumulative distribution function $F(\cdot)$ of a random variable having the probability distribution function $f : [a, b] \rightarrow \mathbb{R}$. It is assumed that we know the lower and the upper bound for f , i.e., the real numbers γ, ϕ such that $0 \leq \gamma \leq f(t) \leq \phi \leq 1$ a.e. t on $[a, b]$. Some related results are also established.

2. SOME INEQUALITIES FOR EXPECTATION AND DISPERSION

We start with the following result for expectation.

Theorem 1. *Let X be a random variable having the probability density function $f : [a, b] \rightarrow \mathbb{R}$. Assume that there exists the constants γ, ϕ such that $0 \leq \gamma \leq f(t) \leq \phi \leq 1$ a.e. t on $[a, b]$. Then we have the inequality*

$$(2.1) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2,$$

where $E(X)$ is the expectation of the random variable X .

Proof. If we put $g(t) = t$ in (1.1), we obtain

$$(2.2) \quad \left| \frac{1}{b-a} \int_a^b tf(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b t dt \right| \\ \leq \frac{1}{2} (\phi - \gamma) \left[\frac{1}{b-a} \int_a^b t^2 dt - \left(\frac{1}{b-a} \int_a^b t dt \right)^2 \right]^{\frac{1}{2}}$$

and as

$$\int_a^b tf(t) dt = E(X), \\ \int_a^b f(t) dt = 1, \quad \frac{1}{b-a} \int_a^b t dt = \frac{a+b}{2}$$

and

$$\frac{1}{b-a} \int_a^b t^2 dt - \left(\frac{1}{b-a} \int_a^b t dt \right)^2 = \frac{(b-a)^2}{12},$$

then by (2.2) we deduce (2.1). ■

To point out a result for the p -moments of the random variable X , $p \in \mathbb{R} \setminus \{-1, 0\}$, we need the following p -Logarithmic mean

$$M_p(a, b) := \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}},$$

where $0 < a < b$.

Theorem 2. Let X and f be as in Theorem 1 and $E_p(X)$ be the p -moment of X , i.e.,

$$E_p(X) := \int_a^b t^p f(t) dt,$$

which is assumed to be finite.

Then we have the inequality

$$(2.3) \quad |E_p(X) - M_p^p(a, b)| \leq \frac{1}{2} (\phi - \gamma) \left[M_{2p}^{2p}(a, b) - M_p^{2p}(a, b) \right]^{\frac{1}{2}}.$$

The proof is obvious by the inequality (1.1) in which we choose $g(t) = t^p$, $p \in \mathbb{R} \setminus \{-1, 0\}$ and use the definition of p -logarithmic means.

If we consider the Logarithmic mean

$$M_{-1}(a, b) := L(a, b) = \frac{b-a}{\ln b - \ln a}, \quad 0 < a < b$$

and define the (-1) -moment of the random variable X by

$$E_{-1}(X) := \int_a^b \frac{f(t)}{t} dt,$$

then we can also state the following theorem.

Theorem 3. Let X and f be as in Theorem 1. Then we have the inequality:

$$(2.4) \quad |E_{-1}(X) - M_{-1}^{-1}(a, b)| \leq \frac{1}{2} (\phi - \gamma) \left[M_{-2}^{-2}(a, b) - M_{-1}^{-2}(a, b) \right]^{\frac{1}{2}},$$

provided the (-1) -moment of X is finite.

The proof is obvious by the inequality (1.1) and we omit the details.

The following theorem also holds.

Theorem 4. Let X and f be as above. If

$$\sigma_\mu(X) := \left[\int_a^b (t - \mu)^2 f(t) dt \right]^{\frac{1}{2}}, \quad \mu \in [a, b],$$

then we have the inequality

$$(2.5) \quad \left| \sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right| \\ \leq \frac{1}{2} (\phi - \gamma) (b-a)^2 \left[\frac{1}{3} \left(\mu - \frac{a+b}{2} \right)^2 + \frac{1}{180} (b-a)^2 \right]^{\frac{1}{2}} \\ \leq \frac{1}{3\sqrt{5}} (\phi - \gamma) (b-a)^3.$$

Proof. If we put $g(t) = (t - \mu)^2$ in (1.1) we get

$$(2.6) \quad \left| \frac{1}{b-a} \int_a^b f(t) (t - \mu)^2 dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b (t - \mu)^2 dt \right| \\ \leq \frac{1}{2} (\phi - \gamma) \left[\frac{1}{b-a} \int_a^b (t - \mu)^4 dt - \left(\frac{1}{b-a} \int_a^b (t - \mu)^2 dt \right)^2 \right]^{\frac{1}{2}},$$

and as

$$\begin{aligned}
\int_a^b f(t) dt &= 1, \\
\frac{1}{b-a} \int_a^b (t-\mu)^2 dt &= \frac{(b-\mu)^3 + (\mu-a)^3}{3(b-a)} \\
&= \frac{(b-\mu)^2 - (b-\mu)(\mu-a) + (\mu-a)^2}{3} \\
&= \left(\mu - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}, \\
\frac{1}{b-a} \int_a^b (t-\mu)^4 dt - \left(\frac{1}{b-a} \int_a^b (t-\mu)^2 dt\right)^2 &= \frac{(b-\mu)^5 + (\mu-a)^5}{5(b-a)} - \left[\frac{(b-\mu)^3 + (\mu-a)^3}{3(b-a)}\right]^2 \\
&= \frac{(b-\mu)^4 - (b-\mu)^3(\mu-a) + (b-\mu)^2(\mu-a)^2 - (b-\mu)(\mu-a)^3 + (\mu-a)^4}{5} \\
&\quad - \left[\frac{(b-\mu)^2 - (b-\mu)(\mu-a) + (\mu-a)^2}{3}\right]^2 \\
&= \frac{1}{45} \left[9(b-\mu)^4 - (b-\mu)^3(\mu-a) + (b-\mu)^2(\mu-a)^2 \right. \\
&\quad \left. - (b-\mu)(\mu-a)^3 + (\mu-a)^4 - 5(b-\mu)^4 - 5(b-\mu)^2(\mu-a)^2 \right. \\
&\quad \left. - 5(\mu-a)^4 + 10(b-\mu)^3(\mu-a) + 10(b-\mu)(\mu-a)^3 - 10(b-\mu)^2(\mu-a)^2 \right] \\
&= \frac{1}{45} \left[4(b-\mu)^4 + 4(\mu-a)^4 - 8(b-\mu)^2(\mu-a)^2 + 2(b-\mu)^3(\mu-a)^2 \right. \\
&\quad \left. + (\mu-a)(b-\mu) \left[(b-\mu)^2 + (\mu-a)^2 \right] \right] \\
&= \frac{1}{45} \left[4 \left[(b-\mu)^2 - (\mu-a)^2 \right]^2 \right. \\
&\quad \left. + 2(b-\mu)^2(\mu-a)^2 + (\mu-a)(b-\mu) \left[(b-\mu)^2 + (\mu-a)^2 \right] \right] \\
&: = A.
\end{aligned}$$

However,

$$\begin{aligned}
(b-\mu)^2 - (\mu-a)^2 &= (b-a)(b+a-2\mu) = 2(b-a) \left(\frac{b+a}{2} - \mu \right), \\
(b-\mu)(\mu-a) &= \frac{1}{4}(b-a)^2 - \left(\mu - \frac{a+b}{2} \right)^2, \\
(b-\mu)^2 + (\mu-a)^2 &= \frac{1}{2}(b-a)^2 + 2 \left(\mu - \frac{a+b}{2} \right)^2.
\end{aligned}$$

Denote $\delta := b - a$ and $x = \mu - \frac{a+b}{2}$. Then we get

$$\begin{aligned} 45A &= 4(2\delta x)^2 + 2\left(\frac{1}{4}\delta^2 - x^2\right)^2 + \left(\frac{1}{4}\delta^2 - x^2\right)\left(\frac{1}{2}\delta + 2x^2\right) \\ &= 16\delta^2 x^2 + \left(\frac{1}{4}\delta^2 - x^2\right)\left[2\left(\frac{1}{4}\delta^2 - x^2\right) + \frac{1}{2}\delta + 2x^2\right] \\ &= \delta^2\left(15x^2 + \frac{1}{4}\delta^2\right). \end{aligned}$$

Then

$$\begin{aligned} A &= \frac{(b-a)^2}{45}\left[15\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \\ &= (b-a)^2\left[\frac{1}{3}\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2\right]. \end{aligned}$$

Using the inequality (2.6), we deduce the desired inequality (2.5). ■

The best inequality we can obtain from (2.5) is that one for which $\mu = \frac{a+b}{2}$ and therefore, we can state the following corollary.

Corollary 1. *With the above assumptions and denoting $\sigma_0(X) := \sigma_{\frac{a+b}{2}}(X)$, we have the inequality:*

$$(2.7) \quad \left| \sigma_0^2(X) - \frac{(b-a)^2}{12} \right| \leq \frac{1}{12\sqrt{5}}(\phi - \gamma)(b-a)^3.$$

The following theorem also holds.

Theorem 5. *Let X and f be as above. If*

$$A_\mu(X) := \int_a^b |t - \mu| f(t) dt, \quad \mu \in [a, b],$$

then we have the inequality

$$(2.8) \quad \left| A_\mu(X) - \frac{1}{b-a} \left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2} \right)^2 \right] \right| \\ \leq \frac{1}{2}(\phi - \gamma)(b-a) \left[\frac{(b-a)^2}{48} + \left(\frac{\mu - \frac{a+b}{2}}{b-a} \right)^2 \left[\frac{1}{2}(b-a)^2 + \left(\mu - \frac{a+b}{2} \right)^2 \right] \right]^{\frac{1}{2}}.$$

for all $\mu \in [a, b]$.

Proof. If we put $g(t) = |t - \mu|$ in (1.1), we have

$$(2.9) \quad \left| \frac{1}{b-a} \int_a^b |t - \mu| f(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b |t - \mu| dt \right| \\ \leq \frac{1}{2}(\phi - \gamma) \left[\frac{1}{b-a} \int_a^b |t - \mu|^2 dt - \left(\frac{1}{b-a} \int_a^b |t - \mu| dt \right)^2 \right]^{\frac{1}{2}}$$

and as

$$\begin{aligned}
\int_a^b f(t) dt &= 1, \\
\frac{1}{b-a} \int_a^b |t-\mu| dt &= \frac{1}{b-a} \left[\int_a^\mu (\mu-t) dt + \int_\mu^b (t-\mu) dt \right] \\
&= \frac{1}{b-a} \left[\frac{(b-\mu)^2 + (\mu-a)^2}{2} \right] \\
&= \frac{1}{b-a} \left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2} \right)^2 \right], \\
\frac{1}{b-a} \int_a^b (t-\mu)^2 dt &= \frac{(b-\mu)^3 + (\mu-a)^3}{3(b-a)} \\
&= \frac{(b-a)^2}{12} + \left(\mu - \frac{a+b}{2} \right)^2, \\
\frac{1}{b-a} \int_a^b |t-\mu|^2 dt - \left(\frac{1}{b-a} \int_a^b |t-\mu| dt \right)^2 &= \frac{(b-a)^2}{12} + \left(\mu - \frac{a+b}{2} \right)^2 - \left[\frac{(b-a)}{4} + \frac{1}{b-a} \left(\mu - \frac{a+b}{2} \right)^2 \right]^2 \\
&= \frac{(b-a)^2}{48} + \frac{1}{2} \left(\mu - \frac{a+b}{2} \right)^2 - \frac{1}{(b-a)^2} \left(\mu - \frac{a+b}{2} \right)^4 \\
&= \frac{(b-a)^2}{48} + \left(\frac{\mu - \frac{a+b}{2}}{b-a} \right)^2 \left[\frac{1}{2} (b-a)^2 - \left(\mu - \frac{a+b}{2} \right)^2 \right].
\end{aligned}$$

Finally, using (2.9) we deduce the desired inequality. ■

The best inequality we can get from (2.8) is embodied in the following corollary.

Corollary 2. *The best inequality we can get from (2.8) is for $\mu = \mu_0 := \frac{a+b}{2}$, obtaining*

$$(2.10) \quad \left| A_{\mu_0}(X) - \frac{b-a}{4} \right| \leq \frac{1}{8\sqrt{3}} (\phi - \gamma) (b-a)^2.$$

Proof. Consider the mapping $g(\mu) := \frac{(b-a)^2}{48} + \frac{1}{2} \left(\mu - \frac{a+b}{2} \right)^2 - \frac{1}{(b-a)^2} \left(\mu - \frac{a+b}{2} \right)^4$.

We have

$$\begin{aligned}
\frac{dg(\mu)}{d\mu} &= \left(\mu - \frac{a+b}{2} \right) - \frac{4}{(b-a)^2} \left(\mu - \frac{a+b}{2} \right)^3 \\
&= \left(\mu - \frac{a+b}{2} \right) \left[1 - \frac{4}{(b-a)^2} \left(\mu - \frac{a+b}{2} \right)^2 \right].
\end{aligned}$$

We observe that $\frac{dg(\mu)}{d\mu} = 0$ if $\mu = a$ or $\mu = \frac{a+b}{2}$ or $\mu = b$ and as

$$\frac{dg(\mu)}{d\mu} < 0 \text{ for } \mu \in \left(a, \frac{a+b}{2}\right) \text{ and } \frac{dg(\mu)}{d\mu} > 0 \text{ for } \mu \in \left(\frac{a+b}{2}, b\right),$$

we deduce that $\mu = \frac{a+b}{2}$ is the point realizing the global minimum on (a, b) and as $g(\mu_0) = \frac{(b-a)^2}{48}$, the inequality (2.10) is indeed the best inequality we can get from (2.8). ■

Another inequality which can be useful for obtaining different inequalities for dispersion is the following weighted Grüss type result (see for example [8] or [6]).

Lemma 1. *Let $g, p : [a, b] \rightarrow \mathbb{R}$ be measurable functions and such that $\alpha \leq g \leq \beta$ a.e., $p \geq 0$ a.e. on $[a, b]$ and $\int_a^b p(x) dx > 0$. Then*

$$(2.11) \quad 0 \leq \frac{\int_a^b p(x) g^2(x) dx}{\int_a^b p(x) dx} - \left(\frac{\int_a^b p(x) g(x) dx}{\int_a^b p(x) dx} \right)^2 \leq \frac{1}{4} (\beta - \alpha)^2,$$

provided that all the integrals in (2.11) exist and are finite.

Using the above lemma we shall be able to prove the following result for dispersion.

Theorem 6. *Let X be a random variable whose probability density function f is defined on the finite interval $[a, b]$ and $\sigma(X) < \infty$. Then we have the inequality*

$$(2.12) \quad 0 \leq \sigma_\mu^2(X) - (E(X) - \mu)^2 \leq \frac{1}{4} (b - a)^2$$

for all $\mu \in [a, b]$, or, equivalently,

$$(2.13) \quad 0 \leq \sigma(X) \leq \frac{1}{2} (b - a).$$

Proof. Let us choose in (2.11), $g(x) = x - \mu$, $p(x) = f(x)$. Then, obviously, $\sup_{x \in [a, b]} g(x) = b - \mu$, $\inf_{x \in [a, b]} g(x) = a - \mu$, $\int_a^b f(x) dx = 1$, and then by (2.11), we get

$$0 \leq \int_a^b (x - \mu)^2 f(x) dx - \left(\int_a^b (x - \mu) f(x) dx \right)^2 \leq \frac{1}{4} (b - a)^2$$

and the inequality (2.12) is proved. ■

The following inequality connecting $\sigma_\mu(X)$ and $A_\mu(X)$ also holds.

Theorem 7. *Let X be as in Theorem 6 and assume that $\sigma_\mu(X)$, $A_\mu(X) < \infty$ for all $\mu \in [a, b]$. Then we have the inequality*

$$(2.14) \quad 0 \leq \sigma_\mu^2(X) - A_\mu^2(X) \leq \frac{1}{2} \left| \mu - \frac{a+b}{2} \right|$$

for all $\mu \in [a, b]$.

Proof. Choose in Lemma 1, $p(x) = f(x)$, $g(x) = |x - \mu|$, $\mu \in [a, b]$. Then

$$\begin{aligned}\beta &= \sup_{x \in [a, b]} g(x) = \max\{\mu - a, b - \mu\} = \frac{b - a + |\mu - a - b + \mu|}{2}, \\ \alpha &= \inf_{x \in [a, b]} g(x) = \min\{\mu - a, b - \mu\} = \frac{b - a - |\mu - a - b + \mu|}{2},\end{aligned}$$

which gives us

$$\beta - \alpha = 2 \left| \mu - \frac{a + b}{2} \right|.$$

Applying (2.11), we deduce (2.14). ■

3. SOME INEQUALITIES FOR THE CUMULATIVE DISTRIBUTION FUNCTION

The following theorem contains an inequality which connects the expectation $E(X)$, the cumulative distributive function $F(X) := \int_a^x f(t) dt$ and the bounds γ and ϕ of the probability density function $f : [a, b] \rightarrow \mathbb{R}$.

Theorem 8. *Let X , f , $E(X)$, $F(\cdot)$ and γ, ϕ be as above. Then we have the inequality:*

$$(3.1) \quad \left| E(X) + (b - a)F(x) - x - \frac{b - a}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b - a)^2,$$

for all $x \in [a, b]$.

Proof. We have the following equality established by Barnett and Dragomir in [3]

$$(3.2) \quad \begin{aligned}(b - a)F(x) + E(X) - b &= \int_a^b p(x, t) dF(t) \\ &= \int_a^b p(x, t) f(t) dt,\end{aligned}$$

where

$$p(x, t) := \begin{cases} t - a & \text{if } a \leq t \leq x \leq b \\ t - b & \text{if } a \leq x < t \leq b \end{cases}.$$

Now, if we apply the inequality (1.1) for $g(t) = p(x, t)$, we get

$$(3.3) \quad \begin{aligned}&\left| \frac{1}{b - a} \int_a^b p(x, t) f(t) dt - \frac{1}{b - a} \int_a^b p(x, t) dt \cdot \frac{1}{b - a} \int_a^b f(t) dt \right| \\ &\leq \frac{1}{2} (\phi - \gamma) \left[\frac{1}{b - a} \int_a^b p^2(x, t) dt - \left(\frac{1}{b - a} \int_a^b p(x, t) dt \right)^2 \right]^{\frac{1}{2}}.\end{aligned}$$

Now, we observe that

$$\begin{aligned}\frac{1}{b - a} \int_a^b p(x, t) dt &= x - \frac{a + b}{2}, \\ \int_a^b f(t) dt &= 1,\end{aligned}$$

and

$$\begin{aligned}
D & : = \frac{1}{b-a} \int_a^b p^2(x,t) dt - \left(\frac{1}{b-a} \int_a^b p(x,t) dt \right)^2 \\
& = \frac{1}{b-a} \left[\frac{(b-x)^3 + (x-a)^3}{3} \right] - \left(x - \frac{a+b}{2} \right)^2 \\
& = \frac{(b-x)^2 - (b-x)(x-a) + (x-a)^2}{3} - \left(x - \frac{a+b}{2} \right)^2.
\end{aligned}$$

As a simple calculation shows that

$$\begin{aligned}
& (b-x)^2 - (b-x)(x-a) + (x-a)^2 \\
& = 3 \left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4} (b-a)^2,
\end{aligned}$$

then we get

$$D = \frac{1}{12} (b-a)^2.$$

Using (3.3), we deduce (3.1). ■

Remark 1. If in (3.1) we choose either $x = a$ or $x = b$, we get

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2,$$

which is the inequality (2.1).

Remark 2. If in (3.1) we choose $x = \frac{a+b}{2}$, then we get the inequality

$$(3.4) \quad \left| E(X) + (b-a) \Pr \left(X \leq \frac{a+b}{2} \right) - b \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2.$$

The following theorem also holds.

Theorem 9. Let X , f , γ , ϕ and $F(\cdot)$ be as above. Then we have the inequality:

$$\begin{aligned}
(3.5) \quad & \left| E(X) + \frac{b-a}{2} F(x) - \frac{b+x}{2} \right| \\
& \leq \frac{1}{2\sqrt{3}} (\phi - \gamma) \left[\frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \\
& \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2,
\end{aligned}$$

for all $x \in [a, b]$.

Proof. We use the following identity proved by Barnett and Dragomir in [3]

$$\begin{aligned}
(3.6) \quad (b-a) F(x) + E(X) - b & = \int_a^x (t-a) dF(t) + \int_x^b (t-b) dF(t) \\
& = \int_a^x (t-a) f(t) dt + \int_x^b (t-b) f(t) dt
\end{aligned}$$

for all $x \in [a, b]$.

Applying the pre-Grüss inequality (1.1), we get for $x \in [a, b]$

$$\begin{aligned}
 (3.7) \quad & \left| \frac{1}{x-a} \int_a^x (t-a) f(t) dt - \frac{1}{x-a} \int_a^x (t-a) dt \cdot \frac{1}{x-a} \int_a^x f(t) dt \right| \\
 & \leq \frac{1}{2} (\phi - \gamma) \left[\frac{1}{x-a} \int_a^x (t-a)^2 dt - \left[\frac{1}{x-a} \int_a^x (t-a) dt \right]^2 \right]^{\frac{1}{2}} \\
 & = \frac{1}{4\sqrt{3}} (\phi - \gamma) (x-a)
 \end{aligned}$$

and, similarly, we have

$$\begin{aligned}
 (3.8) \quad & \left| \frac{1}{b-x} \int_x^b (t-b) f(t) dt - \frac{1}{b-x} \int_x^b (t-b) dt \cdot \frac{1}{b-x} \int_x^b f(t) dt \right| \\
 & \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-x), \quad x \in (a, b).
 \end{aligned}$$

From (3.7) and (3.8) we can write

$$(3.9) \quad \left| \int_a^x (t-a) f(t) dt - \frac{x-a}{2} F(x) \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (x-a)^2$$

and

$$(3.10) \quad \left| \int_x^b (t-b) f(t) dt + \frac{b-x}{2} (1-F(x)) \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-x)^2,$$

for all $x \in [a, b]$.

Summing (3.9) and (3.10) and using the triangle inequality, we deduce that

$$\begin{aligned}
 (3.11) \quad & \left| \int_a^x (t-a) f(t) dt + \int_x^b (t-b) f(t) dt - \frac{b-a}{2} F(x) + \frac{b-x}{2} \right| \\
 & \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) [(x-a)^2 + (b-x)^2] \\
 & = \frac{1}{2\sqrt{3}} (\phi - \gamma) \left[\frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right].
 \end{aligned}$$

Using the identity (3.6), we deduce the desired inequality (3.5). ■

Remark 3. If we choose in (3.5), either $x = a$ or $x = b$, we get the inequality

$$(3.12) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2$$

and thus recapture (2.1).

Remark 4. If we choose in (3.5), $x = \frac{a+b}{2}$, then we get

$$(3.13) \quad \left| E(X) + \left(\frac{b-a}{2} \right) \Pr \left(X \leq \frac{a+b}{2} \right) - \frac{a+3b}{4} \right| \leq \frac{1}{8\sqrt{3}} (\phi - \gamma) (b-a)^2,$$

which is the best inequality that can be obtained.

REFERENCES

- [1] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, On new estimation of the remainder in generalized Taylor's formula, *Math. Ineq. & Appl.*, **3,2**(1999), 343-361.
- [2] P. CERONE and S.S. DRAGOMIR, Three Point Quadrature Rules Involving, at Most, a First Derivative, *RGMA Res. Rep. Coll.*, **4,2**(1999), Article 8. <http://melba.vu.edu.au/~rgmia/v2n4.html>
- [3] N.S. BARNETT and S.S. DRAGOMIR, An Inequality of Ostrowski's Type for Cumulative Distribution Functions, *RGMA Re. Rep. Coll.*, **1,1**(1998), Article 1., <http://melba.vu.edu.au/~rgmia/v1n1.html>
- [4] S.S. DRAGOMIR, New Esttimation of the Remainder in Taylor's Formula using Grüss' Type Inequalities and Applications, *Math. Ineq. & Appl.*, **2,2**(1999), 183-194.
- [5] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, *accepted*.
- [6] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of r -Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, (*in press*)
- [7] S.S. DRAGOMIR, Some discrete inequalities of Grüss type and applications in guessing theory, *submitted*.
- [8] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.*, (*accepted*).

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