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INEQUALITIES FOR A WEIGHTED MULTIPLE INTEGRAL

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ABSTRACT. In the article, using Taylor's formula for functions of several variables, the author establishes some inequalities for the weighted multiple integral of a function defined on an m -dimensional rectangle, if its partial derivatives of $(n + 1)$ -th order remain between bounds. From which Iyengar's inequality is generalized and related results in references could be deduced.

1. MAIN RESULTS

For given points $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ and $a_i < b_i$, $i = 1, 2, \dots, m$, denote the m -rectangles by

$$(1) \quad Q_m = \prod_{i=1}^m [a_i, b_i], \quad Q_m(t) = \prod_{i=1}^m [a_i, c_i(t)],$$

where $c_i(t) = (1 - t)a_i + tb_i$, $i = 1, 2, \dots, m$, $t \in [0, 1]$.

Let $\nu = (\nu_1, \dots, \nu_m)$ be a multi-index, that is, $\nu_i = \text{integer} \geq 0$, with $|\nu| = \sum_{i=1}^m \nu_i$. Let f be a function of several variables defined on Q_m , and its partial derivatives of $(n+1)$ -th order remain between the upper and lower bounds $M_{n+1}(\nu)$ and $N_{n+1}(\nu)$ as follows

$$(2) \quad N_{n+1}(\nu) \leq D^\nu f(x) \leq M_{n+1}(\nu), \quad x \in Q_m,$$

where we define

$$(3) \quad D^\nu f(x) = \partial^{n+1} f(x) \Big/ \prod_{i=1}^m \partial x_i^{\nu_i}.$$

Let $w(x) \geq 0$ be an integrable function of several variables defined on the m -rectangle Q_m , which is not identically zero for $x \in Q_m$. Define

$$(4) \quad h_{s,\nu}(t) = \int_{Q_m(t)} w(x) \prod_{i=1}^m (x_i - s_i)^{\nu_i} dx,$$

where $s = (s_1, s_2, \dots, s_m) \in \mathbb{R}^m$, $t \in [0, 1]$.

In this article, using Taylor's formula for functions with several variables, we obtain some inequalities for a weighted multiple integral $\int_{Q_m} w(x)f(x) dx$ with weight $w(x) \geq 0$ on the m -rectangle Q_m in terms of the values of the partial

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derivatives of the function f at points a and b and the bounds $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ of $D^\nu f(x)$, that is

Main Theorem. *Let $f \in C^{n+1}(Q_m)$ and $N_{n+1}(\boldsymbol{\nu}) \leq D^\nu f(x) \leq M_{n+1}(\boldsymbol{\nu})$ hold for any $x \in Q_m$ and $|\boldsymbol{\nu}| = n + 1$, where $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ are constants depending on n and $\boldsymbol{\nu}$. Let $w(x)$ be an integrable function of several variables over Q_m , which is not identically zero. Then, for any $t \in (0, 1)$,*

(i) *if n is an even, we have*

$$\begin{aligned}
(5) \quad & \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(1) - N_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^m (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^m (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
& \leq \int_{Q_m} w(x)f(x) dx - \sum_{k=0}^n \sum_{|\boldsymbol{\nu}|=k} \frac{D^\nu f(b)}{\prod_{i=1}^m (\nu_i!)} [h_{b,\boldsymbol{\nu}}(1) - h_{b,\boldsymbol{\nu}}(t)] - \sum_{k=0}^n \sum_{|\boldsymbol{\nu}|=k} \frac{D^\nu f(a)}{\prod_{i=1}^m (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
& \leq \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(1) - M_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^m (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^m (\nu_i!)} h_{a,\boldsymbol{\nu}}(t);
\end{aligned}$$

(ii) *if n is an odd,*

$$\begin{aligned}
(6) \quad & \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(1) - M_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^m (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^m (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
& \leq \int_{Q_m} w(x)f(x) dx - \sum_{k=0}^n \sum_{|\boldsymbol{\nu}|=k} \frac{D^\nu f(b)}{\prod_{i=1}^m (\nu_i!)} [h_{b,\boldsymbol{\nu}}(1) - h_{b,\boldsymbol{\nu}}(t)] - \sum_{k=0}^n \sum_{|\boldsymbol{\nu}|=k} \frac{D^\nu f(a)}{\prod_{i=1}^m (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
& \leq \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(1) - N_{n+1}(\boldsymbol{\nu})h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^m (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^m (\nu_i!)} h_{a,\boldsymbol{\nu}}(t).
\end{aligned}$$

2. PROOF OF MAIN THEOREM

Let $t \in (0, 1)$ be a parameter, and write

$$(7) \quad \int_{Q_m} w(x)f(x) dx = \int_{Q_m(t)} w(x)f(x) dx + \int_{Q_m \setminus Q_m(t)} w(x)f(x) dx.$$

The well-known Taylor's formula for a multivariable function states that

$$(8) \quad f(x) = \sum_{k=0}^n \frac{1}{k!} \left(\sum_{i=1}^m (x_i - a_i) \frac{\partial}{\partial x_i} \right)^k f(a) + R_n(x),$$

$$(9) \quad f(x) = \sum_{k=0}^n \frac{1}{k!} \left(\sum_{i=1}^m (x_i - b_i) \frac{\partial}{\partial x_i} \right)^k f(b) + r_n(x),$$

where

$$(10) \quad R_n(x) = \frac{1}{(n+1)!} \left(\sum_{i=1}^m (x_i - a_i) \frac{\partial}{\partial x_i} \right)^{n+1} f(a + \theta(x - a)), \quad \theta \in (0, 1),$$

$$(11) \quad r_n(x) = \frac{1}{(n+1)!} \left(\sum_{i=1}^m (x_i - b_i) \frac{\partial}{\partial x_i} \right)^{n+1} f(b + \mu(x - b)), \quad \mu \in (0, 1).$$

Since

$$(12) \quad \left(\sum_{i=1}^m q_i \right)^k = k! \sum_{|\nu|=k} \prod_{i=1}^m \frac{q_i^{\nu_i}}{\nu_i!},$$

integrating on both sides of (8) over $Q_m(t)$ gives us

$$\begin{aligned} & \int_{Q_m(t)} w(x) f(x) dx \\ &= \sum_{k=0}^n \frac{1}{k!} \int_{Q_m(t)} w(x) \left(\sum_{i=1}^m (x_i - a_i) \frac{\partial}{\partial x_i} \right)^k f(a) dx + \int_{Q_m(t)} w(x) R_n(x) dx \\ &= \sum_{k=0}^n \sum_{|\nu|=k} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^m \left((x_i - a_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(a) dx \\ (13) \quad & + \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^m \left((x_i - a_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(a + \theta(x - a)) dx \\ &= \sum_{k=0}^n \sum_{|\nu|=k} \frac{1}{\prod_{i=1}^m (\nu_i!)} \frac{\partial^k f(a)}{\prod_{i=1}^m \partial x_i^{\nu_i}} \int_{Q_m(t)} w(x) \prod_{i=1}^m (x_i - a_i)^{\nu_i} dx \\ & + \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^m (x_i - a_i)^{\nu_i} \cdot \frac{\partial^{n+1} f(a + \theta(x - a))}{\prod_{i=1}^m \partial x_i^{\nu_i}} dx \\ &= \sum_{k=0}^n \sum_{|\nu|=k} \frac{D^\nu f(a)}{\prod_{i=1}^m (\nu_i!)} h_{a,\nu}(t) \\ & + \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^m (x_i - a_i)^{\nu_i} D^\nu f(a + \theta(x - a)) dx. \end{aligned}$$

Using inequality (2) and computing directly yields

$$\begin{aligned}
& \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})}{m} h_{a,\boldsymbol{\nu}}(t) \\
(14) \quad & \leq \sum_{|\boldsymbol{\nu}|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^m (x_i - a_i)^{\nu_i} D^{\boldsymbol{\nu}} f(a + \theta(x - a)) dx \\
& \leq \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})}{m} h_{a,\boldsymbol{\nu}}(t).
\end{aligned}$$

The combination of (13) and (14) leads to

$$\begin{aligned}
& \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})}{m} h_{a,\boldsymbol{\nu}}(t) \\
(15) \quad & \leq \int_{Q_m(t)} w(x) f(x) dx - \sum_{k=0}^n \sum_{|\boldsymbol{\nu}|=k} \frac{D^{\boldsymbol{\nu}} f(a)}{\prod_{i=1}^m (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
& \leq \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})}{m} h_{a,\boldsymbol{\nu}}(t).
\end{aligned}$$

Integrating (9) on the domain $Q_m \setminus Q_m(t)$, we arrive at

$$\begin{aligned}
& \int_{Q_m \setminus Q_m(t)} w(x) f(x) dx \\
&= \sum_{k=0}^n \frac{1}{k!} \int_{Q_m \setminus Q_m(t)} w(x) \left(\sum_{i=1}^m (x_i - b_i) \frac{\partial}{\partial x_i} \right)^k f(b) dx + \int_{Q_m \setminus Q_m(t)} r_n(x) dx \\
&= \sum_{k=0}^n \sum_{|\nu|=k} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m} w(x) \prod_{i=1}^m \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b) dx \\
&\quad - \sum_{k=0}^n \sum_{|\nu|=k} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^m \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b) dx \\
(16) \quad &+ \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m} w(t) \prod_{i=1}^m \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x - b)) dx \\
&\quad - \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(t) \prod_{i=1}^m \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x - b)) dx \\
&= \sum_{k=0}^n \sum_{|\nu|=k} \frac{D^\nu f(b)}{\prod_{i=1}^m (\nu_i!)} [h_{b,\nu}(1) - h_{b,\nu}(t)] \\
&\quad + \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m} w(t) \prod_{i=1}^m \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x - b)) dx \\
&\quad - \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(t) \prod_{i=1}^m \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x - b)) dx.
\end{aligned}$$

Similar to the deduction of (14), if n is an odd, we have

$$\begin{aligned}
& \sum_{|\nu|=n+1} \frac{N_{n+1}(\nu)}{\prod_{i=1}^m (\nu_i!)} h_{b,\nu}(t) \\
(17) \quad &\leq \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(t) \prod_{i=1}^m \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x - b)) dx \\
&= \sum_{|\nu|=n+1} \frac{1}{\prod_{i=1}^m (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^m (x_i - b_i)^{\nu_i} D^\nu f(b + \mu(x - b)) dx \\
&\leq \sum_{|\nu|=n+1} \frac{M_{n+1}(\nu)}{\prod_{i=1}^m (\nu_i!)} h_{b,\nu}(t);
\end{aligned}$$

if n is even, the reversed inequalities in (17) hold. Note that $Q_m(1) = Q_m$.

Substituting (17) into (16) we have that, if n is an odd number, then

$$\begin{aligned}
 & \sum_{|\nu|=n+1} \frac{N_{n+1}(\nu)h_{b,\nu}(1) - M_{n+1}(\nu)h_{b,\nu}(t)}{\prod_{i=1}^m (\nu_i!)} \\
 (18) \quad & \leq \int_{Q_m \setminus Q_m(t)} w(x)f(x) dx - \sum_{k=0}^n \sum_{|\nu|=k} \frac{D^\nu f(b)}{\prod_{i=1}^m (\nu_i!)} [h_{b,\nu}(1) - h_{b,\nu}(t)] \\
 & \leq \sum_{|\nu|=n+1} \frac{M_{n+1}(\nu)h_{b,\nu}(1) - N_{n+1}(\nu)h_{b,\nu}(t)}{\prod_{i=1}^m (\nu_i!)};
 \end{aligned}$$

if n is an even number, then the inequalities in (18) are reversed.

By addition of inequalities (15) and (18), the Main Theorem was proved. ■

Remark 1. It is noted that we also can consider the similar estimates for the weighted multiple integral $\int_{Q_m} w(x)f(x) dx$ on the m -dimensional ball centered at a with radius $|b - a|$, that is, $Q_m = B_a(|b - a|)$, $a, b \in \mathbb{R}^m$.

Remark 2. In the Main Theorem, if we take $m = 1$, we can obtain the results in [14]; if we set $m = 1$ and $w(x) = 1$, then we get the results in [12]; if we let $w(x) = 1$, we have the results in [13]. In particular, if we take $w(x) = 1$, $m = 1$ and $n = 0$, the Iyengar inequality [6] is deduced, which has been generalized by many mathematicians in [1, 2, 3, 4, 5, 8, 11, 15] (also see [7, 9, 10]).

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